Series approach to the randomly diluted elastic network

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Series expansions in powers of the concentration p for elastic and other susceptibilities of randomly diluted elastic networks have been generated for a bond-bending model on a honeycomb lattice up to 13th order, and for the central-force model on a triangular lattice up to 22nd order, in p. Critical exponents for both models and the critical threshold of the central-force problem have been estimated by Padé-approximant-analysis techniques. We obtain exponent estimates that are consistent with scaling relations and other calculations. For the bond-bending model, the effective splay elastic constant scales like $L^{-\phi_{sp}/\nu}$ with $\phi_{sp}=1.20\pm0.015$. For central-force elastic percolation, we find $\beta+\gamma=1.9\pm0.2$ and $\nu=1.1\pm0.2$.

I. INTRODUCTION

Recently, much effort has been expended to further our knowledge of randomly diluted elastic networks. This type of problem has been discussed by de Gennes.[1] He argued that the elastic properties of gels could be modeled by an isotropic elastic potential energy isomorphic to that used to describe resistive properties of the analogous resistor networks. Thus these two systems belong to the same universality class. Feng and Sen[2] pointed out later that for the randomly diluted elastic network consisting of central-force springs between nearest neighbors, the threshold concentration, $p_{\rm rigid}$, above which the bulk modulus is nonzero, is significantly larger than the usual percolation threshold, p_c. That $p_{\rm rigid}$ is larger than p_c reflects the fact that connections via a single bond are not rigid if only central forces are considered. Furthermore, their calculations of the bulk modulus indicated that this model belongs to a different universality class from that of the random resistor network. Since then, various people have studied various additional models, including the bond-bending model,[3,4] the granular-disk model,[5,6] and the Swiss Cheese model.[7] Claims have been made that these systems belong to four different universality classes. Recent calculations[8] seem to indicate a trend towards reunification of these classes, but the situation is still under debate.

Here we confine our attention to the model described by the following Hamiltonian:

$$H=\frac{1}{2}k_{\rm CF}\sum_b \epsilon_b|\mathbf{u}_b\cdot\hat{\mathbf{R}}_b|^2 + \frac{1}{2}k_{\rm bb}\sum_{\langle b,b'\rangle}\epsilon_b\epsilon_{b'}|\mathbf{u}_b\times\hat{\mathbf{R}}_b-\mathbf{u}_{b'}\times\hat{\mathbf{R}}_{b'}|^2 , \quad (1)$$

where $\mathbf{u}_b=\mathbf{u}_{s_1}-\mathbf{u}_{s_2}$ is the displacement associated with a bond b (connecting points $\mathbf{s}_1$ and $\mathbf{s}_2$), $\hat{\mathbf{R}}_b$ is a unit vector along the direction of bond b, ϵ_b is an indicator variable: $\epsilon_b=1$ if bond b is occupied and $\epsilon_b=0$ otherwise, and $\langle b,b'\rangle$ indicates a sum over pairs of nearest-neighboring bonds. Here $k_{\rm CF}$ and $k_{\rm bb}$ are spring constants such that when $k_{\rm bb}=0$, the model describes central-force springs and when $k_{\rm CF}=0$ one has only bond-bending forces. In contrast to the isotropic case treated by de Gennes, the model of Eq. (1) is rotationally invariant for all values of $k_{\rm CF}$ and $k_{\rm bb}$. As we have mentioned, for the central-force model[2] ($k_{\rm bb}=0$) the rigidity threshold at $p_{\rm rigid}=p_{\rm CF}$ is significantly larger than the percolation threshold, p_c. For the bond-bending model[3,4] ($k_{\rm bb}\neq0$) in two dimensions, all bonds in a given cluster are rigid with respect to one another and thus $p_{\rm rigid}=p_c$. However, $p_{\rm rigid}$ is greater than p_c in three or higher dimensions for the bond-bending model,[9–11] unless higher-order torsional interactions are included.

For these elastic models one defines the critical exponents associated with the rigidity transition at the threshold concentration $p_{\rm rigid}$ in close analogy to those of percolation and the random resistor network.[12] In terms of the bulk elastic constants and the properties of rigid clusters we write

$$\chi(p)\sim|p-p_{\rm rigid}|^{-\gamma}, \quad (2a)$$

$$P(p)\sim|p-p_{\rm rigid}|^{\beta}, \quad p>p_{\rm rigid}, \quad (2b)$$

$$B(p)\sim|p-p_{\rm rigid}|^{f_B}, \quad p>p_{\rm rigid}, \quad (2c)$$

$$\mu(p)\sim|p-p_{\rm rigid}|^{f_\mu}, \quad p>p_{\rm rigid}, \quad (2d)$$

$$\xi(p)\sim|p-p_{\rm rigid}|^{-\nu}, \quad (2e)$$

where $\chi(p)$ is the mean number of bonds in a totally rigid cluster, $P(p)$ is the probability that a bond be in an

infinite totally rigid cluster, $\xi(p)$ is the correlation length associated with total rigidity, and $B(p)$ and $\mu(p)$ are respectively the bulk and shear modulus. We will later introduce similar definitions with respect to a type of partial rigidity known as splay rigidity.[13] As we shall see, for the bond-bending model the quantities in Eqs. (2a), (2b), and (2e) reduce to those of percolation (with $p_{\text{rigid}}=p_c$) in which case the exponents β, γ, and ν are identical to those of percolation. For the model with only central forces, where the rigidity threshold no longer coincides with p_c, the critical exponents may be expected to differ from those of percolation. Accordingly, where confusion may arise, we will indicate exponents for the central-force model by subscripts CF, as in β_{CF}.

The exponents f_B and f_μ have been studied for quite some time.[2] It is believed[2] that $f_B=f_\mu$, although this equality has not yet been derived analytically from a microscopic theory. Also, it has been suggested[4,10,14,15] that, for the bond-bending model, one has the relation

$$f_B=t+2\nu \ , \tag{3}$$

where t is the exponent for the conductivity of the analogous random resistor network defined by $\sigma(p) \sim \sigma_0|p-p_c|^t$, where σ_0 is the conductance of an occupied bond (which occurs with probability p) and the vacant bonds occurring with probability $1-p$ have zero conductance. However, the justification for the relation Eq. (3) is inadequate and there is still some argument concerning a scaling picture which encompasses the exponents f_B, f_μ, and ν. We have made a systematic study of these exponents in an attempt to throw further light on the scaling picture.

At present there is disagreement in the literature as to whether f_B is the same for both the central-force model and the bond-bending model. Although most calculations[16–18] suggest that exponents are different for the two models, a recent one[8] suggests that they are the same. The disagreement may be due to anomalous corrections to scaling, the treatment of which, in turn influences the exact location of the critical threshold, and thereby the determination of the critical exponents. Our results, although not very precise, support the idea that these two models have different exponents, β, γ, and ν. We do not address the question of how the transport exponents f_B and f_μ of the two models compare with one another. Instead we concentrate on the nature of the corrections to scaling in these models, the location of the critical threshold for the central force model, and the nature of elastic correlations for $p<p_{\text{rigid}}$.

Our analysis utilizes series expansions which have proved so useful[19–21] for the study of critical exponents for the usual percolation and random resistor network models. The series expansion approach to the resistor network problem is based on the definition of the resistive and conductive susceptibilities χ_r and χ_c. These diverge at p_c with exponents γ_r and γ_c, given by

$$\gamma_r=\gamma+\phi \ , \quad \gamma_c=\gamma-\phi \ , \tag{4}$$

where ϕ is a crossover exponent for resistance. For resistor networks it has been known for some time[12] that

$$t=(d-2)\nu+\phi \ . \tag{5}$$

Thus, a series determination of γ_r (or γ_c) is equivalent to a determination of t, since γ and ν are known rather accurately.

The derivation of this relation will be discussed in detail in Sec. II, where we will also define various "elastic susceptibilities" and discuss relations [analogous to Eq. (5)] between the exponents governing their divergences. In Sec. III we describe the generation of series expansions for these susceptibilities, and present these expansions in Tables I and II. We also give some details of previous results for these models. The analysis of the expansions is described in Sec. IV. Some comparisons between the series results, the scaling relations, and the results of simulations are given in Sec. V.

TABLE I. Coefficients for the bond-bending model for the honeycomb lattice.

n	$2a_b^{(2)}(n)$	$a_s^{(2)}(n)$	$a_s^{(3)}(n)$	$a_{gs}^{(2)}(n)$	$a_{\text{el}}(n)$	$a_{\text{sp}}(n)$
1	3	3	9	3.0000	3.0000	0.0000
2	12	6	36	18.0000	10.5000	6.0000
3	24	12	114	66.0000	32.9286	22.0000
4	48	24	324	198.0000	100.0714	64.0000
5	96	48	864	534.0000	290.4775	168.0000
6	162	75	1935	1314.0000	602.9286	337.4987
7	300	144	4464	3078.0000	1576.3828	785.2938
8	546	261	9873	6876.0000	3805.2981	1661.4468
9	984	468	212 34	14 886.0000	8916.4618	3434.4906
10	1482	663	410 13	30 795.0000	16 570.8697	5715.0242
11	2952	1419	880 47	63 927.0000	42 338.7244	13 408.5268
12	4716	2166	170 130	127 002.0000	82 381.4444	22 400.3594
13	7968	3663	329 877	250 689.5300	175 893.5441	45 389.1441

TABLE II. Coefficients for the central-force model for the triangular lattice.

n	$a_b^{(2)}(n)$	$a_b^{(3)}(n)$	$a_b^{(4)}(n)$	$a_{\xi b}^{(2)}(n)$
1	3	3	3	0
2	0	0	0	0
3	12	48	156	6
4	0	0	0	0
5	24	216	1392	42
6	0	0	0	0
7	48	720	7392	162
8	0	0	0	0
9	96	2112	30912	498
10	0	0	0	0
11	420	9360	160 980	2052
12	−324	−5484	−76 212	−948
13	1284	349 92	728 340	8502
14	−1116	−270 36	−507 348	−612
15	3216	113 520	289 833 6	282 54
16	−3000	−997 20	−240 528 0	−237 66
17	154 20	536 904	147 242 28	129 276
18	−286 62	−876 090	−216 658 50	−198 492
19	583 68	224 915 4	686 966 56	561 030
20	−847 20	−336 254 4	−102 662 112	−834 828
21	214 788	929 625 6	316 890 112	230 417 4
22	−436 230	−178 739 10	−587 620 352	−424 287 0

II. ELASTIC SUSCEPTIBILITIES AND CROSSOVER EXPONENTS

We define two elastic susceptibilities, $\chi_{\rm el}$ and $\chi_{\rm sp}$, which are analogous to the resistive susceptibility χ_r for the randomly diluted resistor network.[12] Here $\chi_{\rm el}$ describes the response to extension and $\chi_{\rm sp}$ the response to splay distortion.

In the case of resistor networks, for $p<p_c$, when there are only finite clusters, χ_r is defined by[12,20,22]

$$\chi_r = \sum_j{}'[R_{ij}]_p \ . \tag{6}$$

Here $[R_{ij}]_p$ is the configurationally averaged resistance between sites i and j, and the prime indicates that the terms when $R_{ij}=\infty$ (i.e., those from configurations where site i and j are not connected) are to be omitted. The configurational average $[R_{ij}]_p$ is found by summing R_{ij} over all clusters, weighing each term by the probability that the given cluster occur. The crossover exponent for resistivity ϕ is then defined by the relation for the resistance, R_{ij}, between typical points i and j in the same cluster,

$$R_{ij}\sim|\mathbf{r}_i-\mathbf{r}_j|^{\phi/\nu} \ , \tag{7}$$

using which one can derive Eq. (4) via a scaling argument.

Inasmuch as the elastic analogs of V and I for resistors are respectively the displacement $\mathbf{u}$ and the force $\mathbf{F}$, one defines $\chi_{\rm el}\equiv[u/F]_p$ as follows. One assigns fixed displacements to sites i and j in an arbitrary cluster and allows the other sites to relax to their new equilibrium positions. One then obtains the forces $\mathbf{F}_i$ and $\mathbf{F}_j$ needed to maintain $\mathbf{u}_i$ and $\mathbf{u}_j$. Since $\mathbf{F}_i$ and $\mathbf{F}_j$ do not depend on the translational part $(\mathbf{u}_i+\mathbf{u}_j)$ or the rotational part $[\hat{\mathbf{r}}_{ij}\times(\mathbf{u}_i-\mathbf{u}_j)]$ of the displacements, one sets $\mathbf{u}_i=-\mathbf{u}_j=u_{ij}\hat{\mathbf{r}}_{ij}$, where $\hat{\mathbf{r}}_{ij}$ is a unit vector from site i to site j. Likewise, the induced forces can produce no net force or net torque, so that $\mathbf{F}_i=-\mathbf{F}_j=F_{ij}\mathbf{r}_{ij}$. In Fig. 1 we illustrate these forces for an arbitrary pair of points i and j on an arbitrary cluster. Thus we are led to define the elastic susceptibility $\chi_{\rm el}$ via

$$\chi_{\rm el}=\sum_j{}'[u_{ij}/F_{ij}]_p \ , \tag{8}$$

where the prime again denotes omission of terms from configurations with no restoring force. The analog of Eq. (7) is

$$u_{ij}/F_{ij}\sim|\mathbf{r}_i-\mathbf{r}_j|^{\phi_{\rm el}/\nu} \ , \tag{9}$$

from which it follows that

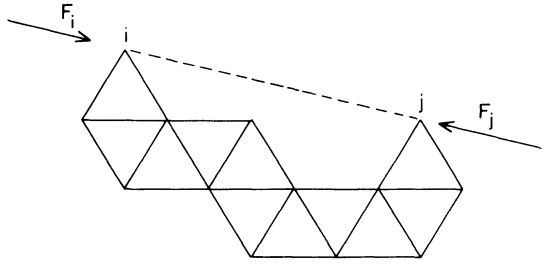

FIG. 1. Forces $\mathbf{F}_i$ and $\mathbf{F}_j$ on an arbitrary cluster where sites i and j have fixed displacements along the axis joining the two sites. Since the net force on the cluster must be zero, $\mathbf{F}_j=-\mathbf{F}_i$.

$$\chi_{\rm el} \sim (p_{\rm rigid} - p)^{-\gamma-\phi_{\rm el}} . \tag{10}$$

The splay elastic susceptibility $\chi_{\rm sp}$ and crossover exponent $\phi_{\rm sp}$ are defined similarly as follows. One rotates bond b about its center through angle θ_b and bond b' about its center by the angle $\theta_{b'}$. Since we require that there be no net rotation, we take $\theta_b = -\theta'_b \equiv \theta_{b,b'}$. In order to maintain the system in equilibrium with these displacements, one must apply forces to the system as illustrated in Fig. 2, and in particular let τ_b and $\tau_{b'}$ be the torques applied to the bonds b and b'. Since we take the total torque on the system to be zero, we set $\tau_b = -\tau_{b'} = \tau_{b,b'}$. Then we define $\chi_{\rm sp}$ as

$$\chi_{\rm sp} = \sum_b{}' [\theta_{b,b'}/\tau_{b,b'}] , \tag{11}$$

where the prime on the summation has the same meaning as Eq. (8). For the splay response one can also write the analog of Eq. (9), namely,

$$\theta_{b,b'}/\tau_{b,b'} \sim |\mathbf{r}_b|^{\phi_{\rm sp}/\nu} , \tag{12}$$

in which case we have

$$\chi_{\rm sp} \sim (p_{\rm rigid} - p)^{-\gamma-\phi_{\rm sp}} . \tag{13}$$

In writing Eqs. (10) and (13) we have assumed that both splay and total rigidity have the same threshold for rigidity. This is obviously the case for systems (in two dimensions) having both central forces and bond-bending forces. In the absence of bond-bending forces the situation is less clear.[13] Although it is possible to construct models for which the two thresholds differ,[23] most results[17,18] seem to indicate that for the model of Eq. (1) with only central forces, the two thresholds do coincide. On the other hand, a recent study[8] finds two distinct thresholds.

We now use scaling arguments[15,24] to relate $\phi_{\rm el}$ and $\phi_{\rm sp}$ to f_B, the elastic bulk modulus exponent. For resistor networks, one has the spatial Fourier transform of the voltage-voltage correlation function [denoted $\langle V(q)V(0)\rangle$] at $p < p_c$:

$$\langle V(q)V(0)\rangle = |p_c - p|^{-\gamma} f(q\xi, \sigma_0 |p_c - p|^{\phi}) , \tag{14}$$

where f is some scaling function. For $p > p_c$ we have[20]

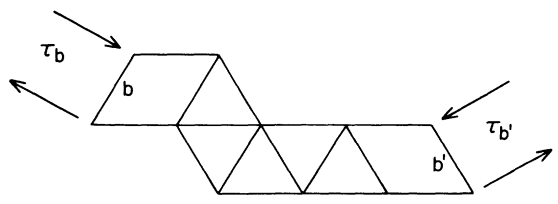

FIG. 2. Torques, τ_b and τ_b, on an arbitrary cluster when two bonds are given equal and opposite angular (splay) displacements. Since the net torque on the cluster must be zero, $\tau_b = -\tau_{b'}$.

$$\langle V(q)V(0)\rangle = \frac{P(p)^2}{\Sigma(p) q^2} , \tag{15}$$

where $P(p)$ is the fraction of sites in the infinite cluster and $\Sigma(p)$ is the bulk conductivity. We assume that the form of Eq. (14) also holds for $p > p_c$, in which case it assumes the form written explicitly in Eq. (15). Then it follows that

$$\Sigma(p) \sim \frac{P(p)^2 |p_c - p|^{\gamma}}{q^2 f(q\xi, \sigma_0 |p_c - p|^{\phi})} , \tag{16}$$

Since $\Sigma(p)$ is proportional to σ_0, we then have

$$\Sigma(p) \sim \sigma_0 \xi^2 |p_c - p|^{\gamma+2\beta+\phi} \tag{17}$$

$$\sim \sigma_0 |p_c - p|^{\gamma+2\beta+\phi-2\nu} \tag{18}$$

and using the scaling relation $\gamma + 2\beta = \alpha = 2$ and the hyperscaling relation $d\nu = 2 - \alpha$ we obtain Eq. (3).

For the bond-bending model of elastic networks, we consider the angle-angle correlation function[15,24]

$$\langle \delta\theta\, \delta\theta \rangle \sim \langle (\nabla \times \mathbf{u})(\nabla \times \mathbf{u}) \rangle \tag{19a}$$

$$\sim \frac{[P(p)]^2}{\mu(p)} \sim |p_c - p|^{-\gamma} g(q\xi, k_{\rm bb} |p_c - p|^{\phi_{\rm sp}}) , \tag{19b}$$

where g is a scaling function, $\mu(p)$ is the shear modulus. We get

$$\mu(p) \sim \frac{[P(p)]^2 |p_c - p|^{\gamma}}{g(q\xi, k_{\rm bb} |p_c - p|^{\phi_{\rm sp}})} \sim k_{\rm bb} |p_c - p|^{\gamma+2\beta+\phi_{\rm sp}} . \tag{20}$$

So assuming B and μ to have the same critical behavior, we have

$$f_B = d\nu + \phi_{\rm sp} . \tag{21}$$

If we consider the $\langle uu \rangle$ correlation function, we obtain another relation

$$f_B = (d-2)\nu + \phi_{\rm el} , \tag{22}$$

where $\phi_{\rm el}$ is the exponent characterizing the central-force spring constant $k_{\rm CF}$.

III. SERIES EXPANSIONS

In this section we present the series. In resistor networks or for usual percolation, two topologically equivalent diagrams give the same resistance R_{ij} or the same contribution to quantities such as the mean cluster size. In elastic networks, however, we need to keep track of the shape of the diagram because topologically identical diagrams (of different shape) can give different values for $\chi_{\rm el}$ and $\chi_{\rm sp}$.

The series up to order p^{13} for the bond-bending model on the honeycomb lattice are presented in Table I. To do this we generated all diagrams on this lattice with up to 13 bonds. We have constructed series for the elastic susceptibility and for several quantities that are equivalent to those for the usual bond percolation on this lattice. The quantities we have studied are defined as follows. First we have the mean number of bonds in a cluster

defined by

$$\chi_b^{(2)} \equiv \sum_{\Gamma_r} P(\Gamma_r)[n_b(\Gamma_r)]^2 \equiv \sum_n a_b^{(2)}(n)p^n , \tag{23a}$$

where the sum is over all rigid clusters Γ_r, $P(\Gamma_r)$ is the associated probability per site that the cluster Γ_r occur, and $n_b(\Gamma_r)$ is the number of bonds in the cluster Γ_r. Here a rigid cluster is one which has a nonzero elastic energy under any system of displacements other than a uniform translation or an overall rotation. We refer to such a cluster as being "totally" rigid. Furthermore, we define moments of the cluster size by

$$\chi_s^{(2)} \equiv \sum_{\Gamma_r} P(\Gamma_r)[n_s(\Gamma_r)]^2 \equiv 1 + \sum_n a_s^{(2)}(n)p^n , \tag{23b}$$

$$\chi_s^{(3)} \equiv \sum_{\Gamma_r} P(\Gamma_r)[n_s(\Gamma_r)]^3 \equiv 1 + \sum_n a_s^{(3)}(n)p^n , \tag{23c}$$

where $n_s(\Gamma_r)$ is the number of sites in the rigid cluster Γ_r. Finally we write

$$\xi^2\chi_s^{(2)} = \sum_{\Gamma_r} P(\Gamma_r) \sum_{i,j\in\Gamma r} r_{ij}^2 \equiv \sum_n a_{\xi s}^{(2)}(n)p^n , \tag{23d}$$

where r_{ij} is the vector displacement between sites i and j. The mean cluster size has a dominant critical exponent of γ, $\chi_s^{(3)}$ has the exponent $2\gamma+\beta=\gamma=\Delta$, and $\xi^2\chi_s^{(2)}$ the exponent $2\nu+\gamma$. The series for $\chi_b^{(2)}$ are two terms shorter than the honeycomb bond series for usual isotropic percolation quoted by Essam,[25] but the new series agree as far as they go with the old series. This gives a necessary and satisfying check on the reliability of our diagram enumerations and of the algorithm to calculate the coefficients. We have therefore obtained the higher moment series and that involving the correlation length. In addition we have generated two additional series for the elastic susceptibilities $\chi_{\rm el}$ and $\chi_{\rm sp}$ that were defined above for the bond-bending model. For the bond-bending model, we know that the percolation threshold is the same as that for usual bond percolation on the honeycomb lattice, $p_c=0.6527$.

For the central-force model, on the triangular lattice, we have generated four additional series through to order p.[22] Here we have generated series for moments of the cluster size as defined through

$$\chi_b^{(k)} = \sum_{\Gamma_r} P_{\rm CF}(\Gamma_r)[n_b(\Gamma_r)]^k \equiv \sum_n a_b^{(k)}(n)p^n , \tag{23e}$$

where $P_{\rm CF}(\Gamma)$ is the probability (within the central-force model) that the cluster Γ be totally rigid. For the central-force model it is essential to consider clusters of bonds because a decomposition into clusters is only possible with respect to bonds and not to sites.[13] Note that for the central-force model, $P_{\rm CF}(\Gamma)$ is different from $P(\Gamma)$ for percolation, because adding a dangling bond to a rigid cluster, for instance, does not change the size of the rigid cluster. This is illustrated in Fig. 3, where we contrast the probabilities of clusters in percolation and in the central-force problem. In contrast to the case for percolation, since rigidity is nonlocal,[9] there is no closed-form expression for $P_{\rm CF}(\Gamma)$. Because for the central-force model we are dealing with bond clusters, we define the correlation length via

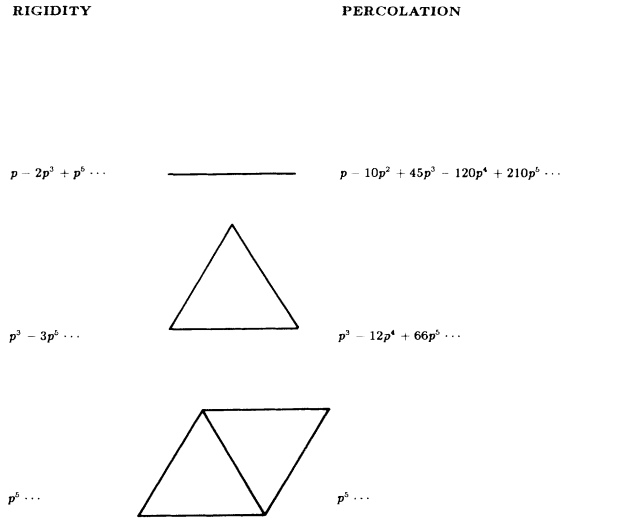

FIG. 3. Probabilities for rigid clusters in the central-force model (left) and for percolation (right). Note that for percolation the probability of having a specified cluster can be expressed in closed form. For rigidity percolation (which is nonlocal) we give the probabilities (which cannot be expressed in closed form) only up to order p^5.

$$\xi_{\rm CF}^2\chi_b^{(2)} = \sum_{\Gamma_r} P_{\rm CF}(\Gamma_r) \sum_{b,b'\in\Gamma_r} r_{b,b'}^2 \equiv \sum_n a_{\xi b}^{(2)}(n)p^n , \tag{23f}$$

where the sum is over bonds and $r_{b,b'}$ is the distance between the centers of the bonds b and b'. The series coefficients for the central-force model on a triangular lattice are presented in Table II. The exponents associated with these series are defined as for the bond-bending problem, but they may assume different values. Whether or not the exponents are the same for the central-force and bond-bending models is a question that we are trying to settle in this calculation. There are a few previous calculations that are relevant to this question. For instance, Lemieux *et al.*[16] obtained $\nu_{\rm CF}=1.05\pm0.15$, which is significantly different from that (4/3) of ordinary percolation.[26] Marshall and Harris[18] find exponents different from those of usual percolation, e.g., $\nu_{\rm CF}=1.14\pm0.1$, $\gamma_{\rm CF}=1.6\pm0.3$, and $\beta_{\rm CF}=0.46\pm0.4$. (For percolation,[26] $\gamma=43/18$ and $\beta=5/14$.) Roux and Hansen[8] find that the ratio between the transport exponent, f, and ν for the central-force problem is the same as that for the bond-bending problem. This would seem to suggest that the other exponents should also be the same but is certainly no proof of this.

The critical threshold $p_{\rm CF}$ for this model has been variously estimated by Lemieux *et al.*[16] and Day *et al.*,[17] who find $p_{\rm CF}=0.65\pm0.005$, by Marshall and Harris,[18]

who find $p_{CF}=0.64\pm0.002$, by Roux and Hansen,[8] who find $p_{CF}=0.642\pm0.002$, and by Burton and Lambert,[27] who find $p_{CF}=0.6375\pm0.0025$.

IV. ANALYSIS

A. Methods

We have analyzed the individual series presented above with two different methods,[28,29] one based on the assumption that there are nonanalytic confluent corrections to scaling and another[30] based on the assumption that there are logarithmic confluent corrections. Nonanalytic confluent corrections to scaling[34] have several origins, including irrelevant operators. They are definitely present in both isotropic and directed two-dimensional percolation,[28] and thus must be allowed for in the series discussed in this paper. The possibility of logarithmic corrections to scaling in percolation has been raised by Andelman and Berker.[32] This conjecture was based on their presence in the four-state Potts model, and was proposed in order to explain discrepancies between series expansion estimates of percolation critical exponents and the exact results. Stauffer[33] showed from a reanalysis of simulation data that the logarithmic corrections are very small and Adler and Privman[31] showed that they are absent or extremely small in percolation series for static quantities. As summarized in Ref. 28, the discrepancies between series expansion estimates of percolation critical exponents and the exact result are in fact caused by the nonanalytic corrections to scaling. There is, however, evidence from some Monte Carlo calculations[34] for a bond-bending model that there may be logarithmic corrections to the elastic critical behavior in two dimensions. Logarithmic corrections may also be present in resistor network series.[35]

In addition to the individual analyses we have studied various combinations of the series for the central-force model. These combinations eliminate the need for prior knowledge of the exact percolation threshold, and were studied using both methods of analysis. Given the problems with the determination of the threshold for this model this type of analysis is potentially extremely useful.

The analyses based on the assumption of nonanalytic corrections to scaling assume that the series being studied, denoted by $\chi(p)$ in general, has the form

$$\chi(p)\sim a(p_c-p)^{-h}[1+a(p_c-p)^{\Delta_1}+\dots]\ ,\quad p<p_c\ , \tag{24}$$

where h is the critical exponent that we wish to determine, and p_c is the critical threshold. In the first method of analysis, denoted below as $M1$,[29] we study the logarithmic derivative of

$$B(p)=h\chi+(p_c-p)\left[\frac{d\chi}{dp}\right]\ , \tag{25}$$

which has a pole at p_c with residue $-h+\Delta_1$. For a given value of p_c we obtain graphs of Δ_1 versus input h for all Padé approximants. We select the point in the three-dimensional p_c, Δ_1, and h space where all Padé approximants give as closely as possible the same residue. In the second method, denoted below as $M2$,[28] we first transform the series in p into series in the variable y, where

$$y=1-(1-p/p_c)^{\Delta_1}\ , \tag{26}$$

and then take Padé approximants to

$$G(y)=\Delta_1(y-1)\frac{d}{dy}\ln(\chi)\ , \tag{27}$$

which should converge to $-h$. Here we plot graphs of h versus the input Δ_1 for different values of p_c and again choose p_c and Δ_1 such that all Padé approximants give as closely as possible the same values of h. Both those methods have proven very useful for many problems,[35] but in general require the simultaneous determination of three critical quantities. For cases such as bond bending, where the threshold is exactly known we carry out the analyses at the exact p_c only. If Δ_1 in Eq. (26) is fixed to be unity, the second method reduces to the usual d log Padé method.

In addition to these analyses of individual series we have used a method given in Ref. 36 and recently elaborated by Meir.[37] This method involves term by term dividing two series with the same critical threshold and then studying the divided series. This divided series should have critical behavior with a threshold at $p=1$ and a dominant critical exponent equal to the difference between the exponents of the two original series plus 1. The division is expected to introduce an analytic correction to scaling (i.e., $\Delta_1=1$). If this correction has a large enough amplitude, it could provide a convergence region for the evaluation of the dominant exponent. It is to be hoped that the amplitude of the introduced analytic correction is sufficient to swamp the nonanalytic correction of the individual series which is still present. This method avoids the problems associated with uncertainties in p_c and is ideal for the central-force series.

The assumption of logarithmic corrections entails fitting to the form

$$\chi\sim(p_c-p)^{-h}|\ln(p_c-p)|^{\theta}\ ,\quad p<p_c\ . \tag{28}$$

We fitted this form with the method of Adler and Privman.[30] The analysis of the logarithmic form involves taking Padé approximants to the series

$$g(p)=-(p_c-p)|\ln(p_c-p)|\{(\chi'/\chi)-[h/(p_c-p)]\}\ . \tag{29}$$

We can show that the limit of $g(p)$ as $p\to p_c$ is θ. We take Padé approximants to g at the exact or most reliable estimate of p_c to obtain graphs of θ as a function of h.

B. Results for the bond-bending model

For this problem we know the critical threshold exactly. Therefore we have carried out the $M1$ and $M2$ analyses directly on the six bond-bending series of Table I. We expect that the first four series will have a behavior of the

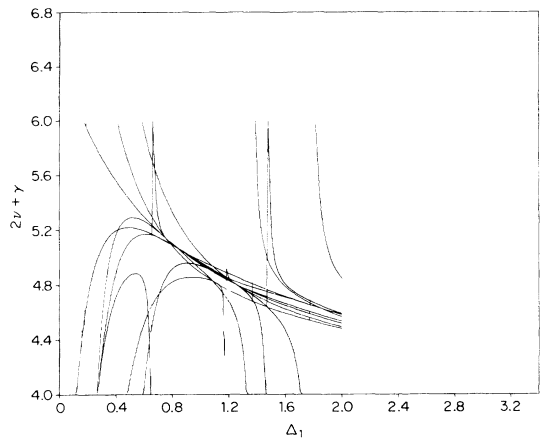

FIG. 4. Graph of Padé approximants to $2\nu+\gamma$ as a function of Δ_1 and p_c for the $\xi^2\chi_s^{(2)}$ series using $M2$.

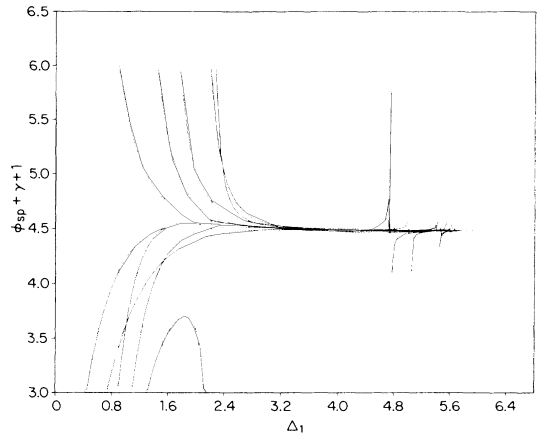

FIG. 5. Graph of Padé approximants to $\phi_{sp}+\gamma+1$ as a function of Δ_1 at p_c for the χ'_{sp} series using $M2$.

form of Eq. (24) and the last two either this behavior or that of Eq. (28). The exponent h is defined to be γ, γ, $2\gamma+\beta$, $2\nu+\gamma$, $\phi_{el}+\gamma$, and $\phi_{sp}+\gamma$, respectively. We have concentrated on the analysis of χ_{el} and χ_{sp}, since the γ and β values are those of usual percolation and are known exactly. Some selected plots of the Padé approximants that give Δ_1 as a function of input h value from $M1$, h as a function of input Δ_1 from $M2$, and h as a function of input θ from the logarithmic analysis are given in the figures.

From the $\chi_b^{(2)}$ series we found $\gamma=2.35-2.38$ from $M1$ and $\gamma=2.2-2.6$ from $M2$. When we add an additional two terms to this series we find a much tighter estimate of $\gamma=2.35-2.48$ with a correction exponent of $\Delta_1=1.05\pm0.3$. We cannot reliably distinguish this from an analytic correction. These values are consistent with the exact value of $2.38\overline{8}$, and give an idea of the error estimates for series of this length on the honeycomb lattice.

The new moment series also give exponents in good agreement with the exact values and as a sample of these results we present the graph from $M2$ for $\xi^2\chi_s^{(2)}$ in Fig. 4. We obtain $2\nu+\gamma=5.0\pm0.2$ and $\Delta_1=1.0\pm0.3$, consistent with the exact exponent of $5.05\overline{5}$.

For χ_{el} and χ_{sp} the analysis is rather less straightforward. Analysis with the usual D log Padé method [which is equivalent to assuming $\Delta_1=1$ in Eq. (24)] gave estimates of about 8.0 and 3.6 for the exponents $\phi_{el}+\gamma$ and $\phi_{sp}+\gamma$, respectively. The first of these is considerably above any other estimate that could be deduced for $\phi_{el}+\gamma$ from scaling arguments and other calculations.

For these series we are less certain that the behavior is that of Eq. (24), as discussed above. Therefore we have tested for behavior both as in Eq. (24) and with logarithmic corrections, as in eq. (28). As a control, we also tested χ_b^2, χ_s^2, and χ_s^3 as well as the longer version of χ_b^2 that is in Ref. 25 for logarithmic corrections. As expected, we found no evidence for their presence, and no significant improvement in convergence was obtained by allowing θ to be nonzero.

For the χ_{sp} series the initial constant is zero and hence we have studied both χ'_{sp} and χ_{sp}/p. The results of our analysis are presented in Table III. An overall estimate of $\phi_{sp}+\gamma=3.45\pm0.15$ and $\Delta_1=3.2\pm1.2$ can be quoted if there is no logarithmic correction, although the χ'_{sp} series are slightly better converged, thereby favoring the higher end of these ranges. If we assume no logarithmic correction (this assumption gives values close to the exact results for the other series), we obtain $\phi_{sp}+\gamma=3.6\pm0.1$ for the χ'_{sp} series and $\phi_{sp}+\gamma=3.45\pm0.3$ for the χ_{sp}/p series. For the χ_{sp} series we found that convergence in the logarithmic test was similar to that for the $\chi_s^{(2)}$ series. (θ estimates resembled those the χ_{sp}/p series but with a poorer convergence.) We therefore may conclude that there is no significant logarithmic behavior in the χ_{xp} series. This conclusion is not surprising in view of the similarity between splay and resistive correlations.[38] A graph of Padé approximants from $M2$ from the χ'_{xp} series is given in Fig. 5.

For the χ_{el} series the convergence is in general far poorer than for the other five series. Despite their ab-

TABLE III. Values of the critical exponents for the bond-bending model on the honeycomb lattice as deduced from the series coefficients of Table I.

Method	Series	Dominant exponent	Correction exponent
$M1$	χ'_{sp}	$\phi_{sp}+\gamma=3.5\pm0.1$	$\Delta_1>2.0$
$M2$	χ'_{sp}	$\phi_{sp}+\gamma=3.5\pm0.1$	$\Delta_1=3.8\pm0.6$
$M1$	χ_{sp}/p	$\phi_{sp}+\gamma=3.4\pm0.1$	$\Delta_1=2.6\pm0.6$
$M2$	χ_{sp}/p	$\phi_{sp}+\gamma=3.2\pm0.2$	$\Delta_1>2.0$
"best value"		$\phi_{sp}=1.20\pm0.15$	

sence in the other cases we decided that in view of the Monte Carlo results, it is still possible that logarithmic corrections do influence χ_{el}. Therefore we have fitted these series to the forms of both Eqs. (24) and (28). There is no indication of any convergence to the logarithmic form; enforcing a zero logarithmic correction would give an exponent $\phi_{el}+\gamma$ of 5.5 ± 0.2. A logarithmic correction of $\theta=-0.6$ gives $\phi_{el}=\gamma=6.4\pm1.0$. We note here that the form of correction of Eq. (28) is much simpler than that of Ref. 34. In Ref. 34 excellent fits were found with three adjustable parameters, and therefore our estimate may correspond to some effective fit with the more complex form. The results of the analysis via $M2$ are given in Fig. 6. We find that $\phi_{el}+\gamma=4.7\pm0.15$ and $\Delta_1=4.4\pm0.4$ from this analysis. From the figure one can observe that the usual D log Padé analysis which is equivalent to $\Delta_1=1.0$ is indeed consistent with an exponent estimate of about 8. The $M1$ analysis gives a similar Δ_1 estimate but a $\phi_{el}=\gamma=5.2\pm0.1$. We have also considered a derivative of this series. From $M1$ we find $\Delta_1=4.8\pm0.8$ and $\phi_{el}+\gamma=5.2\pm0.2$. From $M2$ we obtain $\Delta_1=4.8\pm0.8$ and $\phi_{el}+\gamma=4.8\pm0.2$.

For this model we conclude that our results for β, γ, and ν are consistent with those of ordinary percolation. We quote the overall results for the bond-bending splay elasticity exponent as $\phi_{sp}+\gamma=3.50\pm0.15$, or $\phi_{sp}=1.11\pm0.15$. We find that the estimate for $\phi_{el}+\gamma$ is strongly dependent on the form assumed for the corrections to scaling.

C. Results for the central-force model

For the central-force problem we analyzed the four series whose coefficients are given in Table II. These series should have the critical behavior $\sim|p_c-p|^{-h}$, where h is γ_{CF}, $\gamma_{CF}+\Delta_{CF}$, $\gamma_{CF}+2\Delta_{CF}$, and $\gamma_{CF}+2\nu_{CF}$, respectively. For most critical phenomena one has the scaling relation.

$$2\Delta-\gamma=2\beta+\gamma=d\nu\ , \tag{30}$$

which we may test with our data. We carried out intensive analyses for trial threshold choices in the range $0.61\le p\le0.65$. Our initial calculations involved studying the four χ series with both $M1$ and $M2$ at intervals of 0.005 throughout the above range. These calculations are summarized in Table IV. In these analyses superior convergence was definitely seen for the middle to lower end of the p_{CF} range. An example of the convergence for $M2$ at $p_{CF}=0.63$ is given in Fig. 7.

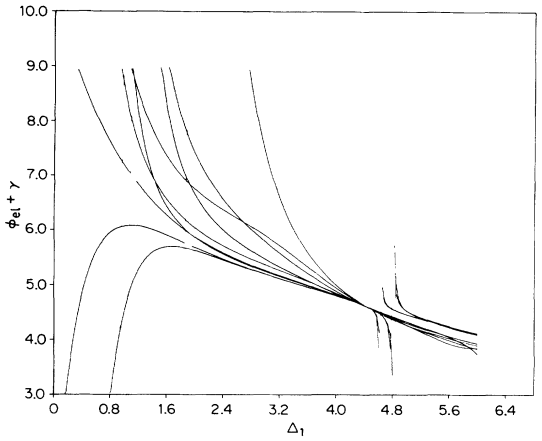

FIG. 6. Graph of Padé approximants to $\phi_{el}+\gamma$ as a function of Δ_1 at p_c for the χ_{el} series using $M2$.

There is a notable lack of dependence of the critical exponents on the value of the correction term exponent, Δ_1, relative to the amount of dependence on the p_{CF} choice, throughout the p_{CF} range. We analyzed both the series themselves and their first derivatives. These gave similar results in all cases thus minimizing the possibility that analytic parts are interfering with the Padé analyses.

We also carried out analyses of the term-by-term divided series. Here we do not have to make an initial trial choice of p_{CF} and thus we should be able to make quite a tight choice of exponent and perhaps determine an accurate critical threshold. The results of the analyses of the divided series are given in Table V. We note that since many of the terms in these series are zero, we were faced with dividing zero by itself. We set this equal to both 0 and 1 and found to our relief that this choice had no real

TABLE IV. Summary of critical exponents for the central-force model on a triangular lattice.

p_c	γ[a]	$2\gamma+\beta$[b]	Δ[c]	β[d]	$3\gamma+2\beta$[e]	Δ[f]	Δ[g]	$2\nu+\gamma$[h]	2γ[i]
0.61	0.5	1.9	1.4	0.9	3.25	1.35	1.4	1.95	1.45
0.62	0.65	2.04	1.39	0.74	3.475	1.435	1.4	2.3	1.65
0.625	0.7	2.20	1.50	0.8	3.725	1.525	1.51	2.4	1.7
0.63	0.8	2.44	1.64	0.84	3.87	1.43	1.5	2.5	1.7
0.635	0.975	2.6	1.625	0.65	3.9	1.4	1.46	2.55	1.575
0.64	1.05	2.75	1.65	0.6	4.2	1.5	1.56	2.88	1.75
0.645	1.10	2.8	1.7	0.6	4.45	1.65	1.7	3.15	2.05
0.65	1.14	2.9	1.8	0.7	4.525	1.6	1.5	3.25	2.15

[a]Source: $\chi_b^{(2)}$r.
[b]Source: $\chi_b^{(3)}$.
[c]Third column minus second column.
[d]Fourth column minus second column.
[e]Source: $\chi_b^{(4)}$.
[f]Sixth column minus third column.
[g](Sixth column minus second column)/2.
[h]Source: $\zeta^2\chi_b^{(2)}$.
[i]Ninth column minus first column.

TABLE V. Estimates of central-force critical exponents from divided series. All estimates are from term-by-term division of the indicated series by $\chi_b^{(2)}$. Derivatives are taken after division.

Series	Exponent	"0/0=0"	"0/0=1"	"0/0=0"	"0/0=1"	"0/0=1"
		without derivative		first derivative		second derivative
$\chi_b^{(3)}$	$\Delta+1$	2.8	2.8–3.1	3.0	3.0	
$\chi_b^{(4)}$	$2\Delta+1$	4.65	4.6–4.8	4.7	4.8	
$\xi^2\chi_b^{(2)}$	$2\nu+1$		2.9–3.0		3.0–3.2	3.25

effect on our results. We quote $\Delta_{CF}=1.9\pm0.2$ and $\nu_{CF}=1.1\pm0.2$. These values correspond to the exponents seen right at the top of the range of p_{CF} values that gives indications of convergence.

V. CONCLUSIONS

For the bond-bending model our main interest has been in finding numerical evidence to support the scaling relations developed in Sec. II. From the scaling relations Eqs. (19) and (20) and the result of Ref. 31 that $f_B=3.96\pm0.04$ one can predict that for the bond-bending model

$$\phi_{sp}=f_B-d\nu=1.29\pm0.04\ . \tag{31}$$

This value is slightly higher than our value $\phi_{sp}=1.11\pm0.15$. Better agreement can be obtained if a logarithmic correction of the form of Eq. (28) is allowed. Good agreement with Eq. (31) is obtained for $\theta=-0.6\pm0.6$. These results are a little ambiguous but seem to support the presence of a small, negative logarithmic correction. This comparison also indicates that the scaling relation of Eq. (21) probably holds for this system.

One would also predict from Eq. (22) that

$$\phi_{el}=f_B=3.96\pm0.04\ . \tag{32}$$

The values obtained from the regions of best convergence of the confluent analyses for $\phi_{el}+\gamma(\phi_{el}+\gamma=5.5\pm0.2)$ are not consistent with this prediction. The scaling relation predicts an exponent that is midway between the Padé result and the best convergence, and if we impose the logarithmic correction seen in the χ_{sp} case we do obtain something close to 6.35 for the central value but there is an error of ±1.0. A value of ϕ_{el} close to the scaling result has also been found for a logarithmic correction of $\theta=-0.6$. The fact that this value produces good results for both elastic crossover exponents lends support to the hypothesis that the scaling relation of Eq. (22) is also correct.

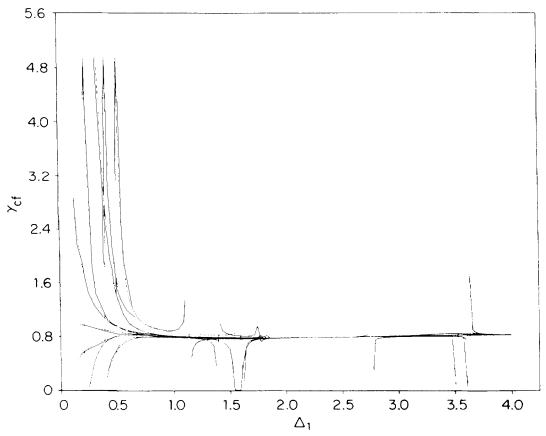

FIG. 7. Graph of Padé approximants to γ_{CF} as a function of $\Delta_{1,CF}$ at $p_{CF}=0.63$ for the χ_b^2 series using $M2$.

The overall conclusion for the bond-bending model is that the values predicted by the scaling relations for ϕ_{el} and ϕ_{sp} are consistent with a logarithmic correction of $\theta\approx-0.6$ or with a nonanalytic confluent correction of $\Delta_1\approx2.5$. The fact that there is consistency with two different types of behaviors could suggest that one is giving effective correction exponent estimates or that a third type of behavior is actually present. Further efforts in the direction of studying different types of complicated correction behavior would be most desirable. However, it is unlikely that we could obtain a consistent set of exponents if the scaling relations were incorrect.

For the central-force model we were mainly concerned to determine the location of p_{CF} and whether or not the exponents are those of usual percolation. We find results that are not completely internally consistent. From our threshold-biased analyses (Table IV) we find indications of a p_{CF} estimate of about 0.63, which is below that of other calculations. The threshold-independent analysis (Table V) gives exponent estimates that correspond best to those at the upper range of the threshold range. The values of the exponents given in Table IV from the upper range clearly differ from those of usual percolation and are barely consistent with the extreme of the range of Marshall and Harris.[18] The threshold-independent estimates are, however, in reasonable agreement with those of Ref. 18.

This numerical evidence suggests that the static exponents of the central-force problem do differ from those of usual or bond-bending percolation, e.g., $\Delta=1.9$ and $\nu=1.05$. On the other hand, Roux and Hansen[8] find a dynamic exponent that is the same for bond-bending and central-force percolation. It should be noted that we find exponents that differ from the usual ones for all thresholds within any range ever cited for the problem. Then we would naturally expect the values f_B and f_μ for the central-force model not to coincide with those of the bond-bending model. Although most numerical studies show that these models have different dynamical ex-

ponents, Roux and Hanson[8] obtain the same critical exponents for the two models. The relatively large range of threshold estimates over which some convergence is seen seems to suggest that there could be a crossover between different behaviors and the series are too short to discriminate between them and decide which is the correct one. In this case there could be two fixed points in the phase space, one corresponding to the short-range behavior and one to the larger scale behavior that is the same as that of usual percolation. The β estimates are larger than those of usual percolation, suggesting that the finite series are trying to describe a cluster structure that is more compact than usual percolation.

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