Real-space renormalization-group study of hard-core dirty bosons

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We investigate the critical phenomena of hard-core bosons in a disordered medium. Such a system is mapped onto a quantum spin-$\frac{1}{2}$ XY model with transverse random field. The system then is studied through a quantum real-space renormalization-group method. We find that randomness is always relevant in a one-dimensional (1D) system, in agreement with exact results. In two and three dimensions, there is a critical amount of disorder, below which the superfluid phase is stable. In 2D, the dynamic exponent $z=1.7$ for compressible states, and is close to the value of $z=d$ as predicted by Fisher *et al.* z is smaller for incompressible states. The correlation length exponent ν is insensitive to z, and roughly equals 1.4. Unlike the superfluid–Mott-insulator transition without disorder, which has two distinct universality classes, we find there is only one universality class for the superfluid–Bose-glass transition.

I. INTRODUCTION

The effect of randomness on the superfluid (SF) properties of bosonic systems is a challenging problem of great interest. It is well known that disorder can result in the localization of single-particle states. For an ideal-gas system, all particles will Bose condense into the lowest-energy state at $T=0$. Since the lowest-energy state is localized by any finite disorder, superfluidity is therefore destroyed. Repulsive interactions between bosonic particles, however, prevent this condensation into a single localized state. Instead, interaction allows only a finite number of bosons to occupy each localized state. Thus, naively, one expects that, as the chemical potential is raised, a "Bose-glass"– (BG-) SF transition, corresponding to filling up of localized states and Bose condensation into the first extended state, should take place at zero temperature.[1] This phase transition, as a result of the interplay of quantum-mechanical effects, disorder, and interaction, may display very different critical phenomena from the usual SF transition at finite temperature in a pure system. This problem, as interesting theoretically as it is, is by no means a purely academic question. The most direct experimental realization of disordered boson systems is perhaps the recent experiments on ^{4}He in Vycor glass and in other porous media.[1,2] In these systems the porous medium provides the random potential experienced by the ^{4}He atoms. Also, in disordered (conventional) superconductors, disorder may reduce the coherence length, and hence the size of Cooper pairs, to the scale of interparticle distance.[3] One may view, therefore, strongly disordered superconductors as disordered (composite) boson systems. Yet another example is the disordered negative-U Hubbard model, with the "medium"-T_c compound BaBiO as a possible realization.[4]

The dirty-boson problem was studied a few years ago by Ma, Halperin, and Lee.[5] In their work the authors studied a model of hard-core bosons on a lattice with random potential at zero temperature. It was established that the SF phase in the ground state (GS) can indeed be destroyed in the strongly disordered limit. However, this transition, which was shown to be driven by disorder-enhanced quantum-mechanical fluctuations, has no classical counterpart. Some scaling relations were derived. Fisher *et al.*[6] studied the dirty-boson problem more recently. They considered a system of bosons on a lattice, with soft on-site interaction. This system exhibits a Mott-insulator– (MI-) SF transition without disorder. With disorder, the transition can be a direct BG-SF one, or there can be an intermediate MI phase. Based on general scaling arguments, they deduced some specific properties of this zero-temperature BG-SF critical point.

Let δ denote the "distance" from criticality. Then the correlation length ξ diverges near criticality as $\xi \propto \delta^{-\nu}$, which defines the critical exponent ν. Besides ξ there is also a characteristic time τ, diverging as $\tau \propto \delta^{-\nu z}$, which defines the dynamic exponent z. The critical exponent ν is bounded by the (rigorously proven) inequality $\nu \geq 2/d$.[7] Since the system is compressible in both the SF and BG phases, it is reasonable to expect that it is so at criticality. By further assuming that the compressibility is completely due to collective excitations (i.e., phonons), Fisher *et al.*[6] deduced that z should be exactly equal to the dimensionality of the system. Their results, while physically appealing, should be systematically examined on a more solid ground. Furthermore, by considering whether or not the transition can be reached by tuning the chemical potential μ, they argued that the critical behavior of the MI-SF transition without disorder should depend upon whether or not there is a "particle-hole" symmetry in the system. It would be of interest to know whether that is also the case for the BG-SF transition with disorder. Finally, the exponent ν, which cannot be obtained

by scaling arguments, needs to be calculated independently.

The conventional field-theoretical renormalization-group (RG) analysis, which is one of the most powerful theoretical means developed in the past two decades to study critical phenomena, however, has not been applied successfully to this problem so far. Essentially, this is due to the fact that the BG-SF phase transition, which is a pure quantum critical phenomena and has no classical counterpart, does not have a proper Gaussian (mean-field) theory upon which one can build a perturbative expansion.[6] This motivates us to perform a real-space renormalization-group (RSRG) study of this problem. In this paper we investigate the lattice hard-core-boson model previously studied by Ma, Halperin, and Lee[5] using a RSRG method. The belief is that for the generic BG-SF phase transition, the hard-core model will have the same critical behavior as that of the more general soft-core systems. We wish to calculate ν, verify the scaling prediction $z=d$, and explore the critical properties of the BG-SF transition at and away from the particle-hole symmetric point to decide if they are in the same universality class. Since the hard-core model is equivalent to a spin-$\frac{1}{2}$ model, the calculation will also be relevant to disordered quantum spin systems. In our calculation two RG procedures are used, corresponding to zero or finite compressibility. In one dimension (1D), the SF phase is found to be always unstable, in agreement with exact results.[8,9] In 2D and 3D, there is a BG-SF transition. The exponent ν is approximately 1.4 in 2D and 1.0 in 3D, and is not very sensitive to which procedure is used. The exponent z, however, is found to be highly sensitive to whether the compressibility is zero or finite, and is equal in 2D to 1 and 1.7, respectively, in semiquantitative agreement with scaling prediction. The rest of the paper is organized as follows. In Sec. II we describe our RSRG procedures for the disordered hard-core lattice boson model, clarifying along the way certain unexplained features of the RSRG calculation of the pure quantum spin model by Jullien and co-workers.[10] This part is mainly technical, and the readers may choose to skip to the next section if they wish. In Sec. III we present and discuss our results. We conclude our study in Sec. IV.

While this work was in progress,[11] other groups have also performed numerical calculations for the dirty-boson problem.[12–15] However, the emphasis of these calculations is on the phase diagram[12,14] and critical exponents in one dimension.[13,15]

II. PROCEDURE

Hard-core bosons on a lattice may be described by the Hamiltonian[5]

$$H=-t\sum_{\langle i,j\rangle} b_i^\dagger b_j+\text{H.c.}+\sum_i (W_i-\mu)b_i^\dagger b_i \ , \tag{1}$$

where $b_i^\dagger, b_i$ are the usual boson creation and annihilation operators on the lattice site i. The hard-core condition is reflected by the restriction on the eigenvalue of the number operator $n_i=b_i^\dagger b_i$ so that it can only take the values 0 and 1. $\langle i,j\rangle$ indicates the nearest neighbor. W_i is a random on-site potential such that

$$\langle W_i\rangle=0, \quad \langle W_i W_j\rangle=W^2\delta_{ij} \ .$$

This is equivalent to a quantum spin-$\frac{1}{2}$ XY model with transverse random fields:

$$H=-\frac{J}{2}\sum_{\langle i,j\rangle}(S_i^x S_j^x+S_i^y S_j^y)-\sum_i h_i S_i^z \ , \tag{2}$$

with

$$S_i^+\leftrightarrow b_i^\dagger, \quad S_i^-\leftrightarrow b_i \ ,$$

$$J\leftrightarrow t, \quad h_i\leftrightarrow\mu-W_i \ .$$

Superfluidity in model (1) corresponds to spin long-range order in the XY plane in (2).

The pure quantum XY model, corresponding to (2) with $h_i=h$, was studied sometime ago through RSRG by Jullien and co-workers[10] using a block-spin method. Here we generalize their work to include disorder. Along the way we also clarify some questions which appeared in their calculations. Our method is closely analogous to Ma's work[16] on a dirty-fermion problem. In our approach the lattice is divided into blocks of a chosen size. Each block is governed by a block Hamiltonian. The interaction between neighboring blocks is due to the coupling between the spins at the edge of the blocks. We use two low-energy states on each block to define the block-spin variable. The renormalized (transverse) block field is given by the difference in energy in these two states, and the renormalized coupling between two blocks is obtained from the interaction energy between them. The randomness generated in the block couplings and block fields is then treated approximately, so that we can restrict our RG transformation to a finite parameter space, as we will describe in detail below.

To study the XY model (2), an essential point is to treat correctly the conservation of the z component of the total spin. To reflect this symmetry of the Hamiltonian in the RG procedure, the quantum states represented by the block spins must be chosen to be eigenstates of the z component of the total spin on the block. Or, in the boson language, these states must be the eigenstates of the block-particle-number operator. This is a rather delicate and crucially important point, and we will come back to it again. We would like to point out now, however, that the mysterious nonconvergence of the RG iteration reported in Ref. 10 in their study of the pure XY model can be understood quite clearly in the boson representation of this problem. Hereafter, we will use the spin and boson representations interchangeably, depending on whichever is more convenient to describe the physics.

We now illustrate our RG procedure in detail with the 2D triangular lattice as an example. Analogous to Ref. 10, our RG procedure consists of the following steps.

(i) Break the lattice into N/n_s blocks of spins. Each block is governed by a block Hamiltonian H_i^0 and interacts with its neighbors through the coupling between the blocks ($n_s\equiv$size of a block, $N=$size of the lattice).

(ii) Select the subspace with $\sum_{j=1}^{n_s} S_{i,j}^z=\frac{1}{2}Q$ and $\frac{1}{2}Q+1$ [or $\sum_{j=1}^{n_s} b_{i,j}^\dagger b_{i,j}=q$ and $q+1$ in the boson representation,

where Q and q are integers, with $q=\frac{1}{2}(Q+n_s)]$ for each block, and the subscripts i,j indicate site j of block i. Find the lowest eigenvalue and corresponding eigenvector of H_i^0 in each of these subspaces. These two states, together with renormalized spin operators $\mathbf{S}'$ defined to have the same function on them as the original spin operators have on the site spins, define the block spin. Note that the same Q or q value is chosen for all the blocks. We call this the fixed-q procedure. The physical content of this step and which q to choose will be discussed later.

For example, consider $n_s=3$ on the 2D triangular lattice (see Fig. 1):

$$H_i^0=-J(S_{i,1}^{\perp}S_{i,2}^{\perp}+S_{i,1}^{\perp}S_{i,3}^{\perp}+S_{i,2}^{\perp}S_{i,3}^{\perp}) -(h_{i,1}S_{i,1}^{z}+h_{i,2}S_{i,2}^{z}+h_{i,3}S_{i,3}^{z}) .$$

In the S^z representation,

$$H_i^0=-J\begin{bmatrix} x_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & x_2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & x_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -x_1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -x_2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -x_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -x_0 \end{bmatrix},$$

where

$$x_0=(h_1+h_2+h_3)/J,\quad x_1=(h_1+h_2-h_3)/J ,$$
$$x_2=(h_1-h_2+h_3)/J,\quad x_3=(-h_1+h_2+h_3)/J .$$

There are four subspaces corresponding to $Q=3,1,-1,-3$.

(iii) Renormalize the fields and coupling constant: The renormalized field of the block i is given by the energy difference of the two states:

$$h'=E_Q-E_{Q+1} . \tag{3}$$

The renormalized coupling constant is obtained through rewriting the interaction between the blocks. In the above example, it is calculated as

$$J'_{ij}=\langle +'_i-'_j|J(S_{i,2}^{+}S_{j,1}^{-}+S_{i,3}^{+}S_{j,1}^{-})|-'_i+'_j\rangle . \tag{4}$$

Now both the block field h'_i and block coupling J'_{ij} are random variables depending on the original site random fields, and their distribution functions are very complicated in general. However, if we assume that the block fields are sufficiently described by two parameters, the mean and variance of the fields, then these block fields may be represented by a Gaussian distribution with renormalized mean and width. Although J'_{ij} is random, it remains always ferromagnetic. Assuming critical properties are unaffected by the fluctuations in block coupling J'_{ij}, we can use its mean to represent the renormalized coupling constant. This is in the same spirit as Ma's calculation on the Hubbard-Anderson.[16] As explained below, one need not be concerned with the flow of the average field in the present RG procedure; we thus keep track of only the renormalization of the variance of the random fields.

(iv) Repeat the procedure to find the fixed point(s) of the RG transformation and calculate the critical exponents associated with them.

Before we proceed, we would like to make following remarks.

(a) As the RG iteration proceeds, one will finally reach a stage that the whole system is represented by a single spin variable (assume $n_s^n=N$, for some integer n). The corresponding lowest-energy state, written in the Hilbert space of the original Hamiltonian, is the (approximate) GS. The wave function constructed in this way will, presumably, be closer to the true GS of the system as n_s, the size of the block in our procedure, be chosen larger and larger (when $n_s=N$, it becomes the exact GS). Thus this block RG procedure may be viewed as a systematic way to construct a trial wave function for the GS of the system.

(b) Assuming that the random field can be characterized by its mean h and its variance $\tilde{h}$, the Hamiltonian is described by the three parameters J, h, and $\tilde{h}$. Out of these we can take h/J and $\tilde{h}/J$ to be our two dimensionless variables and study their RG flows. However, in the fixed-q procedure, the choice of q predetermines and so fixes the density $n(q)$ of the system, with $n(q)=q/(n_s-1)$. Thus the flow of h/J will not converge unless one chooses h to exactly equal the correct chemical potential for the density $n(q)$. These are the unstable fixed points of Ref. 10. The flow of h/J therefore contains no physics. Worse, there is nothing to guarantee that $n(q)$ is commensurate with the block size, in the sense that the product $n_s n(q)=q+\frac{1}{2}$. For example, if $q=1$, $n_s=5$, $n(q)=\frac{5}{4}$. This means that the correct chemical potential for the whole system is not the correct chemical potential for the block. This is the origin of the oscillatory behavior of the h/J flow in Ref. 10, which merely corresponds to unavoidable successive overshooting and undershooting of the density. We note here that a case where commensurability is guaranteed is when n_s is odd and $q=(n_s-1)/2$, which gives a density of $\frac{1}{2}$ bosons per site. This will prove to be useful later. Since the

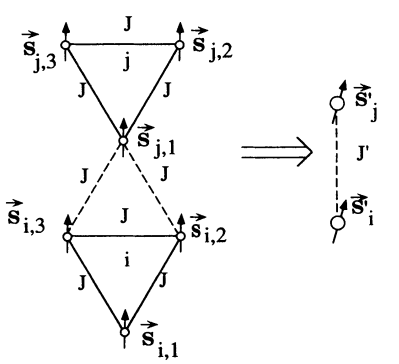

FIG. 1. Renormalization on triangle lattice where block size $n_s=3$. See text for details.

scaling of h/J is meaningless because of the procedure fixing the density, we simply drop this parameter from the problem and keep only the variance of the field. We have then effectively a one-parameter ($\tilde{h}/J$) theory. Depending on whether $\tilde{h}/J$ scales to 0 or ∞, the system is then in the SF or BG phase. The phase diagram obtained this way corresponds to changing $\tilde{h}/J$ while keeping the density constant, with the density given by the choice of q.

(c) In our actual calculation, the random average in step (iv) is carried out numerically by averaging over a finite number of random configurations. More specifically, for a given block of size n_s, we use a random generator to generate typically a few thousand independent configurations of random fields on the blocks according to a (symmetric) Gaussian distribution. After performing step (iii), we obtain a set of block fields $\{h_i'\}$ and a set of coupling $\{J_{ij}'\}$ between two blocks. We use the average and standard deviations of these block fields to approximate the mean and variance of the block random field and use the average of these couplings to obtain the renormalized coupling constant between the block spins. In this way the renormalized Hamiltonian is confined at each iteration to the same parameter space as that of the original Hamiltonian.

By choosing the value of q to be the same for all blocks in step (ii), we have forced the density to be uniform. Clearly, in the presence of the random potential, this cannot be the case. Allowing the density to adjust to the random potential will "screen" the latter and stabilize the SF phase. This will change the critical value of $\tilde{h}/J$, but may not be crucial to the critical phenomena. A more unfortunate consequence of (ii) is that since q is discrete, so then is the density $n(q)$, and we have forced the system to be incompressible. While this may or may not be a serious problem for all the critical exponents, scaling arguments indicate that it should drastically affect the exponent z. Fortunately, at least for the aforementioned special density of $\frac{1}{2}$ bosons per site, a procedure which gives a finite compressibility can be used. This modified RG procedure is based on the observation that the particle density can also be fixed statistically, instead of being fixed by fixing q as one did in step (ii). Thus we modify our RG procedure by replacing step (ii) with step (ii').

(ii') For a given block Hamiltonian H_i^0 with quenched disorder, find the lowest eigenvalue for each subspace characterized by $\sum_{j=1}^{n_s} b_{i,j}^\dagger b_{i,j} = q$ in the boson representation ($\sum_{j=1}^{n_s} S_{i,j}^z = \frac{1}{2}Q$). Find the minimum among them. Denote the corresponding q value by $q_{\min}$. We find numerically that the q value corresponding to the subspace which has the next lowest eigenvalue is either $q_{\min}+1$ or $q_{\min}-1$. We choose the lowest-energy state of these two subspaces with adjacent q values, to represent the block spin.

Obviously, $q_{\min}$, and hence the density of such a block state, fluctuates according to the random configuration of the on-site potentials in the block. The finite compressibility is due to the fact that upon the addition of a small uniform field, there may be blocks with the appropriate random fields so that the value of $q_{\min}$ may change, or the q value of the state with the next lowest energy may change (from $q_{\min}+1$ to $q_{\min}-1$, say). In other words, by allowing q to change from block to block, the density is now continuous and can change continuously with the chemical potential.

The dependence of $q_{\min}$ on a uniform field means that, in general, we *do* have to keep track of the renormalization of h/J. Fortunately, for the particle-hole symmetric case, corresponding to the case where there is on the average $\frac{1}{2}$ particle per site (or $\langle S_j^z \rangle = 0$ in spin language), the average field is zero by symmetry. If one starts at this point so that the original random field has zero mean, the probability for (3) to be positive or negative with the same magnitude is equal. Thus, at stage (iv), when the random average is carried out, the system is again renormalized at this symmetric point, and h/J remains zero under RG iteration.

This modified RG method, utilizing the "conservation" of the particle-hole symmetry, also introduces an artificial dependence on the choice of the size of the block. In the zero-disorder limit, $q_{\min}$ is the same on all blocks. If the block size n_s is chosen to be even, there is no q value such that the states determined by step (ii) are particle-hole symmetric. For example, if $n_s=2$, then $q_{\min}=1$, and the other q state needed to define the block spin must be arbitrarily chosen to be either $q=0$ or 2, thus breaking the particle-hole symmetry. h/J will become nonzero under renormalization and, in fact, oscillates about 0 with successive iterations, as previously discussed. The pure fixed point at $h=0$ is unstable and pathological. With disorder, $q=0$ and 2 will be chosen with equal probability in the above example according to the random fields of the blocks, and h/J will remain zero under renormalization. As a result, the SF phase will be given by a stable fixed point with finite $\tilde{h}/J$ for n_s even (see Sec. III).

Suppose now that one starts a RG iteration away from the particle-hole symmetric point. The random field has a nonvanishing mean h. Since both the SF and BG phases are gapless,[6] h will renormalize to zero. h/J, on the other hand, may still have some nontrivial fixed point. According to Fisher *et al.*,[6] the general lattice boson system can be described by the effective classical action

$$S=\frac{1}{2}\int_\omega \sum_i (r_i+i\omega g_i)|\psi_i(\omega)|^2 + \frac{1}{2}\int_{k,\omega}(k^2+\omega^2)|\psi(k,\omega)|^2+u\int_\tau \sum_i |\psi|^4 ,$$

where $\psi_i(\omega)$ is the local order parameter at site i and ω is the frequency. This system has a SF-MI transition without disorder and a SF-BG transition, but possibly also a SF-MI transition with disorder. Note the lack of space-time isotropy due to the term linear in ω. The coefficient g_i is equal to $\partial r_i/\partial\mu$, where r_i is the mean-field local transition temperature in the equivalent $(d+1)$-dimensional statistical-mechanics problem. In the pure system, r_i and hence also g_i are, of course, independent of i. Therefore, for densities where the SF-MI transition

cannot be tuned by μ, $g_i \equiv 0$. Otherwise, $g_i \neq 0$. In the former case, the action has space-time isotropy, and the SF-MI transition is in the same universality class as the $(d+1)$-dimension XY model. In the latter case, the resulting space-time anisotropy is relevant, and the (generic) MI-SF transition corresponds simply to a transition from commensurate to incommensurate densities (the vacuum or filled state to an unfilled state in the hard-core case). In the presence of disorder, g_i changes from site to site and is never identically zero. Nevertheless, where the critical disorder for the SF-BG transition is a local maximum or minimum, and hence where the transition cannot be tuned by changing μ, it has a zero mean g. It is not known whether the $g=0$ and the more generic $g \neq 0$ systems or, for that matter, the $g_i \equiv 0$ model considered in Ref. 5 belong to the same or different universality classes. A related point is that for the generic system, the compressibility is also just the "specific heat" of the equivalent statistical-mechanics problem. This imposes *a priori* a relation between the compressibility and specific-heat exponents which is not necessarily present in the case of $g=0$. For the hard-core-boson Hamiltonian (1), it is clear by symmetry that the critical disorder is a maximum or minimum for the particle-hole symmetric ($\frac{1}{2}$-filled, $h/J=0$) system. In fact, since the $\frac{1}{2}$-filled system has the lowest kinetic energy without disorder, the critical disorder should be a maximum. Thus it is of interest to investigate whether this system has different exponents from the transition away from particle-hole symmetry. One can probe this in the fixed-q procedure [step (ii)] by seeing if the critical exponents depend on whether the choice of q is particle-hole symmetric. More reliably, one should study the RG flow on the $(h/J, \tilde{h}/J)$ plane using the modified method [step (ii')], to see if h/J is relevant at the critical fixed point with $h/J=0$. Because of the reason we discussed before, even this modified method will become ill defined as one approaches the pure limit $\tilde{h}/J \to 0$ with finite h/J. Away from the pure limit, however, the fluctuation in random field accommodates enough density fluctuation so that the density can in fact follow the renormalization of h. Thus one can obtain within this RG scheme the RG flow of h/J near the critical fixed point and in the disordered phase.

III. RESULTS

The RG procedures described above are carried out on one-, two-, and three-dimensional systems. For a 1D chain, we find no (nontrivial) fixed point, and the system is found to be in the disordered phase for any amount of randomness. This agrees with our exact result obtained previously[8] and also with the more general scaling studies of Giamarchi and Schulz[9] and Nagaosa.[9]

In two dimensions the calculation is performed on both the triangular and square lattices. The blocks we used in our calculations are shown in Fig. 2(a). Renormalization with fixed q (incompressible) and renormalization at the particle-hole symmetric point, but allowing density to fluctuate locally (compressible), are both performed. However, for block $S3$ we calculate only in the case of $q=1$. Calculation in three dimensions is performed on the cubic lattice, with the block shown in Fig. 2(b), for the case of uniform q with $q=1$ only. Since the dirty-boson system has a finite compressibility, it would seem the calculations with uniform q are useless. However, incompressibility occurs if there are long-ranged interactions, with the range of the interaction determining how the static density-density correlation function vanishes at long wavelength. This in turn gives z through scaling.[17] Thus we can apply our results for fixed q to the appropriate incompressible system. Alternatively, we can view the calculations as calculating critical exponents for different z's.

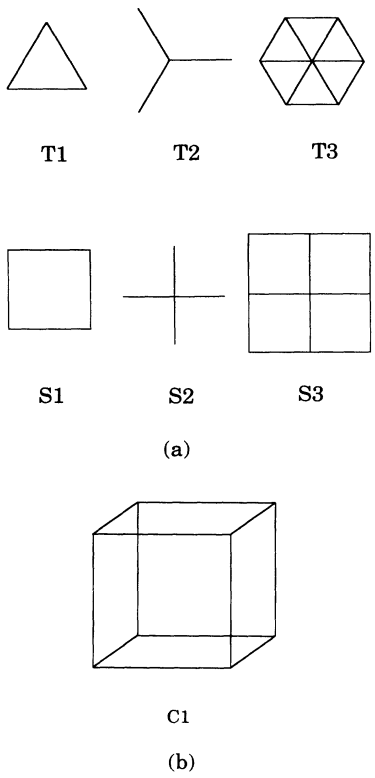

FIG. 2. (a) Blocks used in our calculations on the 2D triangular and square lattices. (b) Blocks used in our 3D calculation on the cubic lattice.

In 2D and 3D, we find the SF phase to be stable against weak disorder, and there is a critical (unstable) fixed point at a finite value of $\tilde{h}/J$. For the fixed-q procedure [step (ii)], the RG flow is shown schematically in Fig. 3. As remarked and explained earlier, if the block size is even, the stable fixed point for the SF phase is not at $\tilde{h}/J=0$, but at $\tilde{h}/J$ finite. The physics around the BG-SF transition point, however, is not expected to have and has no apparent dependence on whether the size of the block is odd or even.

The critical exponents ν and z are calculated at the nontrivial fixed point $(\tilde{h}/J)^*$ as follows.[18]

(1) By linearizing the RG iteration function $(\tilde{h}/J)' = f(\tilde{h}/J)$ as

$$\left[\frac{\tilde{h}}{J}\right]' = \left[\frac{\tilde{h}}{J}\right]^* + a\Delta\left[\frac{\tilde{h}}{J}\right], \tag{5}$$

we obtain ν according to

$$\nu=\frac{\ln(n_s^{1/d})}{\ln a} . \tag{6}$$

(2) The dynamic critical exponent z relates the energy rescaling to length rescaling. Since the Hamiltonian is scale invariant except for the overall energy (frequency) rescaling, z is obtained from the RG recursion relation for J at the fixed point (F.P.):

$$z=\frac{\ln(J/J')_{\text{F.P.}}}{\ln(n_s^{1/d})} . \tag{7}$$

Table I shows the results of the RG procedure with fixed q on 2D lattices, and Table II gives the results of the modified RG method with (ii) replaced by (ii').

Because of the approximations we used in the calculation, especially the truncation of the random distribution of the renormalized block field and coupling, we expect that the results should improve with increasing block size and also with increasing connectivity inside the block, so as to minimize boundary effects. Hence we believe $T3$ to give the most reliable results. The value of ν calculated by either method is essentially the same, roughly equal to 1.4, independent of whether the procedure gives zero or finite compressibility. However, as expected from scaling arguments, the z value is considerably larger for the latter case, with z equal to 0.9 and 1.7, respectively. Since 1.7 is quite close to 2 and since the value of z slowly increases with increasing block size n_s, this result may be viewed as a partial confirmation of the scaling prediction of $z=d$ (i.e., 2) in 2D.

For 3D our calculation gives $\nu\approx 1$ for the fixed-q procedure which corresponds to an incompressible system with $z\approx 1.2$. Unfortunately, because of computational limitations, we are unable to perform the calculation for compressible states. Assuming that the relatively weak dependence of ν on the compressibility in 2D is not merely coincidence, we postulate $\nu\approx 1$ for that case also. Both our 2D and 3D values of ν satisfy the modified Harris criteria.[7]

We would like to remark that a third independent critical exponent η, defined by the power decay of the SF correlation function at the criticality through

$$\overline{\langle b_i^{\dagger} b_j\rangle}\propto|\mathbf{R}_i-\mathbf{R}_j|^{-(d+z-2+\eta)} , \tag{8}$$

TABLE I. Results of RG procedure with fixed q on 2D lattices.

Block	Basis (q)	$(h/J)^*$	ν	z
$T1$	1	1.7	1.6	0.9
$T2$	2	1.4	1.6	0.8
$T3$	3	2.5	1.5	0.9
$T3$	4	2.3	1.5	0.9
$T3$	5	1.5	1.7	0.9
$S1$	2	1.0	1.8	0.9
$S2$	2	1.2	1.6	0.8
$S2$	3	0.9	1.8	0.8
$S3$	7	0.7	2.1	0.9
$C1$	6	1.6	1.0	1.2

TABLE II. Results of modified RG method with (ii) replaced by (ii').

Block	$(h/J)^*$	ν	z
$T1$	3.8	1.6	1.6
$T2$	3.7	1.9	1.6
$T3$	4.9	1.4	1.7
$S1$	2.6	1.5	1.7
$S2$	2.5	1.6	1.6

or in spin language,

$$\overline{\langle S_i^x S_j^x\rangle}\propto|\mathbf{R}_i-\mathbf{R}_j|^{-(d+z-2+\eta)} , \tag{9}$$

can in principle also be calculated. Suppose the spins on a block transform in the RG iteration according to

$$S_{i,p}^x=\xi_{i,p}^x S_i^{x'},\quad p=1,2,\ldots,n_s , \tag{10}$$

and similarly for y and z components. $\xi_{i,p}^x$'s can be calculated in a way similar to (4).[8] If one further ensemble averages the spin renormalization

$$S_{i,p}^x=\frac{1}{n_s}\sum_p \xi_{i,p}^x S_i^{x'}\equiv\xi^x S_i^{x'} , \tag{11}$$

then the critical exponent η can be computed as

$$d+z-2+\eta=-\frac{2\ln\xi^x}{\ln n_s^{1/d}} . \tag{12}$$

However, this is more like treating the randomness as annealed and, therefore, is unreliable for the present situation. Equation (10) itself, with no ensemble averaging, must be used for determining η. This would require keeping track of the complete distributions of the renormalized fields and couplings, and not just the first and second moments, throughout the RG calculation. This is planned to be addressed in a later work.

The fixed-point value $(\tilde{h}/J)^*$ of course also gives us the critical randomness for the SF-BG transition. For the fixed-q procedure, the values of $(\tilde{h}/J)^*$ for different q's and blocks are listed in Table I. Here the critical value is that for the SF-BG transition at constant density given by $n(q)=q/(n_s-1)$. In the modified RG method [step (ii')], one has instead a separatrix on the $(h/J,\tilde{h}/J)$ plane (see Figs. 3 and 4), so that the value of $\tilde{h}/J$ on the separatrix gives the critical disorder for the SF-BG transition at constant chemical potential. However, for $h/J=0$, particle-hole symmetry implies that the average density must be independent of $\tilde{h}/J$ and equal to that for the particle-hole symmetric choice of $q=(n_s-1)/2$. Since for (ii') the local density adjusts to the randomness and, in fact, partially screens out the latter, the net result is to have the fixed point $(\tilde{h}/J)^*$ at a much larger value and presumably closer to the true one than the one obtained from the procedure with fixed local density [(ii)]. Our most reliable estimate ($T3$ and $S1$) is for the transition at average density equal to $\frac{1}{2}$ to occur at $\tilde{h}/ZJ\approx 0.8$ (Z is the coordination number) for the triangular lattice and $\tilde{h}/ZJ\approx 0.6$ for the square lattice.

As we discussed at the end of Sec. II, the critical properties of the BG-SF phase transition may depend upon

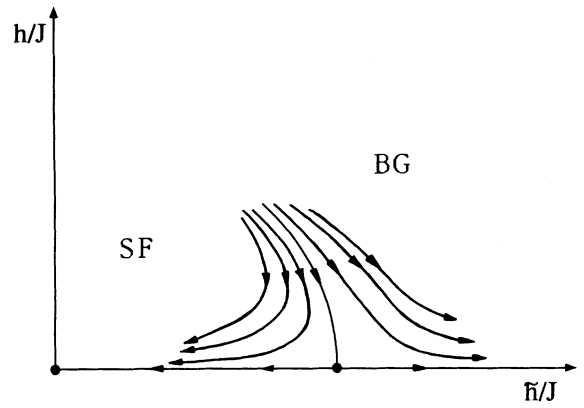

FIG. 3. Schematic RG flow for procedure (ii') for n_s odd.

whether or not there is a particle-hole symmetry in the system or, more precisely, whether the transition can be tuned by the chemical potential. Our results with fixed density (see Table I) show no clear difference in the values of z and ν between the particle-hole symmetric choice and other choices of q, suggesting that this is not the case.

To confirm this we study the RG flow through the modified RG method with finite h. The following scenarios are possible.

(1) The only nontrivial fixed point outside the pure limit is at $h/J=0$ and is stable in the h/J direction. In this case all SF-BG transitions belong to the same universality class.

(2) Same as (1), but the fixed point is unstable in the h/J direction. The generic SF-BG transition is in the same universality class as the generic SF-MI transition of the pure system, but that with particle-hole symmetry is in a different universality class.

(3) There are additional nontrivial fixed points with $(h/J)^*\neq 0$. By symmetry, they always come in pairs, one on either side of the axis. Consider one additional pair and that they are unstable in the h/J direction. Now there are a finite range of densities surrounding the particle-hole symmetric one whose SF-BG transition is in a different universality class from that (the generic pure SF-MI universality class) of other densities.

(4) Same as (3), but these fixed points are stable, while

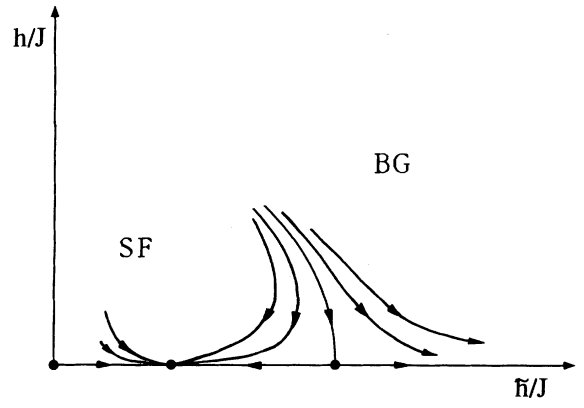

FIG. 4. Schematic RG flow for procedure (ii') for n_s even.

the particle-hole symmetric fixed point is unstable in the h/J direction. In this case the SF-BG transition has two universality classes, depending on whether there is particle-hole symmetry, and both are different from the generic pure SF-MI one.

Scenarios (3) and (4) would imply there are partially filled densities which are special but not due to symmetry, a rather unlikely possibility (additional pairs of fixed points or even fixed lines will be even more unlikely). There is no *a priori* reason to rule out (2), although it seems physically strange. It should be emphasized that the fixed point at $h/J=0$ is a consequence of symmetry, and its presence is not in question, only its stability. Let us add a small positive uniform field h to the particle-hole symmetric fixed-point Hamiltonian. For a given block, the eigenstates of the block Hamiltonian will be unchanged, while the eigenvalues for each q subspace will be shifted by $\frac{1}{2}Qh=(q-\frac{1}{2}n_s)h$. Suppose for the ith block the lowest-energy states without the uniform field correspond to q and $q+1$, and h_i' is the renormalized block field; then, if q and $q+1$ remain the lowest-energy states with the uniform field, the new renormalized block field is simply given by $h_i'(h)=h_i'+h$. Since h_i' has a zero mean, the renormalized uniform field is then $h'=h$. Since $J'<J$, this would imply that h/J is relevant. However, the above does not take into account that in some blocks the values of q for the lowest-energy states will change. Specifically, the important case is when, without the field, $q+1$ is the ground state, q the first excited state, but with the field, $q+2$ becomes the first excited state. For such a block, the renormalized block field is now $h_i'(h)=-h_i'+O(h)$. Because typically $h_i'\approx\tilde{h}$, the renormalized uniform field, when taking these special blocks into account, is

$$h'=h(1-P_2\tilde{h})\,, \tag{13}$$

where P_2 is a (q-averaged) joint density of states of having q as the first excited state and $q+2$ as the next excited state in zero uniform field. If $h'/J'<h/J$, then the particle-hole symmetric fixed point will be stable.

Our numerical calculations (see Figs. 3 and 4) show that this is in fact the case, although, because of statistical error, h/J does not completely renormalize to zero, but shows small random fluctuations about zero. Furthermore, we find no signs of additional fixed points, thus providing further proof for discarding scenarios (3) and (4). Thus we conclude that the SF-BG transition has only one universality class.

IV. DISCUSSION

We investigated the lattice hard-core dirty-boson problem through the RSRG method. Our method reproduces the exact result[8] in 1D system, which shows, for hard-core bosons with *no additional interaction,* instability of the SF phase against any amount of disorder. It is also in agreement with the more general perturbative scaling studies of Ref. 9, which shows that the SF phase is stable (unstable) against weak disorder if the exponent of power-law decay of the order-parameter correlation function in the pure system is less than (greater than) $\frac{1}{3}$, since

it equals $\frac{1}{2}$ for the hard-core model. In 2D and 3D, we find a nontrivial fixed point which separates the SF and disordered (BG) phases. Thus our results show that the lower critical dimension d_{LCD} for the zero-temperature BG-SF phase transition is below 2 ($1 \leq d_{\mathrm{LCD}} < 2$). The aforementioned calculations by Giamarchi and Schulz[9] and Nagaosa[9] for more general 1D boson systems point to $d_{\mathrm{LCD}} = 1$.

In 2D the critical exponents ν and z were calculated with two RG procedures, corresponding to zero and finite compressibility. Our results tend to agree with the scaling prediction $z = d$ ($=2$) for the generic BG-SF phase transition,[6] which is believed to have a finite compressibility at the transition. It should be noted that the scaling prediction is based on the assumption that, even at the transition, the compressibility is completely due to the phonon mode. This does not have to be so: The SF can disappear by the phonon remaining robust, but with vanishing speed of sound, or by the appearance of low-energy single-particle-like excitations.[19] In the latter case, there is no reason for $z = d$, as the action for the phonon mode is no longer well defined. Our calculation can be viewed as support of the former scenario. However, we also cannot rule out z close to but less than 2. We hope to resolve this question in the future by going to larger blocks and carrying out the RG calculation actually on a large but finite lattice, thereby eliminating the errors incurred in approximating the distributions of fields and couplings by first and second moments in step (iii). This will also allow us to calculate the exponent η.

The incompressible case may be relevant to bosons with long-range interactions, and we obtain a smaller value of z (≈ 1), also in agreement with scaling predictions.[17,20] Viewing our calculations as calculating ν for different values of z, we find ν to be roughly 1.4 in both cases. This gives us confidence in its correctness, since, although we can relate our two procedures to compressibility, any feature artificially put in by the RG procedure is undesirable. Dirty bosons with Coulomb interactions in 2D have been used to model the superconductor-insulator transition in thin films.[20] It would be interesting to compare our value of ν to the appropriate experimental data.[21] One possibility is to study the transition temperature $T_c \propto \delta^{\nu z}$, where δ is the deviation from criticality, thus allowing ν to be determined if we assume z is correctly given by scaling. In a recent experiment, Liu *et al.*[22] found that the low-temperature conductivity of Bi films obeys scaling through a characteristic temperature that vanishes with an exponent close to 1.4 or 1.5 as one approaches the transition from the superconducting side. While the scaling behavior is not completely understood (and the data on the insulating side scale with a different exponent), if we take this exponent to be $z\nu$ and assume the scaling prediction $z = 1$, then the experimental value of $\nu \approx 1.4$. While the equality between our rough calculated value and rough experimental value of ν must be completely fortuitous, they are at least not way off from each other. For 3D our calculation suggests $\nu \approx 1$. Our values of ν satisfy $\nu \geq 2/d$.[7]

Fisher *et al.*[6] have shown that the pure SF-MI transition has two distinct universality classes, depending on whether the transition can be tuned by the chemical potential. It is possible that this is the case for the SF-BG-transition also. Within the present context, this implies that the critical phenomena may depend on whether there is "particle-hole" symmetry, since it will determine if, in the effective action, the term of first order in frequency has a zero average coefficient. The model used in Ref. 5 was precisely criticized because it neglects this term completely, thus implicitly assuming this symmetry, and may not describe the generic SF-BG transition. The present calculation shows that the critical phenomena are independent of this symmetry and that the SF-BG transition has only one universality class. Thus, provided only the mean value of the coefficient of this linear term is important, we believe the model considered in Ref. 5 to be adequate. However, the calculation of Ref. 5 also artificially put in a gap in the insulating phase and so is applicable only to the SF-MI transition (about which we can say nothing) and not to the SF-BG transition.

The hard-core-boson model is the limiting case of on-site repulsive boson models, when the strength of the repulsion, U, goes to infinity. Clearly, in an exact renormalization procedure, the hard-core condition is not preserved under renormalization. Hence, even if U is infinite initially, it becomes finite under renormalization. In our approximate procedure, where only two states are kept, this flow of U is ignored, and the RG flow corresponds to the actual flow projected onto the hard-core plane. However, this does not *per se* prevent one from obtaining reliable critical exponents. Thus, although we study specifically the hard-core-boson model in this paper, our results should apply to the more general soft-core model also.

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