Lower critical dimensions for superconducting long-range order in type-II superconductors

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It is shown that phase fluctuations destroy off-diagonal long-range order (phase coherence) in the Meissner phase below two dimensions, below four dimensions in the conventional Abrikosov flux-lattice phase, and below three dimensions in the limit of infinite Ginzburg ratio κ. The destruction of long-range order in the flux lattice is due to phase changes induced by shear motions of the flux lines. The phase coherence decays over a long length scale (of order millimeters) in three-dimensional systems.

I. INTRODUCTION

The form of the phase diagram of a type II superconductor in a magnetic field has again been of interest, since the discovery of high-temperature superconductors. At the level of mean-field theory there are just two phases besides the normal phase. For fields $H < H_{c1}(T)$ there is the Meissner phase in which all flux is excluded from the sample, while for fields $H_{c1}(T) < H < H_{c2}(T)$ there is the mixed phase in which Abrikosov showed that the flux lines penetrated the sample to form a regular flux-line lattice.[1] The thermal fluctuations about the mean-field solution are very much larger in high-temperature superconductors than in conventional superconductors because of their much shorter correlation lengths and the effects of these fluctuations have caused controversies. For example, it has been argued that the fluctuations could lead to the melting of the flux lattice to a flux-line liquid state.[2–8]

In this paper the effects of the fluctuations on the phase coherence or off-diagonal long-range order (ODLRO) are examined both in the Meissner and mixed states. It will be shown that phase fluctuations destroy ODLRO below two dimensions in the Meissner phase. In the mixed phase, the phase fluctuations induced by the thermal excitation of the shear modes of motion of the flux lattice are shown to destroy ODLRO below *four* dimensions. I shall take the loss of ODLRO to be of physical significance (even though it is not directly measurable) and draw the conclusion that its loss implies that the lower critical dimension of the Meissner phase is two, and the lower critical dimension of the conventional mixed phase is four. Hence, barring exotic possibilities (see below), the only genuine phases in a pure bulk type-II superconductor are the normal and Meissner phases, just as in a type-I superconductor.

Previously the effects of phase fluctuations in the mixed state were examined using the approximation where the magnetic induction **B** is assumed to be spatially uniform.[4,9,10] This is only valid within superconductors if the Ginzburg parameter κ is infinite. Actually, this limit has direct physical relevance to rotating neutral superfluids such as ^{4}He and possibly neutron star matter.[11] The previous claims[4] that *three* is the lower critical dimension for ODLRO for this case are confirmed, which removes doubts as to whether further *ad hoc* approximations made in the original papers, such as truncation of the order parameter ψ to the lowest-Landau-level solution of the linearized Ginzburg-Landau equation, were responsible for the unusual value for the lower critical dimension.

Houghton, Pelcovits, and Sudbø[12] (HPS) also recently pointed out that phase fluctuations apparently destroyed ODLRO in the mixed phase below four dimensions, but they regarded this as merely a consequence of studying an inherently non-gauge-invariant quantity such as the phase of the order parameter ψ and proposed instead a gauge-invariant definition of ODLRO, which is not destroyed by the phase fluctuations accompanying shear motions of the flux lattice. In Sec. II, I show that if the definition of ODLRO employed by HPS is used in the Meissner phase, ODLRO is apparently destroyed by fluctuations in the transverse component of the superfluid velocity in any dimension. Furthermore, in Sec. III, I will argue that fluctuations of the transverse component of the superfluid velocity also destroy ODLRO, as defined by them, within the mixed phase in any dimension. As a consequence of this, I have constructed another gauge-invariant definition of ODLRO, and a gauge invariant phase (which turns out to be the phase of ψ within the conventional London gauge[11]) and show that its fluctuations do indeed destroy ODLRO below four dimensions in the mixed phase. It is similar to the definition of ODLRO given by HPS, but involves instead only the longitudinal component of the superfluid velocity.

If superconducting order (ODLRO) is absent in the mixed state in two dimensions (films) and three dimensions (bulk), what is it then that is being observed experimentally in type-II materials? Since ODLRO is destroyed by the phase fluctuations associated with the long-wavelength shear modes of motion of the flux lines, any pinning of the flux lines by disorder, etc., will inhibit and slow the phase fluctuations. Once the flux lines are pinned, long-wavelength shear modes become impossible, and the phase fluctuations responsible for the destruction of superconducting order are removed. Hence, our calculations are only directly relevant to situations in which disorder, which is always present, can be regarded as a weak perturbation. Furthermore, our calculations show that for bulk systems the decay of ODLRO takes place over a length scale, which is typically of order millimeters. Thus, even in the complete absence of disorder there will be few easily observed experimental consequences of the loss of ODLRO. However, for thin films ODLRO decays on the length scale of the average separation of the flux lines. Its loss is likely to result in the formation of a vortex liquid state.

It is possible that the absence of ODLRO below four dimensions does not necessarily rule out a genuine phase transition to a state lacking ODLRO but possessing, say, a flux lattice or even hexatic order. The possibility of a flux lattice without ODLRO has been envisaged by Fisher and Lee,[13] but described by them as "exotic." There is, of course, an example where the loss of ODLRO is not accompanied by the absence of a phase transition, viz., ^{4}He films, which undergo, at low temperatures, a phase transition to a state with no ODLRO but a nonvanishing superfluid density. My chief aim in this paper is to demonstrate the absence of ODLRO below four dimensions. What happens in its absence is essentially left to future studies.

The plan of the paper is as follows. In Sec. II, the Ginzburg-Landau formalism is set up and the mean-field equations written down. The equations that describe the fluctuations about the mean-field solution are obtained in a gauge-invariant form and solved for the Meissner phase. A gauge-invariant phase is constructed, and a gauge-invariant definition of ODLRO is given. Within the Meissner phase it is shown that the thermal fluctuations of the gauge-invariant phase cause a loss of ODLRO below two dimensions, while fluctuations would destroy ODLRO, as defined by HPS, in all dimensions. It is concluded that the gauge-invariant phase defined by HPS is unphysical. Section III starts with a discussion of the two Goldstone modes associated with small displacements of the flux line lattice, which are essentially the transverse and longitudinal elastic modes of motion of the lattice. The tilt modulus c_{44} of the flux lattice is obtained exactly from the Ginzburg-Landau formalism. It is then argued that ODLRO as defined by HPS is destroyed by the transverse components of the superfluid velocity in any dimension within the mixed phase. I show next that thermal excitation of the shear modes destroys my definition of ODLRO only below four dimensions. In Sec. IV, the case of infinite κ is investigated. It differs from the case of finite κ in that in this limit there is only one Goldstone mode, which is associated with the shear elastic mode of the flux lattice and its dispersion curve is found. Its unusual form is the origin of the destruction of ODLRO below three dimensions in the infinite κ limit. Finally, in Sec. V, I discuss further the possible experimental implications of the loss of ODLRO.

II. GINZBURG-LANDAU MODEL AND FORMULISM

A. The free-energy functional

The Ginzburg-Landau free energy for an isotropic superconductor, written in reduced units is[1]

$$F=\int d^3r\left[-|\psi|^2+\tfrac{1}{2}|\psi|^4+\left|\left(\frac{\nabla}{i\kappa}-\mathbf{A}\right)\psi\right|^2+\mathbf{B}^2\right] . \tag{2.1}$$

Here lengths are measured in units of the zero-field penetration depth λ, magnetic fields in units of $\sqrt{2}H_c$, the vector potential $\mathbf{A}$ in units of $\sqrt{2}H_c\lambda$ and the free-energy density in units of $H_c^2/4\pi$. The magnetic induction $\mathbf{B}=\nabla\times\mathbf{A}$. The external field $\mathbf{H}$ will be taken to be along the z axis. The Ginzburg ratio $\kappa=\lambda/\xi$, where ξ is the zero-field correlation length. Real high-T_c materials are highly anisotropic. This introduces interesting complications, which are planned to be discussed elsewhere.[14]

The superconductor's order parameter ψ is a complex scalar, which in terms of its amplitude f and phase Φ can be written $\psi=f\exp(i\Phi)$. We shall work throughout within a gauge invariant formalism. To this end it is convenient to introduce the gauge-invariant superfluid velocity

$$\mathbf{Q}=\mathbf{A}-\frac{1}{\kappa}\nabla\Phi . \tag{2.2}$$

Then the free energy becomes, in terms of f and $\mathbf{Q}$,

$$F=\int d^3r\left[-f^2+\frac{1}{2}f^4+\frac{1}{\kappa^2}(\nabla f)^2+f^2\mathbf{Q}^2+\mathbf{B}^2\right] . \tag{2.3}$$

The mean-field description of the superconductor results when the free energy of Eq. (2.3) is minimized with respect to f and $\mathbf{A}$. The functions f_0, $\mathbf{A}_0$, and $\mathbf{Q}_0$ which give the minimum F_0 of F are solutions of

$$-\frac{1}{\kappa^2}\nabla^2 f_0-f_0+f_0^3+f_0\mathbf{Q}_0^2=0 , \tag{2.4}$$

$$\nabla\times\nabla\times\mathbf{A}_0=-f_0^2\mathbf{Q}_0 . \tag{2.5}$$

There are three solutions of Eqs. (2.4) and (2.5). One solution corresponds to the normal phase and has $f_0=0$, $\mathbf{Q}_0=0$, $\mathbf{B}=\mathbf{H}$. In the Meissner phase $\mathbf{B}_0=0$, $\mathbf{Q}_0=0$, and $f_0(x,y,z)=1$. The solution of Eqs. (2.4) and (2.5) for the mixed or Abrikosov flux-line lattice phase cannot be obtained analytically, but the essential features of their solution are[1] that the flux lines—the zeros of f—lie on the vertices of a triangular lattice and are parallel to the field direction $\mathbf{H}$, i.e., the z axis. The magnetic induction $\mathbf{B}_0(x,y)$ is parallel to $\mathbf{H}$ and is largest at the cores of the flux lines, where $f_0(x,y)=0$. The lattice constant l (the spacing between flux lines) in nonreduced units is given by the flux quantization condition that the area of the unit cell, $\sqrt{3}l^2/2$ equals $\hat{\Phi}_0/B$, where $\hat{\Phi}_0$ is the flux quantum and $B=\overline{B_0(x,y)}$, the bar indicating here and in what follows a spatial average over the unit cell of the flux lattice.

The component of the superfluid velocity field along the z axis, $Q_{z,0}$ is zero in the mean-field solution. Furthermore, from Eq. (2.2)

$$\nabla\times\mathbf{Q}_0=\nabla\times\mathbf{A}_0-\frac{1}{\kappa}\nabla\times\nabla\Phi_0 \tag{2.6}$$

$$=\left[B_0(x,y)-\frac{2\pi}{\kappa}\sum_i\delta(\rho-\mathbf{R}_i^0)\right]\hat{\mathbf{z}} , \tag{2.7}$$

where $\hat{\mathbf{z}}$ denotes a unit vector along the field direction, $\mathbf{R}_i^0$ denotes the position of the ith site on the triangular flux-line lattice, and $\boldsymbol{\rho}$ is a two-component vector, $\boldsymbol{\rho}=(x,y)$. The curl of a gradient is normally taken to be identically zero, so how do the δ-function terms in Eq. (2.7) arise? According to Stokes' theorem

$$\int_c \mathbf{Q}_0\cdot d\mathbf{l}=\int d\mathbf{S}\cdot\nabla\times\mathbf{Q}_0 . \tag{2.8}$$

Let the contour C encircle a flux line in the xy plane and be shrunk down onto it. Equation (2.4) shows that near a flux line (for convenience sited at the origin), $f_0(\boldsymbol{\rho})\to A\rho$, and $\mathbf{Q}_0$ is $-1/\kappa\rho$ and polar in direction. Hence, the left-hand side of Eq. (2.8) equals $-2\pi/\kappa$ as the contour is shrunk down, so the right-hand side, in order to be non-vanishing, must contain a δ-function term as given in Eq. (2.7). [Brandt[15] makes frequent use of Eq. (2.7) in his work on non-local elasticity moduli.]

From Eq. (2.5) it follows that

$$\frac{\partial B_0}{\partial y}=-f_0^2 Q_{x,0} , \tag{2.9}$$

$$-\frac{\partial B_0}{\partial x}=-f_0^2 Q_{y,0} , \tag{2.10}$$

so Eq. (2.7) can be rewritten as

$$\nabla\cdot\left[\frac{1}{f_0^2}\nabla B_0(x,y)\right]=B_0-\frac{2\pi}{\kappa}\sum_i \delta(\boldsymbol{\rho}-\mathbf{R}_i^0) . \tag{2.11}$$

Because $f_0(x,y)$ and $B_0(x,y)$ have the periodicity of the triangular lattice they have the Fourier expansions[15]

$$f_0(x,y)=\sum_{\mathbf{G}} f_{\mathbf{G}} e^{i\mathbf{G}\cdot\boldsymbol{\rho}} , \tag{2.12}$$

$$B_0(x,y)=B+\sum_{\mathbf{G}\neq 0} b_{\mathbf{G}} e^{i\mathbf{G}\cdot\boldsymbol{\rho}} , \tag{2.13}$$

where $\mathbf{G}$ denotes a triangular-lattice reciprocal-lattice vector. Solutions of Eqs. (2.4) and (2.11) can be obtained to any desired level of accuracy by retaining more Fourier coefficients in (2.12) and (2.13). Fortunately, for our analysis of the fluctuations about the mean-field solution, we have no need for an explicit solution of the mean-field equations.

B. Fluctuations about the mean-field solution

To analyze these fluctuations let us write

$$\begin{aligned} f&=f_0+f_1 , \\ \mathbf{A}&=\mathbf{A}_0+\mathbf{a} , \\ \mathbf{Q}&=\mathbf{Q}_0+\mathbf{q} , \\ F&=F_0+F_1 , \end{aligned} \tag{2.14}$$

then to quadratic order

$$F_1=\int d^3r\left[-f_1^2+3f_0^2+\frac{1}{\kappa^2}(\nabla f_1)^2+f_0^2\mathbf{q}^2 +4f_0f_1\mathbf{Q}_0\cdot\mathbf{q}+f_1^2\mathbf{Q}_0^2+(\nabla\times\mathbf{a})^2\right] . \tag{2.15}$$

The quadratic form of Eq. (2.15) can be diagonalized by finding its real eigenvalues λ and their corresponding eigenvectors by solving the equations

$$-\frac{1}{\kappa^2}\nabla^2 f_1-f_1+3f_0^2f_1+f_1\mathbf{Q}_0^2+2f_0\mathbf{Q}_0\cdot\mathbf{q}=\lambda f_1 , \tag{2.16}$$

$$\nabla\times\nabla\times\mathbf{a}+f_0^2\mathbf{q}+2f_0f_1\mathbf{Q}_0=\lambda f_0^2\mathbf{q} . \tag{2.17}$$

This is a difficult task for the mixed phase, so we shall start by solving these equations for the Meissner phase.

C. The Meissner phase

This phase has $f_0(x,y,z)=1$ everywhere and $\mathbf{Q}_0=0$. Within this phase no flux lines are present, so $\nabla\times\mathbf{a}=\nabla\times\mathbf{q}$. Equations (2.16) and (2.17) give rise to four excitation branches: a hard mode associated with amplitude variations and given by

$$-\frac{1}{\kappa^2}\nabla^2 f_1+2f_1=\lambda f_1, \quad \mathbf{q}=0 \tag{2.18}$$

[Eq. (2.18) can be solved by Fourier transforming; the solution of wavevector $\mathbf{k}$ has eigenvalue $\lambda=2+k^2/\kappa^2$]; a longitudinal superfluid velocity mode in which $\mathbf{q}\|\mathbf{k}$ for which $f_1=0$ and $\lambda=1$, and two degenerate transverse superfluid velocity modes ($\mathbf{q}\perp\mathbf{k}$) for which $f_1=0$ and $\lambda=1+k^2$.

At temperature T the thermal fluctuations about the mean-field solution give the following expressions for the propagators:

$$\langle f_1(\mathbf{k})f_1(-\mathbf{k})\rangle=\frac{1}{2}\frac{k_BT}{2+k^2/\kappa^2} , \tag{2.19}$$

$$\langle q_i(\mathbf{k})q_j(-\mathbf{k})\rangle=\frac{1}{2}k_BT\left[\frac{k_ik_j}{k^2}+\left[\delta_{ij}-\frac{k_1k_j}{k^2}\right]\frac{1}{1+k^2}\right] , \tag{2.20}$$

correct to lowest (Gaussian) order. In Eq. (2.20) the term k_ik_j/k^2 is associated with longitudinal fluctuations of $\mathbf{Q}$ (i.e., with components of $\mathbf{Q}$ parallel to $\mathbf{k}$), while the term $\delta_{ij}-k_ik_j/k^2$ is associated with the transverse components.

D. Definition of a gauge-invariant phase

The phase Φ of the superconducting wave function is not a gauge-invariant quantity: the free-energy F is invariant under the gauge transformation

$$\mathbf{A}\to\mathbf{A}'=\mathbf{A}+\frac{1}{\kappa}\nabla\chi , \tag{2.21}$$

$$\Phi\to\Phi'=\Phi+\chi , \tag{2.22}$$

where χ is an arbitrary function. Since it is our central contention that phase fluctuations destroy ODLRO in both the Meissner and the mixed phase in sufficiently low spatial dimensions, it is important that we define the "phase" with some care.

A gauge-invariant correlation function is

$$\hat{G}(\mathbf{r},\mathbf{r}')=\left\langle \psi^*(\mathbf{r}')\left[\exp i\kappa\int_{\mathbf{r}}^{\mathbf{r}'}\mathbf{A}\cdot d\mathbf{l}\right]\psi(\mathbf{r})\right\rangle$$
$$=\langle f(\mathbf{r}')f(\mathbf{r})\exp i\kappa\int_{\mathbf{r}}^{\mathbf{r}'}\mathbf{Q}\cdot d\mathbf{l}\rangle \ . \quad (2.23)$$

This suggests as a possible definition of the phase difference $\Delta\Phi$ between the points $\mathbf{r}$ and $\mathbf{r}'$,

$$\Delta\Phi=-\kappa\int_{\mathbf{r}}^{\mathbf{r}'\mathbf{Q}\cdot d\mathbf{l}} . \quad (2.24)$$

While this is commonly used as a definition of a gauge invariant phase, and is the one used by HPS,[12] it has unsatisfactory features. In particular, the phase difference defined in Eq. (2.24) varies continuously as the path between the points $\mathbf{r}$ and $\mathbf{r}'$ is altered, and moreover, its thermal fluctuations will turn out to be infinite in all spatial dimensions.

Our gauge-invariant phase Θ will be defined as the solution of the equation

$$\nabla^2\Theta=-\kappa\nabla\cdot\mathbf{Q} \ , \quad (2.25)$$

subject to boundary conditions, at the external boundaries and at the vortex lines (see below).

A vector such as $\mathbf{Q}(\mathbf{r})$ can be written as a sum of its longitudinal and transverse components, i.e.,

$$\mathbf{Q}(\mathbf{r})=Q_L(\mathbf{r})+\mathbf{Q}_T(\mathbf{r}) \ , \quad (2.26)$$

where $\nabla\cdot\mathbf{Q}_T=0$, $\nabla\times\mathbf{Q}_L=0$. Hence, the definition of the phase via Eq. (2.25) depends only on the longitudinal component $\mathbf{Q}_L$ of $\mathbf{Q}$ and not on the transverse components $\mathbf{Q}_T$ [which are responsible for the infinite thermal fluctuations in the HPS phase defined via Eq. (2.24)]. In practice, one finds $\mathbf{Q}_L$ and $\mathbf{Q}_T$ from the Fourier transform of $\mathbf{Q}$, $\mathbf{Q}(\mathbf{k})$, via

$$\mathbf{Q}_L(\mathbf{k})=\mathbf{k}[\mathbf{k}\cdot\mathbf{Q}(\mathbf{k})]/k^2 \ ,$$
$$\mathbf{Q}_T(\mathbf{k})=\mathbf{Q}(\mathbf{k})-\mathbf{Q}_L(\mathbf{k}) \ . \quad (2.27)$$

When vortices are present, as in the mixed phase, the superconductor can be regarded for our purposes as no longer simply connected topologically. $\Theta(\mathbf{r})$ is no longer single valued. In this situation it is very convenient to regard each flux line or vortex as a little tube of zero diameter at which boundary conditions have to be enforced. (I am indebted to Dr. E. H. Brandt for this idea.) For a vortex in the mixed phase, one expects the phase to change by 2π on integrating $\nabla\Theta$ along a path encircling the vortex. Mathematically this feature can be imposed as follows. Points on the ith flux line, defined as a zero of $f(\mathbf{r})$, are specified by the three component vector $\mathbf{r}_i=(x_i(z),y_i(z),z)$. We shall impose the "boundary condition" that

$$\nabla\times\nabla\Theta(\mathbf{r})=2\pi\sum_i\int d\mathbf{r}_i\delta(\mathbf{r}-\mathbf{r}_i) \ . \quad (2.28)$$

Then if one integrates $\nabla\Theta(\mathbf{r})$ around any contour in a plane whose normal is locally parallel to the flux line, it follows from Stokes theorem [see Eq. (2.8)] that the phase will increase by 2π if the contour encloses the flux line.

The general solution of Eq. (2.25) for periodic external boundary conditions in the presence of vortices can be written as

$$\nabla\Theta=-\kappa\mathbf{Q}_L+\nabla\Theta_G \ , \quad (2.29)$$

where $\nabla^2\Theta_G=0$ and

$$\nabla\Theta_G(\mathbf{r})=2\pi\nabla\times\sum_i\int d\mathbf{r}_i\sum_k\exp[i\mathbf{k}\cdot(\mathbf{r}_i-\mathbf{r})]/k^2 \ , \quad (2.30)$$

where $\mathbf{k}=2\pi(n_x,n_y,n_z)/L$ and n_x,n_y,n_z are integers. Equation (2.29) reduces to an equation for the phase given by Brandt[15] on using the relation

$$\frac{1}{|\mathbf{r}-\mathbf{r}_i|}=\int\frac{d^3k}{(2\pi)^3}\frac{4\pi}{k^2}\exp[i\mathbf{k}\cdot(\mathbf{r}_i-\mathbf{r})] \ . \quad (2.31)$$

$\nabla\Theta_G$ is determined by the geometry of the flux lines. It is clearly periodic in L and can be shown to be consistent with the internal boundary conditions of Eq. (2.28) on taking the curl of both sides of Eq. (2.30). We have adopted periodic boundary conditions when dealing with vortices as they result in considerable simplification.

The phase difference $\Delta\Theta$ between two points $\mathbf{r}$ and $\mathbf{r}'$ is defined then as

$$\Delta\Theta=\int_{\mathbf{r}}^{\mathbf{r}'}\nabla\Theta\cdot d\mathbf{l} \ . \quad (2.32)$$

Notice that $\Delta\Theta$ is independent of the path between the points $\mathbf{r}$ and $\mathbf{r}'$ provided the different paths do not encircle a flux line.

The phase defined via Eq. (2.25) actually coincides with the phase of the order parameter ψ within a particular choice of gauge—the London gauge. In this gauge, $\nabla\cdot\mathbf{A}=0$. Combining (2.2) and (2.25) it can be seen that $\nabla^2\Theta=\nabla^2\Phi$. Hence, $\Theta=\Phi$ as both Θ and Φ have to satisfy the same boundary conditions.

The existence of ODLRO can be investigated by finding how the gauge-invariant correlation function

$$G(\mathbf{r},\mathbf{r}')=\langle f(\mathbf{r})f(\mathbf{r}')\exp\Delta\Theta(\mathbf{r},\mathbf{r}')\rangle \quad (2.33)$$
$$=\langle\psi(\mathbf{r})\psi^*(\mathbf{r}')\rangle\equiv\langle f(\mathbf{r})f(\mathbf{r}')\exp\Delta\Phi(\mathbf{r},\mathbf{r}')\rangle \quad (2.34)$$

[provided (2.34) is evaluated in the London gauge], varies as the separation $|\mathbf{r}-\mathbf{r}'|$ tends to infinity. If it tends to zero as the separation increases to infinity, ODLRO is absent. Since the only practical way to perform explicit calculations within a gauge-dependent field theory such as that of Eq. (2.1) is to fix a particular gauge at the outset, it is very convenient that our definition of ODLRO via Eq. (2.34) can be studied directly by calculating the propagator of the order parameter field ψ in the commonly used London gauge.

E. ODLRO in the Meissner state

The calculation of $G(\mathbf{r},\mathbf{r}')$ within the Meissner state is straightforward. Writing $f=f_0+f_1$, Eq. (2.33) becomes, as $\Theta(\mathbf{r})$ is single-valued in the Meissner state and $\Delta\Theta(\mathbf{r},\mathbf{r}')$ path independent,

$$G(\mathbf{r},\mathbf{r}')=[f_0^2+\langle f_1(\mathbf{r})f_1(\mathbf{r}')\rangle]$$
$$\times\exp\{-\tfrac{1}{2}\langle[\Theta(\mathbf{r})-\Theta(\mathbf{r}')]^2\rangle\} \ . \quad (2.35)$$

The second term in the first set of square brackets in (2.35) can be evaluated from Eq. (2.20). As $|\mathbf{r}-\mathbf{r}'|$ tends to infinity, its contribution to $G(\mathbf{r},\mathbf{r}')$ is exponentially small. It follows from Eq. (2.25) that

$$k^2\Theta(\mathbf{k})=i\kappa\mathbf{k}\cdot\mathbf{q}(\mathbf{k}) \ , \tag{2.36}$$

so

$$\tfrac{1}{2}\langle[\Theta(\mathbf{r})-\Theta(\mathbf{r}')]^2\rangle=\kappa^2\sum_{\mathbf{k}}[1-\cos\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')] \times\frac{k_ik_j}{k^4}\langle q_i(\mathbf{k})q_j(-\mathbf{k})\rangle \ ,$$

i.e.,

$$\langle\Delta\Theta^2\rangle=\kappa^2k_BT\sum_{\mathbf{k}}\frac{[1-\cos\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')]}{k^2} \tag{2.37}$$

on using Eq. (2.20) and summing over repeated indices. For $d<2$, the right-hand side of Eq. (2.37) varies as $|\mathbf{r}-\mathbf{r}'|^{2-d}$, as can be seen by simple power counting, but for $d>2$ tends to a constant. Thus, we conclude that for $d<2$, ODLRO is destroyed by phase fluctuations within the Meissner phase and that two is probably its lower critical dimension. Because the phase fluctuations seem to be always more important than amplitude fluctuations in reducing ODLRO, I shall henceforth only study $\langle\Delta\Theta^2\rangle$. ODLRO will be deemed to be present if $\langle\Delta\Theta^2\rangle$ remains finite as the separation of $\mathbf{r}$ from $\mathbf{r}'$ increases to infinity. I know of no formal proof that two is actually the lower critical dimension of the Meissner phase, but it is probably a consequence of the theorem of Mermin and Wagner[16] and Hohenberg.[17]

It is interesting to contrast Eq. (2.37) with the corresponding expression for the fluctuations in the phase defined by HPS via Eq. (2.24). Let us set $\mathbf{r}=(0,0,0)$ and $\mathbf{r}'=(0,0,h)$ and integrate along the z axis between these points. Since in the Meissner phase $\mathbf{Q}_0=0$,

$$\Delta\Phi=-\kappa\int_0^h dz\, q_z(0,0,z) \ , \tag{2.38}$$

and so $\langle\Delta\Phi\rangle=0$ and

$$\begin{aligned}\langle\Delta\Phi^2\rangle&=\kappa^2\int_0^h dz\int_0^h dz'\langle q_z(z)q_z(z')\rangle\\ &=2\kappa^2\sum_{\mathbf{k}}(1-\cos k_zh)\langle q_z(\mathbf{k})q_z(-\mathbf{k})\rangle/k_z^2 \qquad (2.39)\\ &=\kappa^2k_BT\sum_{\mathbf{k}}(1-\cos k_zh)\left[1+\frac{1}{k_z^2}\right]\Big/(1+k^2) \qquad (2.40)\end{aligned}$$

on using Eq. (2.20). After evaluating the integral over $\mathbf{k}$, one finds that the second term in the curly brackets in Eq. (2.40), which arises from the transverse component of the superfluid velocity, gives a contribution to $\langle\Delta\Phi^2\rangle$, which increases linearly with h $(=|\mathbf{r}-\mathbf{r}'|)$ as $h\to\infty$, in all dimensions. Thus, if the HPS definition of a gauge-invariant phase is adopted, then one would deduce that ODLRO in the Meissner phase is destroyed by thermal fluctuations in all spatial dimensions. We conclude that in a bulk system, as opposed to a circuit, their phase is without physical significance.

III. THE MIXED PHASE

A. Elasticity theory

The determination of the excitation modes of the mixed phase requires the solution of the eigenvalue equations (2.16) and (2.17). Since the functions $f_0(x,y)$ and $\mathbf{Q}_0(x,y)$, which appear in these equations, cannot themselves be determined analytically, it is only possible to extract information on the eigenvalues and eigenvectors in a few special limits.

Because of the lattice periodicity of the functions $f_0(x,y)$ and $\mathbf{Q}_0(x,y)$ appearing in the eigenvalue equations, it follows from Bloch's theorem that the eigenvectors must be of the type $\exp(i\mathbf{k}\cdot\mathbf{r})E_{\mathbf{k}}(x,y)$, where $\mathbf{k}=(k_x,k_y,k_z)=(\mathbf{k}_\perp,k_z)$ and $E_{\mathbf{k}}(x,y)$ has the periodicity of the triangle flux lattice. The simple z dependence derives from the translational invariance along the field direction. The eigenvalues λ will depend on the branch and the wave vector $\mathbf{k}$. There are two Goldstone branches, for which $\lambda\to0$ as $|\mathbf{k}|\to0$, which arise from the invariance of the flux lattice free energy under arbitrary translations in the xy plane.

The Goldstone branches are just the elastic modes of the flux lattice, which in the long-wavelength limit can be described by the elastic free energy $F_{\rm el}$ written down by de Gennes and Matricon in 1963.[18] The position of the ith flux line, defined as a zero of $f(\mathbf{r})$, is a function $\mathbf{r}_i(z)$, for a general displacement of the flux lines. A two-dimensional displacement field $\mathbf{u}(\mathbf{R}_i^0,z)$ is given by setting $\mathbf{r}_i(z)=\mathbf{R}_i^0-\mathbf{u}(\mathbf{R}_i^0,z)$. It is very convenient to work in the continuum limit when this displacement becomes a function $\mathbf{u}(x,y,z)$. Only the long-wavelength limit of the excitation modes will be correctly given by the continuum limit. The excess elastic free energy $F_{\rm el}$ associated with small gradients of $\mathbf{u}$ is

$$F_{\rm el}=\tfrac{1}{2}\int d^3r\{c_L(\partial_xu_x+\partial_yu_y)^2+c_{44}[(\partial_zu_x)^2+(\partial_zu_y)^2] +c_{66}[(\partial_xu_x-\partial_yu_y)^2+(\partial_xu_y+\partial_yu_x)^2]\} \ . \tag{3.1}$$

c_L is the compression modulus, c_{44} measures the energy cost of tilting the flux lines, and c_{66} is the shear modulus of the flux lattice. ($c_{44}=2HB$ and $c_L=2B^2\partial H/\partial B$ in our reduced units[19]).

The two excitation modes associated with the elastic free energy $F_{\rm el}$ are revealed by Fourier transformation:

$$F_{\rm el}=\tfrac{1}{2}\sum_{\mathbf{k}}u_i(\mathbf{k})[c_Lk_ik_j+\delta_{ij}(c_{66}k_\perp^2+c_{44}k_z^2)]u_j(-\mathbf{k}) \ , \tag{3.2}$$

where $(i,j)=(x,y)$. Hence,

$$\langle u_i(\mathbf{k})u_j(-\mathbf{k})\rangle=k_BT\left[\frac{P_T}{c_{66}k_\perp^2+c_{44}k_z^2}+\frac{P_L}{c_{11}k_\perp^2+c_{44}k_z^2}\right] , \tag{3.3}$$

where $P_T=(\delta_{ij}-k_ik_j/k_\perp^2)$, $P_L=k_ik_j/k_\perp^2$, and $c_{11}=c_L+c_{66}$. Equation (3.3) shows that one mode of $F_{\rm el}$ is a transverse branch in which $\mathbf{u}\perp\mathbf{k}_\perp$ and the other is a longitudinal branch in which $\mathbf{u}\|\mathbf{k}_\perp$.

The elastic free energy of Eq. (3.1) can be derived from the eigenvalue equations (2.16) and (2.17) by studying the null ($\lambda=0$, $\mathbf{k}=0$) eigenvalues and eigenvectors and then performing a small $\mathbf{k}$ expansion of the Bloch functions $E_{\mathbf{k}}(x,y)$. The null eigenvectors are doubly degenerate because they correspond to arbitrary displacements of the flux lattice in either the x or y directions. They are linear combinations of

$$f_1=u_x\frac{\partial f_0}{\partial x} \quad\text{and}\quad u_y\frac{\partial f_0}{\partial y}\ , \tag{3.4}$$

$$\mathbf{q}=u_x\frac{\partial \mathbf{Q}_0}{\partial x} \quad\text{and}\quad u_y\frac{\partial \mathbf{Q}_0}{\partial y}\ , \tag{3.5}$$

where here u_x and u_y are constants. This can be checked by directly substituting Eqs. (3.4) and (3.5) into the eigenvalue equations and noticing their similarity to Eqs. (2.4) and (2.5) after these equations have been differentiated with respect to either x or y.

It is possible to obtain the local (elastic) moduli starting from the eigenvalue equations (2.16) and (2.17). Each elastic modulus is most easily obtained by consideration of special cases. To obtain the tilt modulus c_{44} one supposes $\mathbf{k}=(0,0,k_z)$ and $u_y=0$. The Bloch function $E_{\mathbf{k}}(x,y)$ can be expanded in powers of k_z. To order k_z this is equivalent to setting

$$f_1=u_x(z)\frac{\partial f_0}{\partial x}+\frac{\partial u_x(z)}{\partial z}R(x,y)\ , \tag{3.6}$$

$$\mathbf{q}=u_x(z)\frac{\partial \mathbf{Q}_0}{\partial x}+\frac{\partial u_x(z)}{\partial z}\mathbf{V}(x,y)\ , \tag{3.7}$$

where $u_x(z)=u_0\cos(k_zz)$. Equations (3.6) and (3.7) for f_1 and $\mathbf{q}$ are then substituted into the eigenvalue equations and as $\lambda\sim k_z^2$, the coefficients of $\partial u_x(z)/\partial z$, i.e., terms of order k_z must vanish [just as we have already shown that the coefficient of $u_x(z)$ vanishes]. This leads after some algebra to the following equations for R and $\mathbf{V}$,

$$-\frac{1}{\kappa^2}\left[\frac{\partial^2 R}{\partial x^2}+\frac{\partial^2 R}{\partial y^2}\right]-R+3f_0^2R+R\mathbf{Q}_0^2+2f_0\mathbf{V}\cdot\mathbf{Q}_0=0\ , \tag{3.8}$$

$$\nabla\times\nabla\times\mathbf{V}+f_0^2\mathbf{V}+2f_0R\,\mathbf{Q}_0 =\left[0,0,-\frac{\partial^2}{\partial x^2}Q_{x,0}-\frac{\partial^2}{\partial y^2}Q_{x,0}-\frac{\partial B_0}{\partial y}\right]. \tag{3.9}$$

These equations have solution $R=V_x=V_y=0$ and $V_z=Q_{x,0}$. Then, correct to order k_z, the eigenvectors are

$$f_1=u_x\frac{\partial f_0}{\partial x},\quad q_x=u_x\frac{\partial Q_{x,0}}{\partial x}\ ,$$
$$q_y=u_x\frac{\partial Q_{y,0}}{\partial x},\quad q_z=\frac{\partial u_x}{\partial z}Q_{x,0}\ , \tag{3.10}$$

$$\nabla\times\mathbf{a}=\left[-\frac{\partial u_x}{\partial z}B_0,0,u_x\frac{\partial B_0}{\partial x}\right]. \tag{3.11}$$

The results given in Eq. (3.11) have an obvious interpretation in terms of the geometry of the tilted flux lattice. On substituting Eqs. (3.10) and (3.11) into Eq. (2.15) for F_1 one obtains

$$F_1=\int d^3r\left[\frac{\partial u_x(z)}{\partial z}\right]^2\left[\frac{1}{\kappa^2}\left[\frac{\partial f_0}{\partial x}\right]^2+f_0^2Q_{x,0}^2+B_0^2\right] =\frac{1}{2}c_{44}\int d^3r\left[\frac{\partial u_x(z)}{\partial z}\right]^2. \tag{3.12}$$

Hence $c_{44}=2HB$, on using an identity given in Ref. 24. That $c_{44}=2HB$ has been known for many years.[19] The same technique can be used to obtain both c_{66} and c_{11}. The appropriate special case for c_{66} would be $u_x(y)=u_0\cos(k_yy)$, $u_y=0$. However, in this case I could not find an explicit solution of the equations corresponding to Eqs. (3.8) and (3.9)—only the right-hand sides differ—but they could be solved numerically by Fourier expanding R and $\mathbf{V}$ in reciprocal lattice vectors. The appropriate special case for obtaining c_{11} is $u_x(x)=u_0\cos(k_xx)$, $u_y=0$.

B. Fluctuations of the phase defined by HPS

I have already shown that within the Meissner state the HPS phase difference $\Delta\Phi$ defined by Eq. (2.24), when the line integral is up the z axis between two points separated by a distance h, is such that $\langle\Delta\Phi^2\rangle$ increases linearly with h rather than tending to a constant, implying that with the HPS definition of the phase there is no ODLRO—in any dimension. I shall now argue that a similar result holds for the mixed state.

In the eigenvalue equations (2.16) and (2.17), if q_z depends only on z, i.e., $q_z(z)\neq 0$ and $q_x=q_y=0$, then the equations are solved if $f_1=0$ and $\lambda=1$, for in this case $\nabla\times\mathbf{a}=\nabla\times\mathbf{q}=0$. Hence,

$$\langle q_z(0,0,k_z)q_z(0,0,-k_z)\rangle=\frac{k_BT}{\rho_s\kappa^2}\ , \tag{3.13}$$

where the "superfluid density" $\rho_s=2\overline{f_0^2(x,y)}/\kappa^2$. This mode is analogous to the longitudinal mode of the superfluid velocity in the Meissner phase. It is not a Goldstone mode as $\lambda=1$ as $k_z\to 0$.

The eigenvalue equations also simplify for the transverse mode in which q_z depends only on x and y, i.e., $q_z(x,y)\neq 0$ and $q_x=0=q_y$. In this situation the flux lines move without tilting so $\nabla\times\mathbf{q}=\nabla\times\mathbf{a}$ and the eigenvalue equation becomes

$$-\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]q_z+f_0^2(x,y)q_z=\lambda f_0^2(x,y)q_z\ , \tag{3.14}$$

with $f_1=0$. In the long-wavelength limit Eq. (3.14) leads to

$$\langle q_z(k_x,k_y,0)q_z(-k_x,-k_y,0)\rangle=\frac{k_BT}{\alpha(k_x^2+k_y^2)+\rho_s\kappa^2}\ , \tag{3.15}$$

where α is a numerical constant of order unity. By analogy with Eq. (2.20) for the Meissner phase, one would expect on combining Eq. (3.13) and Eq. (3.15) that for general small $\mathbf{k}$,

$$\langle q_z(\mathbf{k})q_z(-\mathbf{k})\rangle=k_BT\left[\frac{k_z^2}{k^2}\frac{1}{\rho_s\kappa^2}+\left[1-\frac{k_z^2}{k^2}\right]\frac{1}{\alpha k^2+\rho_s\kappa^2}\right]. \tag{3.16}$$

It is then a straightforward matter to substitute Eq. (3.16) into (2.39) where one finds once again that $\langle\Delta\Phi^2\rangle$ increases linearly with h, implying the absence of ODLRO. Note that if only the longitudinal term in Eq. (3.16) is used in (2.39), then $\langle\Delta\Phi^2\rangle$ only increases with h below two dimensions, just as in the Meissner phase. The general conclusion I draw is that the HPS definition of a phase is always unsatisfactory because it includes the transverse components of $\mathbf{Q}$.

C. ODLRO in the mixed phase

We have just argued that the contribution to $\langle\Delta\Theta^2\rangle$ from the fluctuations in $\mathbf{Q}_L$ increase with separation when $d<2$. However, the dominant contribution to $\langle\Delta\Theta^2\rangle$ comes from $\langle\Delta\Theta_G^2\rangle$, since I shall now show that it starts to increase with separation for $d<4$. From Eq. (2.30),

$$\frac{\partial\Theta_G}{\partial z}(\mathbf{r})=2\pi i\sum_{\mathbf{k}}\sum_i\int dz'\left[k_y\frac{dx_i}{dz'}(z')-k_x\frac{dy_i}{dz'}(z')\right]\times\exp\{i\mathbf{k}\cdot[\mathbf{r}_i(z')-\mathbf{r}]\}/k^2\ . \tag{3.17}$$

At low temperatures where displacements of the flux lines are small it should be possible to approximate $\mathbf{r}_i(z')$ by $(\mathbf{R}_1^0,z')$. Then as

$$\sum_i\int dz'\exp(i\mathbf{k}_\perp\cdot\mathbf{R}_i^0+k_zz')\frac{dx_i}{dz'}(z')=ik_zu_x(\mathbf{k})\frac{\kappa B}{2\pi}\ , \tag{3.18}$$

with a similar expression for $u_y(\mathbf{k})$. Equation (3.17), when Fourier transformed, gives

$$\theta(\mathbf{k})=i\kappa B[k_yu_x(\mathbf{k})-k_xu_y(\mathbf{k})]/k^2 \tag{3.19}$$

where $\Theta_G(\mathbf{r})=\Theta_{G,0}(\mathbf{r})+\theta(\mathbf{r})$, and $\Theta_{G,0}(\mathbf{r})$ is the geometric phase in the undisplaced lattice. Notice that within this approximation, $\theta(\mathbf{k})$ is single valued. The approximation should be valid for calculating phase differences along paths that do not go close to flux lines. Hence

$$\langle\theta(\mathbf{k})\theta(-\mathbf{k})\rangle=\kappa^2B^2k_BT\frac{k_\perp^2}{k^4}\frac{1}{c_{66}k_\perp^2+c_{44}k_z^2} \tag{3.20}$$

on using Eq. (3.3). Equation (3.20) shows that phase fluctuations are largely determined by the transverse (shear) mode of the flux lattice. An expression equivalent to Eq. (3.20) was given by HPS.

The gauge invariant correlation function defined in Eq. (2.33) is, as $|\mathbf{r}-\mathbf{r}'|$ tends to infinity,

$$G(\mathbf{r},\mathbf{r})\approx G_{\mathrm{MF}}(\mathbf{r},\mathbf{r}')\langle\exp i[\theta(\mathbf{r})-\theta(\mathbf{r}')]\rangle=G_{\mathrm{MF}}(\mathbf{r},\mathbf{r}')\exp\left[-\sum_{\mathbf{k}}[1-\cos\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')]\langle\theta(\mathbf{k})\theta(-\mathbf{k})\rangle\right], \tag{3.21}$$

where

$$G_{\mathrm{MF}}(\mathbf{r},\mathbf{r}')=f_0(\mathbf{r})f_0(\mathbf{r}')\exp i[\Theta_0(\mathbf{r})-\Theta_0(\mathbf{r}')]\ .$$

Power counting on the integral in Eq. (3.21) shows that $G(\mathbf{r},\mathbf{r}')$ tends to zero as $|\mathbf{r}-\mathbf{r}'|\to\infty$ in all dimensions less than four, i.e., ODLRO vanishes below four dimensions.

The decay of $G(\mathbf{r},\mathbf{r}')$ is highly anisotropic and is most rapid when $\mathbf{r}-\mathbf{r}'$ is perpendicular to the applied field; suppressing all constants and taking $B\approx H$, when $c_{66}\ll c_{44}$, one finds that in this limit

$$G(\mathbf{r},\mathbf{r}')\approx G_{\mathrm{MF}}(\mathbf{r},\mathbf{r}')\exp(-\rho/L_0)\ , \tag{3.22}$$

where $\rho=|\mathbf{r}-\mathbf{r}'|$ with $z=z'$ and

$$L_0=4\Lambda_T\left[\frac{c_{66}}{c_{44}}\right]^{1/2},\quad\text{with }\Lambda_T=\frac{\hat{\Phi}_0^2}{16\pi^2k_BT}\approx\frac{2\times10^8\ \text{Å K}}{T}\ . \tag{3.23}$$

L_0 is a measure of the decay length of ODLRO for $d=3$. The experimental consequences of its large size (approximately millimeters) are discussed in Sec. V. For two-dimensional systems, i.e., films, the destruction of ODLRO is more rapid and can take place over a length scale typically of order of the flux line spacing l.

IV. ROTATING NEUTRAL SUPERFLUIDS

The analogy between superconductors and rotating neutral superfluids such as ^{4}He has been reviewed by Vinen.[11] In the formalism of this paper, all one has to do to study this problem is to set $\nabla\times\mathbf{A}=B\hat{\mathbf{z}}$, where B is a constant (related to the speed of rotation about the z axis) and neglect the fluctuations of the vector potential entirely. In the context of the superconductor problem this is only valid in the limit of infinite κ. However, the approximation of taking the B field just a constant and neglecting the fluctuations of the vector potential has been widely used over many years. The free energy of the system is

$$F=\int d^3r\left[-|\psi|^2+\tfrac{1}{2}|\psi|^4+\left|\left[\frac{\nabla}{i\kappa}-\mathbf{A}\right]\psi\right|^2\right] . \quad (4.1)$$

This is minimized by solving

$$D_\mu^2\psi_0=\psi_0(1-|\psi_0|^2) , \quad (4.2)$$

where $D_\mu=\partial_\mu/i\kappa-A_\mu$. This equation and its complex conjugate are equivalent to the gauge invariant equations

$$-\frac{1}{\kappa^2}\nabla^2 f_0-f_0+f_0^3+f_0\mathbf{Q}_0^2=0 , \quad (4.3)$$

$$\nabla\cdot(f_0^2\mathbf{Q}_0)=0 , \quad (4.4)$$

where $\mathbf{Q}_0=\mathbf{A}-\nabla\Phi_0/\kappa$. As before we shall expand about the stationary point given by the solution of Eqs. (4.3) and (4.4) by writing

$$f=f_0+f_1 ,$$
$$\mathbf{Q}=\mathbf{Q}_0+\mathbf{q} , \quad (4.5)$$
$$F=F_0+F_1 ,$$

but because $\mathbf{A}$ is fixed, $\mathbf{q}$ must be of the form

$$\mathbf{q}=-\frac{1}{\kappa}\nabla\phi , \quad (4.6)$$

where ϕ is the phase change from the mean-field solution. Note that $\nabla\times\nabla\phi\neq 0$. Equation (4.6) is the source of the differences with the finite-κ limit. The free energy to quadratic order is

$$F_1=\int d^3r\left[-f_1^2+3f_0^2f_1^2+\frac{1}{\kappa^2}(\nabla f_1)^2+f_0^2\mathbf{q}^2+4f_0f_1\mathbf{Q}_0\cdot\mathbf{q}+f_1^2\mathbf{Q}_0^2\right] . \quad (4.7)$$

The quadratic form Eq. (4.7) can be diagonalized by solving the equations

$$-\frac{1}{\kappa^2}\nabla^2 f_1-f_1+3f_0^2f_1+f_1\mathbf{Q}_0^2+2f_0\mathbf{Q}_0\cdot\mathbf{q}=\lambda f_1 , \quad (4.8)$$

$$\nabla\cdot(f_0^2\mathbf{q}+2f_0f_1\mathbf{Q}_0)=\frac{\lambda}{\kappa}f_0^2\phi . \quad (4.9)$$

One simple special solution of Eqs. (4.8) and (4.9) is if ϕ is a function of z only, $\phi(z)$, then $q_x=q_y=0=f_1$, and (4.9) reduces to

$$-\frac{\partial^2\phi}{\partial z^2}=\lambda\phi , \quad (4.10)$$

which has solution $\phi=\phi_0\cos(k_z z)$ and $\lambda=k_z^2$.

Because there is a vortex lattice in the limit of infinite κ, one would expect there to be two Goldstone modes in the eigenvalue equations (4.8) and (4.9), associated with the invariance of the free energy (4.1) under arbitrary translations in the xy plane. However, it has been known for many years that the infinite κ limit is very strange. The tilt modulus c_{44} and the compressibility modulus c_L are infinite[15] and in fact there is only one Goldstone mode[10,9] and that is associated with shear motions of the vortex lattice.[4]

For an arbitrary translation (u_x,u_y) the treatment leading to Eq. (3.5) is unchanged, i.e.,

$$\mathbf{q}=(\mathbf{u}\cdot\nabla)\mathbf{Q}_0=\nabla(\mathbf{u}\cdot\mathbf{Q}_0)-\mathbf{u}\times(\nabla\times\mathbf{Q}_0)+O(\nabla\times\mathbf{u}) . \quad (4.11)$$

For Eq. (4.11) to be compatible with the condition that $\mathbf{q}$ be of the form given by Eq. (4.6), $\mathbf{u}$ has to be restricted to

$$\mathbf{u}=-P\nabla\Omega\times\hat{\mathbf{z}} \quad (4.12)$$

(where $P=1/\kappa B$ in reduced units and $2\pi P=\sqrt{3}l^2/2$ in nonreduced units), which corresponds to a shear motion of the flux lattice. Thus, compressions are incompatible with (4.6) and (4.11), and are not Goldstone modes.

The effective free energy associated with the long-wavelength limit of the Goldstone mode follows by combining Eqs. (4.10) and (4.12)

$$F_{\text{eff}}=\frac{1}{2}\int d^3r\left[\rho_s\left[\frac{\partial\Omega}{\partial z}\right]^2+c_{66}P^2(\nabla_\perp^2\Omega)^2\right] , \quad (4.13)$$

where $\rho_s=\overline{2f_0^2(x,y)}/\kappa^2$ and $\nabla_\perp^2=\partial^2/\partial x^2+\partial^2/\partial y^2$. The shear modulus c_{66} can, in principle, be obtained from the eigenvalue equations (4.8) and (4.9) using the same technique as employed in Sec. III A.

Having determined the form of the soft mode of this system, we can now investigate how the thermal fluctuations of the mode affects ODLRO. The phase Θ is once more given by Eq. (2.29) with $\nabla\Theta_G$ specified by Eq. (2.30). The fluctuations of Θ are dominated again by those of Θ_G and it follows from Eq. (3.19) and Eq. (4.12) that $\theta(\mathbf{k})=(k_\perp^2/k^2)\Omega(\mathbf{k})$.

The existence of ODLRO can be investigated by studying

$$G(\mathbf{r},\mathbf{r}')=\langle\psi(\mathbf{r})\psi^*(\mathbf{r}')\rangle\approx G_{\text{MF}}(\mathbf{r},\mathbf{r}')\langle\exp i[\theta(\mathbf{r})-\theta(\mathbf{r}')]\rangle$$
$$\approx G_{\text{MF}}(\mathbf{r},\mathbf{r}')\exp\left[-\sum_{\mathbf{k}}[1-\cos\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')]\langle\theta(\mathbf{k})\theta(-\mathbf{k})\rangle\right] . \quad (4.14)$$

From (4.13) one deduces that in the long-wavelength limit

$$\langle \theta(\mathbf{k})\theta(-\mathbf{k})\rangle = \frac{k_B T}{\rho_s k_z^2 + c_{66} P^2 k_\perp^2} \frac{k_\perp^2}{k^4} . \quad (4.15)$$

Power counting on the integral in Eq. (4.15) shows that $G(\mathbf{r},\mathbf{r}')$ tends to zero in all dimensions less than three. Right in three dimensions,

$$G(\mathbf{r},\mathbf{r}') \approx G_{\mathrm{MF}}(\mathbf{r},\mathbf{r}') \exp\left[-\frac{k_B T}{4\pi P(\rho_s c_{66})^{1/2}} \ln\left[\frac{\rho}{l}\right]\right] \quad (4.16)$$

up to numerical factors where $\rho = |\mathbf{r}-\mathbf{r}'|$ with $z = z'$. Equation (4.16) implies that the lower critical dimension for ODLRO in rotating neutral superfluids is three.

How do the results for finite κ in a superconductor crossover into the infinite κ results? The answer to this question was provided long ago by Brandt[15,8] in his pioneering studies of the k dependence of the elastic moduli. In the extreme nonlocal limit $c_{44}(\mathbf{k}) = c_{44} k_h^2 / k_\perp^2$ when $k > k_h$. The reciprocal effective penetration depth $k_h = (1 - B/H_{c2})^{1/2}/\lambda$, which tends to zero as $\kappa \to \infty$. Substituting this expression for $c_{44}(\mathbf{k})$ into Eqs. (3.20) and (3.21), one again recovers the result that three is the lower critical dimension for the infinite κ limit, without recourse to the direct calculations of this section.

V. DISCUSSION

I have shown that phase fluctuations destroy ODLRO [as given by Eqs. (2.29) and (2.30)] in dimensions $d < 2$ in the Meissner phase and in dimensions $d < 4$ in the mixed phase. ODLRO itself is not experimentally observable. It would seem natural, however, to assume that the loss of ODLRO would affect the nature of the mixed state or result in its complete destruction. For the sake of simplicity we shall discount the exotic possibilities of Ref. 13 and assume that for dimensions d, $2 < d < 4$, the only genuine phases in a pure bulk type-II superconductor are the Meissner phase and the normal phase, i.e., we shall assume that the loss of ODLRO implies the destruction of the mixed phase. In the H-T phase diagram, the $H_{c2}(T)$ line is supposed, on this scenario, not to be a genuine phase boundary but a crossover line at which substantial diamagnetism first appears. For $d = 3$, the $H_{c1}(T)$ line is a real phase boundary.

Experiments on conventional superconductors indicate that in the region $H_{c1}(T) < H < H_{c2}(T)$ there is a state very similar to the mean-field Abrikosov flux lattice state, while for high-temperature superconductors there is a vortex liquid state, at least near the $H_{c2}(T)$ boundary, but at lower fields there is an irreversibility line. (This has been interpreted as either due to the freezing of the flux-line liquid into a crystalline state or the onset of a vortex glass phase due to the consequences of the disorder or just due to the onset of very long pinning times for the flux lines (for a review see Ref. 20). These facts concerning conventional and high-temperature superconductors seem to be incompatible at first sight with the scenario of no mixed phase, i.e., a phase not separated by a phase boundary from the normal phase.

I shall now argue that for $d = 3$ the consequences of the destruction of ODLRO will have few experimental consequences. This is because the length scale L_0 over which ODLRO decays for $d = 3$, is, except for H very close to $H_{c2}(T)$, an enormous length compared to other relevant length scales. It typically is of the order of a millimeter for both the conventional and high-temperature superconductors. In samples smaller than this the effects of the loss of ODLRO will be unobservable.

Furthermore, real superconductors are never free of impurities, pinning centers, etc., and Larkin[21] showed long ago that any disorder destroys the long-range crystalline positional order of the flux lines for all $d < 4$. Only positional short-range order survives, to a length scale R_c, which is a complicated function of the elastic moduli and the strength of the disorder, but which only in the better samples will ever exceed L_0. When $R_c < L_0$, the loss of ODLRO due to thermal fluctuations will be unimportant compared to the consequences of the disorder. For the purer samples, where $R_c \gg L_0$, the effects of disorder will still be felt because the pinning of the flux lines by the disorder will inhibit and slow the long-wavelength shear modes whose thermal excitation is responsible for the loss of ODLRO. A quantitative treatment of the role of disorder is set aside for future study. In addition, the debate as to whether fluctuations will melt the flux lattice is not really affected by the loss of ODLRO, as lattice thermal fluctuation length scales are much shorter than L_0.[6] The only consequence is that any melting would not be a true phase transition but a crossover between two vortex liquid states characterized by radically different degrees of positional short-range order.

For $d = 2$ (thin films) ODLRO is lost on much shorter length scales, typically of the order of the separation of the flux lines. I would expect, in this case, for there to be only a vortex liquid regime in nonzero fields, unless the exotic scenario of Ref. 13 arises from a crystal lattice without ODLRO. Theoretical evidence for only a vortex liquid regime is provided by the work of Brézin, Fujita, and Hikami,[22] whose large-order perturbative calculation for $d = 2$ (and $\kappa \to \infty$) suggested that the normal phase might be stable everywhere in a nonzero field.

Recently Ikeda, Ohmi, and Tsuneto[23] have attempted to analyze in a systematic way the fluctuations about the mean-field solution without recourse to the approximation of retaining only the lowest Landau level.[10] For the limit of infinite κ, their results are entirely consistent with those in Sec. IV. To get to finite κ they expand about the infinite-κ limit in a series in κ^{-2} and obtain results broadly consistent with those of Sec. III.

The discussion of the lower critical dimension in this paper has concentrated on the destruction of ODLRO by perturbative mechanisms, i.e., the mean-field solution was first set up and I investigated whether the fluctuations about it destroyed ODLRO. However, it sometimes happens that nonperturbative contributions from topological singularities are responsible for determining the lower critical dimension. A simple example is provided by the

Ising ferromagnet in one dimension, where the creation of domain walls separating up and down regions of spin costs a finite amount of energy, which results in the loss of long-range ferromagnetic order at finite temperature. Topological excitations of various kinds exist in superconductors and provide a possible additional mechanism for the loss of ODLRO. Thus the energy cost of creating a free vortex in the Meissner phase is finite in two dimensions because of the screening currents, so at nonzero temperatures free vortices will always be present and will produce zero superfluid density. This gives an additional argument for two being the lower critical dimension for the Meissner phase. The role of topological singularities in the mixed phase is complicated and their study is planned to be undertaken elsewhere.

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