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Short-distance expansion for the spin-spin correlation function of uniaxial dipolar systems

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Motivated by recent work on the critical resistivity of gadolinium, a detailed study has been made of the temperature dependence of the two-point vertex function in the large-momentum regime. The operator-project expansion is used to calculate the temperature dependence in uniaxial systems at their critical dimension. Explicit forms are presented for systems with short-range interactions and with dipolar interactions.

I. INTRODUCTION

The two-point spin-spin correlation function is of fundamental importance in the description of phenomena due to critical fluctuations in the vicinity of the Curie temperature of magnetic phase transitions. Its Fourier (lattice) transform determines the quasielastic scattering cross section, in the Born approximation, for neutron or conduction-electron scattering from the (localized) spin systems. It has been argued that quasielastic short-distance correlations ($R \ll \xi$, where ξ is the correlation length of spin fluctuations) dominate the spin-fluctuation contribution to the electrical resistivity $\rho(T)$ for T sufficiently close to T_c. The resulting theoretical description has provided a very useful framework for understanding resistive anomalies in a wide variety of materials.[1-6] Restricting our attention to ferromagnets, the predictions that (a) $d\rho(T)/dT>0$ and (b) $d\rho(T)/dT$ is proportional to the heat capacity for $T \rightarrow T_c$ have been confirmed at least qualitatively. In the particular case of nickel, a canonical example, even a reasonable determination of the heat-capacity exponent was obtained from resistivity data.[7] However, there is a conflict between theory and experiment in the case of gadolinium where the slope of c-axis resistivity above T_c is negative even very close to T_c.[8] The predictions of a positive slope at T_c and the expansion form of the short-distance correlation functions have been studied based on the assumption that the interactions between spin are isotropic and of short range.

However, it has been pointed out that long-range magnetic dipole interactions should become observable in an experimentally accessible range near T_c in the case of Gd.[9] These dipolar interactions cause a crossover as the temperature is reduced in the paramagnetic region from an isotropic Heisenberg Ruderman-Kittel-Kasuya-Yosida (RKKY) regime to an isotropic dipolar regime with a crossover temperature approximately 6.3 K above T_c. As the temperature is reduced further, there is a second anisotropic crossover temperature approximately 0.5 K above T_c with the asymptotic critical behavior being of Ising dipolar character as a consequence of uniaxial symmetry of the system.[9,10] This anisotropy has been established experimentally in a study of the magnetic susceptibility, χ.[10] The c-axis susceptibility χ_c is a measure of correlations of the order parameter, the c axis or z component of the spin, S^z, and exhibits strong critical fluctuations. On the other hand, the basal plane components of the spin are secondary degrees of freedom and much weaker temperature dependence near T_c is observed in χ_b. In the present work we focus attention on the order parameter in the asymptotic Ising dipolar regime to obtain an expansion which determines the temperature dependence in the short-distance limit of its correlation function, $G^{zz}(q,T)$.

In view of the fundamental importance of this type of problem, the short-distance expansion for isotropic short-range interaction systems has been studied extensively by a variety of theoretical techniques (see the review by Brézin[11]). In this paper, we have chosen to base our work on renormalized perturbation theory coupled with the operator-product expansion of Wilson[12] (see also Polyakov and Kadanoff[13,14]). This method addresses most directly the problem and also has some important technical advantages. In particular, it allows a decoupling of the temperature and momentum dependence in such a way that it is not necessary to make any *a priori* assumptions about the temperature dependence at large momentum (momentum arguments are, hereafter, denoted by $\mathbf{q}$).

The upper critical dimension of the uniaxial dipolar system is $d_c=3$, it is thus possible to use the renormalization-group method without introducing any type of ϵ expansion. At $d=d_c$, physical quantities are expected to exhibit classical Landau theory power laws but with nontrivial logarithmic correction. It has been verified that a uniaxial dipolar ferromagnetic system in d dimension behaves as the corresponding system with only short-range interactions in $d+1$ dimensions within the one-loop order approximation.[15,16] Therefore, for initial orientation, we study in Sec. II the short-distance expansion at the upper critical dimension $d_c=4$ for isotropic short-range interaction systems. In Sec. III the operator-product expansion for the uniaxial dipolar system at $d=d_c=3$ is applied to generate the appropriate short-distance expansion. Finally, a summary of these results, their implications for quasielastic cross sections, and a discussion concerning gadolinium are given in Sec. IV.

II. THE SPIN-SPIN CORRELATION OF FOUR-DIMENSIONAL SHORT-RANGE INTERACTION SYSTEMS

In the application of field theory to critical phenomena, the invariance under different renormalization parameters leads to the renormalization-group equations. These partial differential equations lead to useful results which lie beyond perturbation theory.[11,17] The renormalization at the critical point ($t=0$) is defined with an arbitrary nonzero momentum κ introduced to provide a reference point for nonsingular normalizations. The independence with respect to κ of the bare theory provides the renormalization-group equation of a vertex function $\Gamma^{(N,L)}(q_i;q_j';t,g,\kappa)$, the one-particle-irreducible part of the connected N-point correlation function having momenta q_i and with L composite operator ϕ^2 insertions having momenta q_j', which is given by

$$\left[\kappa\frac{\partial}{\partial\kappa}+\beta(g)\frac{\partial}{\partial g}-\frac{N}{2}\eta(g)-\theta(g)\left[L+t\frac{\partial}{\partial t}\right]\right]\Gamma^{(N,L)}=0 \tag{1}$$

except for $N=0$ and $L\leq 2$, in which case a nonzero term due to an additive renormalization appears. In the only special case of present interest ($N=0$, $L=2$), this extra term is a power series in the renormalized coupling constant, g, and is given to the lowest order by

$$B(g)=\frac{M}{2}+O(g)\ . \tag{2}$$

The Wilson functions at $d=4$ are

$$\beta(g)=ag^2+O(g^3),\quad a=\frac{M+8}{6}\ , \tag{3}$$

$$\eta(g)=cg^2+O(g^3),\quad c=\frac{M+2}{72}\ , \tag{4}$$

$$\theta(g)=-bg+O(g^2),\quad b=\frac{M+2}{6}\ . \tag{5}$$

Here we keep the general expression of an M-spin component system for the purpose of comparison with other works.[11,17] The differentiation of renormalized vertex functions with respect to t generates renormalized vertex functions with one more ϕ^2 composite operator insertion (at zero momentum),

$$\frac{\partial\Gamma^{(N,L)}(q_i;q_j';t,g,\kappa)}{\partial t}=\Gamma^{(N,L+1)}(q_i;q_j,0;t,g,\kappa)\ . \tag{6}$$

Our objective is to determine the leading dependence on reduced temperature $t=(T-T_c)/T_c$ of

$$[G^{zz}(q,T)]^{-1}\propto\Gamma^{(2,0)}(q,-q,t)$$

for $0<t/q^2\ll 1$. Brézin *et al.* obtained the leading power-law corrections for $d<4$ by solving the appropriate renormalization-group equations.[11] We extend their arguments to discuss the corresponding leading correction terms at the critical dimension $d_c=4$. From Eqs. (1) and (6) the two-point vertex function ($N=2$, $L=0$) satisfies

$$\left[\kappa\frac{\partial}{\partial\kappa}+\beta\frac{\partial}{\partial g}-\eta\right]\Gamma^{(2)}(q,-q;t,g,\kappa)$$
$$=\theta t\Gamma^{(2,1)}(q,-q;0;t,g,\kappa)\ . \tag{7}$$

In order to evaluate the temperature-dependent correction terms, we have to study the leading critical behavior of $\Gamma^{(2,1)}$. Its renormalization-group equation is

$$\left[\kappa\frac{\partial}{\partial\kappa}+\beta\frac{\partial}{\partial g}-\eta-\theta\right]\Gamma^{(2,1)}(q,-q;0;t,g,\kappa)$$
$$=\theta t\Gamma^{(2,2)}(q,-q;0,0;t,g,\kappa)\ . \tag{8}$$

The operator-product expansion (see Appendix A) determines the leading behavior of $\Gamma^{(2,2)}$ as

$$\Gamma^{(2,2)}(q,-q;0,0;t,g,\kappa)=C(q)\Gamma^{(0,3)}(0;t,g,\kappa)\ , \tag{9}$$

where the temperature independent $C(q)$ is a Wilson coefficient of the operator-product expansion. Substituting Eq. (9) in the renormalization-group equation of $\Gamma^{(2,2)}$ and using Eq. (1) for $\Gamma^{(0,3)}$, $C(q)$ is seen to satisfy the partial differential equation

$$\left[\kappa\frac{\partial}{\partial\kappa}+\beta\frac{\partial}{\partial g}-\eta+\theta\right]C(q)=0\ . \tag{10}$$

The partial differential equations for $C(q)$ and $\Gamma^{(0,3)}(t)$ can be solved and results used in Eq. (8), in principle. We find it convenient to proceed in a slightly different way. Note that, in the case of pure power laws for $d<4$, $\Gamma^{(2,1)}$ and $t\Gamma^{(2,2)}$ have the same leading t dependence. However, this is not so at $d=4$ and the logarithmic leading t dependence of $\Gamma^{(2,1)}$ is stronger than that of $t\Gamma^{(2,2)}$. Motivated by the arguments of Callan,[18] we assume a modified operator-product expansion directly for $\Gamma^{(2,1)}$ of the form (see also Appendix A)

$$\Gamma^{(2,1)}(q,-q;0;t,g,\kappa)=C(q)\Gamma^{(0,2)}(0;t,g,\kappa)+F(q,t,g,\kappa)\ , \tag{11}$$

which defines $F(q,t,g,\kappa)$. Taking the t derivative of $F=\Gamma^{(2,1)}-C(q)\Gamma^{(0,2)}$ and then using the operator-product expansion, F is seen to be independent of t in the large-q regime. Then substituting Eq. (11) in Eq. (8), and using Eq. (10) and the renormalization-group equation for $\Gamma^{(0,2)}$, we derive

$$\left[\kappa\frac{\partial}{\partial\kappa}+\beta\frac{\partial}{\partial g}-\eta-\theta\right]F(q,g,\kappa)=-C(q)B(g)\ . \tag{12}$$

The differential equations for $\Gamma^{(0,2)}(t)$, $C(q)$, and $F(q)$ can be solved by the method of characteristic equations. This will be illustrated in detail for $\Gamma^{(0,2)}(t)$. Introducing an arbitrary parameter λ, its renormalization-group equation becomes

$$\left[\lambda\frac{\partial}{\partial\lambda}-2\theta\right]\Gamma^{(0,2)}(;0,0;t(\lambda),g(\lambda),\kappa(\lambda))$$
$$=B(g(\lambda))\ , \tag{13}$$

where the characteristic equations are

$$\lambda\frac{\partial\kappa(\lambda)}{\partial\lambda}=\kappa(\lambda)\ ,$$
$$\lambda\frac{\partial g(\lambda)}{\partial\lambda}=\beta(g(\lambda))\ , \tag{14}$$
$$\lambda\frac{\partial t(\lambda)}{\partial\lambda}=-\theta t(\lambda)\ ,$$

with the initial conditions

$$\kappa(\lambda=1)=\kappa,\quad g(\lambda=1)=g\ ,$$

and

$$t(\lambda=1)=t\ .$$

The solution of Eq. (13) is

$$\Gamma^{(0,2)}(;0,0;t,g,k)=\exp\left[-2\int_g^{g(\lambda)}\frac{\theta}{\beta}dg'\right]\Gamma^{(0,2)}(;0,0;t(\lambda),g(\lambda),\kappa(\lambda))-\int_g^{g(\lambda)}\frac{dg'}{\beta(g')}B(g')\exp\left[-2\int_g^{g'}\frac{\theta}{\beta}dg''\right] \tag{15}$$

with the solutions of the characteristic equations given by

$$t(\lambda)=\exp\left[-\int_g^{g(\lambda)}\frac{\theta}{\beta}dg'\right]t(\lambda=1)$$
$$\approx\left[\frac{g(\lambda)}{g}\right]^{b/a}t\ , \tag{16}$$

$$g(\lambda)\approx\frac{g}{1-ag\ln\lambda}\ , \tag{17}$$

and

$$\kappa(\lambda)=\kappa\lambda\ . \tag{18}$$

We choose λ by $t(\lambda)/(\kappa\lambda)^2=1$ so that $\Gamma^{(0,2)}(;0,0;t(\lambda)/(\kappa\lambda)^2,g(\lambda),1)$ becomes a regular function of t in the perturbative limit of small $g(\lambda)$. Then $\Gamma^{(0,2)}$ at small t is

$$\Gamma^{(0,2)}\approx\frac{3M}{M-4}S_d g^{-1}\{[1-ag\ln(t^{1/2}/\kappa)]^{1-2b/a}-1\}\ , \tag{19}$$

where S_d is the geometrical factor $2\pi^{d/2}/[(2\pi)^d\Gamma(d/2)]$. In fact, we have included the geometric factor in the coupling constant g. This result has been given by Brézin *et al.*[11]

In a similar way, with appropriate initial conditions in the perturbative limit, we obtain solutions for $C(q)$ and $F(q)$:

$$C(q)\approx\frac{M+2}{3M}\frac{g}{S_d}[1-ag\ln(q/\kappa)]^{b/a-1} \tag{20}$$

and

$$F(q,g,\kappa)\approx\left[1-\frac{M+2}{M-4}\right][1-ag\ln(q/\kappa)]^{-b/a}$$
$$+\frac{M+2}{M-4}[1-ag\ln(q/\kappa)]^{b/a-1}\ . \tag{21}$$

Using these results in Eq. (11) yields

$$\Gamma^{(2,1)}(q,-q;0;t,g,\kappa)=\frac{M+2}{M-4}[1-ag\ln(q/\kappa)]^{b/a-1}\left[1-\frac{ag}{2}\ln(t/\kappa^2)\right]^{1-2a/b}$$
$$+\left[1-\frac{M+2}{M-4}\right][1-ag\ln(q/\kappa)]^{-b/a}+\cdots\ . \tag{22}$$

Substituting this solution into the equation of $\Gamma^{(2)}$ and using the method of characteristic equations, we find

$$\Gamma^{(2)}(q,-q;t,g,\kappa)=\exp\left[-\int_g^{g(\lambda)}\frac{\eta}{\beta}dg'\right]\Gamma^{(2)}(q,-q;t,g(\lambda),\kappa\lambda)$$
$$-\int_g^{g(\lambda)}dg'\frac{\theta(g')t}{\beta(g')}\Gamma^{(2,1)}(q,-q;t,g',\kappa\lambda')\exp\left[-\int_g^{g'}\frac{\eta}{\beta}dg''\right]\ . \tag{23}$$

Since the two-point vertex function $\Gamma^{(2)}$ has a finite limit as $t\to 0$, the first term on the right-hand side of Eq. (23) is evaluated correctly by using Eq. (6) and solving the differential equation for $\Gamma^{(2)}$ at $t=0$. Finally, the behavior of $\Gamma^{(2)}$ at large q is given by

$$\Gamma^{(2)}=q^2+\frac{M+2}{M-4}[1-ag\ln(q/\kappa)]^{b/a-1}t\left[1-\frac{ag}{2}\ln(t/\kappa^2)\right]^{1-2b/a}$$
$$+\left[1-\frac{M+2}{M-4}\right][1-ag\ln(q/\kappa)]^{-b/a}t+\cdots\ . \tag{24}$$

where temperature-independent corrections to the leading q^2 term has been ignored.

For physical relevance, the $M=1$ case corresponds to the uniaxial system at its upper critical dimension. The logarithmic singular t dependence is directly related to the criticality of the specific heat just as for $d<4$ short-range interaction systems. Moreover, it is seen from Eq. (24) that $dG^{zz}(q,T)/dT>0$ at large q for $T\to T_c$ at $d=4$.

III. THE SPIN-SPIN CORRELATION FUNCTION OF THREE-DIMENSIONAL UNIAXIAL DIPOLAR SYSTEMS

The discussion in the previous section provides useful orientation for the study of $d=3$ uniaxial systems with not only isotropic short-range exchange but also dipolar interactions since both systems are at their respective upper critical dimension. In fact, some results of Larkin and Khmel'nitskii[15] for the $d=3$ dipolar system and $d=4$ short-range interaction system have been shown to be equivalent within the one-loop order calculation.[16] Brézin and Zinn-Justin,[19] however, showed that there are differences between the two systems in results of two-loop order calculation. In this section, we pursue the same arguments, as in Sec. II for three-dimensional (3D) uniaxial dipolar systems within a one-loop approximation for the leading t dependence.

We define renormalized quantities, such as the reduced temperature t, in the same way as for purely short-range interaction systems. Therefore, correlation functions of composite operators in the uniaxial dipolar system are derived by differentiating certain vertex functions with respect to the reduced temperature so that Eq. (6) still holds. We use a simplified Gaussian propagator in a graphical perturbation expansion

$$G_0(q)=\left[t+q_b^2+\alpha^2\left[\frac{q_z}{q_b}\right]^2\right]^{-1}, \tag{25}$$

where α is the renormalized dipolar coupling constant and q_b and q_z are basal plane and z-axis components of the momentum q. Hence, renormalization-group equations are given by the usual arguments. In particular,

$$\left[\kappa\frac{\partial}{\partial\kappa}+\beta\frac{\partial}{\partial g}+\frac{\eta}{2}\left[\alpha\frac{\partial}{\partial\alpha}-2\right]-\theta\right]\Gamma^{(2,1)}=\theta t\Gamma^{(2,2)} . \tag{26}$$

Here g is defined to be the dimensionless coupling constant including a numerical constant due to the angular integrations

$$S'_{d-1}=2\pi^{(d-1)/2}/[(2\pi)^d\Gamma((d-1)/2)]$$

and also a factor of π/α. The operator-product expansion arguments can be applied to this system as well. The general structure of Feynman diagrams of vertex functions are the same as those in short-range interaction systems. Hence, the graphical structures of $C(q)$ and $F(q)$ are just as those of short-range interaction systems of the previous section (see Appendix B). Analysis of these diagrams, within the leading one-loop approximation, verifies that leading behavior is given by

$$\Gamma^{(2,2)}(q,t)=C(q)\Gamma^{(0,3)}(t) , \tag{27}$$

$$\Gamma^{(2,1)}(q,t)=C(q)\Gamma^{(0,2)}(t)+F(q) . \tag{28}$$

Using the Wilson coefficients given by Brézin and Zinn-Justin[19] and the method of characteristic equations, we solve the differential equations for $C(q)$, $F(q)$, and other required vertex functions. For simplicity, final results will be given only for some limiting cases. For $(\alpha q_z)^2\ll q_b$,

$$C(q)\approx\frac{M+2}{3M}\frac{\alpha g}{\pi S'_{d-1}}[1-ag\ln(q_b/\kappa)]^{b/a-1} , \tag{29}$$

$$F(q)\approx\left[1-\frac{M+2}{M-4}\right][1-ag\ln(q_b/\kappa)]^{-b/a} +\frac{M+2}{M-4}[1-ag\ln(q_b/\kappa)]^{b/a-1} , \tag{30}$$

and

$$\Gamma^{(0,2)}\approx\frac{3M}{M-4}\frac{\pi S'_{d-1}}{\alpha}g^{-1} \times\left[\left[1-\frac{ag}{2}\ln(t/\kappa)^2\right]^{1-2b/a}-1\right] , \tag{31}$$

where $a=(M+8)/12$ and $b=(M+2)/12$ in this case.[19] Finally, the two-point vertex function with correction terms is

$$\Gamma^{(2)}(q_b,q_z;t,g,\alpha,\kappa)=q_b^2+\alpha^2\frac{q_z^2}{q_b^2}+\left[\frac{M+2}{M-4}\right]t[1-ag\ln(q_b/\kappa)]^{b/a-1} \times[1-ag\ln(t/\kappa^2)]^{1-2b/a}+\left[1-\frac{M+2}{M-4}\right]t[1-ag\ln(q_b/\kappa)]^{-b/a}+\cdots . \tag{32}$$

Explicit forms for other regions of $\mathbf{q}$ can also be found, e.g., $(\alpha q_z)^2\gg q_b$. $\Gamma^{(2)}$ is given in this region by

$$\Gamma^{(2)}(q_b,q_z;t,g,\alpha,\kappa)=q_b^2+\alpha^2\frac{q_z^2}{q_b^2}+\left[\frac{M+2}{M-4}\right]t[1-ag\ln(\alpha q_z/\kappa^2)]^{b/a-1} \times[1-ag\ln(t/\kappa^2)]^{1-2b/a}+\left[1-\frac{M+2}{M-4}\right]t[1-ag\ln(\alpha q_z/\kappa^2)]^{-b/a}+\cdots , \tag{33}$$

where temperature-independent corrections to the leading q^2 term are again neglected. Therefore, for the physically relevant uniaxial dipolar system ($M=1$), it is seen that the logarithmic singular t dependence of two-point correlation function is related to the criticality of the specific heat and its t-dependent slope is positive at large q for $T \to T_c$.

IV. SUMMARY AND DISCUSSION

The primary objective of this work has been the determination of the temperature dependence in the short-distance regime ($q\xi \gg 1$) of the equal time correlation function of the order parameter (S^z), $G^{zz}(q,T)$, for a three-dimensional uniaxial dipolar ferromagnet in the limit as $T \to T_c^+$. Since the upper critical dimension of this system is $d_c=3$, the corresponding (simpler) problem of a short-range interaction system at its upper critical dimension $d_c=4$ was considered for purposes of orientation. The calculations were carried out using renormalized perturbation theory and the operator-product expansion. Renormalization-group equations were solved within a consistent one-loop approximation. Making use of the fact that the physical correlation function is $[\Gamma^{(2)}(q,t)]^{-1}$ apart from a positive temperature-independent constant, we conclude from Secs. II and III that (a) $dG^{ZZ}(q,T)/dT>0$ and (b) $dG^{zz}(q,T)/dT$ is proportional to the singular part of the specific heat ($|\ln t|^{1/3}$) as $t \to 0$ in the paramagnetic state. These results are similar to those for systems with only short-range interactions below their upper critical dimensions.

We now consider the implications of these results for the "anomalous" c-axis resistivity of Gd. The slope of the contribution to the c-axis resistivity due to electrons scattering from spin fluctuations can be isolated from the total resistivity and is found experimentally to be negative in the paramagnetic state at least down to a reduced temperature of $t=10^{-4}$.[8] Although the uniaxial dipolar character of spin fluctuations does have a profound effect on $G^{zz}(q,T)$, the results of this work show that the observed c-axis resistivity cannot be explained solely in terms of critical fluctuations of the (primary) order parameter, assuming that the conventional theory of quasielastic transport properties near magnetic critical points is indeed correct.

We suggest that the resolution of this anomalous behavior requires a consideration of the secondary degrees of freedom, $\mathbf{S}^{\perp}=(S^x,S^y)$, which also enter the resistivity via the total electron-scattering cross section. Of course, all spin degrees of freedom enter the scattering cross section irrespectively of the particular crystallographic direction along which the resistivity is measured. Even though the temperature dependence of

$$G^{xx}(q,T)+G^{yy}(q,T)=G^{\perp}(q,T)$$

is relatively weak, it still may be strong enough to compete with the weak (large $q\xi$) temperature dependence of $G^{zz}(q,T)$. The calculations are rather complex due to explicit anisotropy factors in the renormalization-group equations (see, for example, Amit and Goldschmidt and Goldschmidt[20,21]) and results will be given subsequently.

Finally, we should emphasize that although the primary motivation for this work was given by the unusual electronic transport properties of Gd, the implications of our results are not limited to that specific case. As mentioned in the Introduction, the spin-spin correlation function also enters the scattering cross section for neutrons. Using appropriate energy analysis, polarized neutrons and isotopically enriched samples as required, $G_{ZZ}(q,T)$ is directly observable, in principle, over a wide range of q in the $q\xi \gg 1$ regime for T sufficiently close to T_c. The study by neutron scattering of $G_{ZZ}(q,T)$ in the large-wave-number regime would provide an important experimental test of the detailed renormalization-group predictions given in this work. This experimental study would be of interest not only for Gd but also for other uniaxial dipolar ferromagnets, including $LiTbF_4$.

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APPENDIX A: GRAPHICAL EXPRESSION FOR THE OPERATOR-PRODUCT EXPANSION COEFFICIENT $C(q)$ AND $F(q,t,g,\kappa)$

For systems with short-range interactions, we consider a model with the standard bare Hamiltonian:

$$-\mathcal{H}=\int d\mathbf{q}(r_0+q^2)\varphi(\mathbf{q})\varphi(-\mathbf{q}) + \int d\mathbf{q}_1 d\mathbf{q}_2 d\mathbf{q}_3 u\,\varphi(\mathbf{q}_1)\varphi(\mathbf{q}_2)\varphi(\mathbf{q}_3) \times\varphi(-\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3)\,, \tag{A1}$$

where φ is an M-component field. The corresponding renormalized theory is generated by a standard procedure.[17]

Following Wilson's operator-product expansion arguments, the renormalized vertex function $\Gamma^{(2,2)}$ with exceptional momenta is expanded

$$\Gamma^{(2,2)}(q,-q;0,0;t,g,\kappa) =C(q)\Gamma^{(0,3)}(0,0,0;t,g,\kappa)[1+O(t/q^2)]\,. \tag{A2}$$

The renormalized vertex functions, to two loops, are

$$\Gamma^{(2,2)}=\frac{M+2}{3}\left[g+g\frac{M+4}{2}g^2\right]D_1-\frac{M+2}{3}g^2(D_2+D_3) -\frac{(M+2)^2}{6}g^2D_4+\frac{(M+2)^2}{6}g^2t(D_5-I_{\rm sp}D_6) \tag{A3}$$

and

$$\Gamma^{(0,3)}=MD_1-\frac{M(M+2)}{2}g(D_4-I_{\rm sp}D_1) +\frac{M(M+2)}{2}gt(D_5-I_{\rm sp}D_6)\,, \tag{A4}$$

where the D_i are the Feynman integrals represented by the diagrams of Fig. 1. These diagrams will be evaluated for $d=4-\epsilon$ and the limit of $\epsilon\to 0$ will be taken in final expression. $I_{\rm sp}$ is defined as $I(q=\kappa,\ t=0)$ where

$$I(q,t)=\int d^dk\, G(k,t)G(k+q,\ t)\ . \tag{A5}$$

It is straightforward to verify that, to this order, the Wilson coefficient $C(q)$, which appears in (35), has the same structure as the four-point function with exceptional momenta

$$\tilde{\Gamma}^{(4)}(q,-q,0,0;t,g,\kappa) =\frac{M+2}{3M}g-\frac{M+2}{3M}g^2[I(q,t)-I_{\rm sp}] \tag{A6}$$

up to corrections of order $(t/q^2)\ln(t/q^2)$. This follows from inspection of the graphical structure of the vertex functions and noting that

$$\begin{aligned}D_2&=\int d^dk\, G(k,t)^3 I(q+k,\ t)\\
&=S_dC_d\int d^dk[G(k,t)]^3(q+k)^{-\epsilon}\left[\frac{t}{(q+k)^2}+\frac{1}{4}\right]^{-\epsilon/2} {}_2F_1(\tfrac{1}{2},\epsilon/2,\tfrac{3}{2};[1+4t/(q+k)^2]^{-1})\\
&\approx S_dC_d\frac{\Gamma(3/2)\Gamma(1-\epsilon/2)}{2^{-\epsilon}\Gamma((3-\epsilon)/2)}\int d^dk[G(k,t)]^3[(k+q)^2]^{-\epsilon/2}\\
&=\frac{S_dC_d\Gamma(3/2)\Gamma(1-\epsilon/2)}{\Gamma((3-\epsilon)/2)2^{-\epsilon}}\left\{t^{-1-\epsilon/2}(q^2)^{-\epsilon/2}\int\frac{d^dk}{(1+k^2)^3}\left[1+O\left[\frac{t}{q^2}\right]\right]\right\}\\
&\approx D_1 I(q,\ t=0)\ ,\end{aligned} \tag{A7}$$

$$\begin{aligned}D_3&=\int\int d^dk\, d^dk'[G(k,t)]^2[G(k',t)]^2G(k+k'+q,\ t)\\
&=\frac{-1}{2}\int d^dk\, G(k,t)^2\frac{\partial}{\partial t}I(k+q,t)\\
&\approx S_dC_d\int d^dk[G(k,t)]^2[(q+k)^2]^{-1-\epsilon/2}O(\epsilon)\\
&=O(D_2t/q^2)\ ,\end{aligned} \tag{A8}$$

where C_d is defined to be

$$C_d=\frac{\Gamma(d/2)\Gamma(\epsilon/2)}{2\Gamma(2)}\ .$$

From Eq. (11), the operator product expansion of $\Gamma^{(2,1)}$ is

$$\Gamma^{(2,1)}(q,-q;0;t,g,\kappa)=C(q)\Gamma^{(0,2)}(;0,0;t,g,\kappa)+F(q,t,g,\kappa)\ . \tag{A9}$$

The graphical expansion of $\Gamma^{(2,1)}$ to two loops is

$$\begin{aligned}\Gamma^{(2,1)}=1&-\frac{M+2}{6}g(D_7-I_{\rm sp})+\frac{M+2}{6}g^2(D_8-I_{4\rm sp})+\left[\frac{M+2}{6}\right]^2g^2(D_9-I_{\rm sp}D_7)\\
&+\frac{(M+2)(M+8)}{36}g^2I_{\rm sp}(D_7-I_{\rm sp})-\frac{(M+2)^2}{18}g^2t(D_{10}-I_{\rm sp}D_1)\ ,\end{aligned} \tag{A10}$$

where $I_{4\rm sp}$ is the integral D_8 at the symmetry point. The corresponding expression of $\Gamma^{(0,2)}$ is

$$\Gamma^{(0,2)}=-\frac{M}{2}(D_7-I_{\rm sp})-\frac{M(M+2)}{6}gI_{\rm sp}(D_7-I_{\rm sp})+\frac{M(M+2)}{12}g(D_9-I_{\rm sp}^2)-\frac{M(M+2)}{6}gt(D_{10}-I_{\rm sp}D_1)\ . \tag{A11}$$

A detailed examination of these graphs is required. In particular,

$$\begin{aligned}D_8&=\int d^dk[G(k,t)]^2I(q+k,\ t)\\
&=\int d^dk[G(k,t)]^2S_dC_d(q+k)^{-\epsilon}\left[\frac{t}{(q+k)^2}+\frac{1}{4}\right]^{(-\epsilon/2)} {}_2F_1(\tfrac{1}{2},\epsilon/2,\tfrac{3}{2};[1+4t/(q+k)^2]^{-1})\\
&=S_dC_d\frac{\Gamma(3/2)\Gamma(1-\epsilon/2)}{2^{-\epsilon}\Gamma((3/2-\epsilon)/2)}\int d^dk[G(k,t)]^2\left[[4t+(k+q)^2]^{-\epsilon/2}+O\left[\frac{4t}{(q+k)^2}\right]\right]\\
&=S_d^2C_d^2\frac{\Gamma(3/2)\Gamma(1-\epsilon/2)\Gamma(\epsilon)2^{-\epsilon}}{\Gamma(3/2-\epsilon/2)\Gamma(2+\epsilon/2)\Gamma(\epsilon/2)}t^{-\epsilon}\left[1+\frac{4t}{q^2}\right]^{-\epsilon}\\
&\quad\times F_1(\epsilon/2,\epsilon,\epsilon;2+\epsilon/2;-q^2/t,(1+4t/q^2)^{-1})\ .\end{aligned} \tag{A12}$$

Here $F_1(a,b_1,b_2;c;x,y)$ is a double hypergeometric Appel function of argument $x=-q^2/t$ and $y=(1+4t/q^2)^{-1}$.[22] The series expansion of this function does not converge for $t/q^2 \ll 1$. Therefore, it is necessary to apply an appropriate transformation:

$$
\begin{aligned}
F_1(a,b_1,b_2;c;x,y)=&\frac{\Gamma(c)\Gamma(b_1-a)}{\Gamma(c-a)\Gamma(b_1)}(1-x)^{-a}F_1\left[a,c-b_1-b_2,b_2;1+a-b_1;\frac{1}{1-x},\frac{1-y}{1-x}\right]\\
&+\frac{\Gamma(c)\Gamma(a+b_2-c)}{\Gamma(a)\Gamma(b_2)}(1-x)_{-b_1}(1-y)^{c-a-b_2}\\
&\times F_1\left[c-a,b_1,c-b_1-b_2;c-a-b_2+1;\frac{1-y}{1-x},(1-y)\right]\\
&+\frac{\Gamma(c)\Gamma(a-b_1)\Gamma(c-a-b_2)}{\Gamma(a)\Gamma(c-a)\Gamma(c-b_1-b_2)}(1-x)^{-b_1}\\
&\times G_2\left[b_1,b_2,a-b_1;c-a-b_2;\frac{1}{1-x},(y-1)\right],
\end{aligned}
\tag{A13}
$$

where G_2 is another type of double hypergeometric function.[22] At small t/q^2, the series expansion of these transformed functions (F_1's and G_2) show good convergence. Finally, we have the leading contributions of the diagram D_8:

$$
\begin{aligned}
D_8=&S_d^2C_d^2\frac{\Gamma(3/2)\Gamma(1-\epsilon/2)\Gamma(\epsilon)2^{-\epsilon}}{\Gamma(3/2-\epsilon/2)\Gamma(2+\epsilon/2)\Gamma(\epsilon/2)}t^{-\epsilon}\left[1+\frac{4t}{q^2}\right]^{-\epsilon}\left[1+\frac{q^2}{t}\right]^{-\epsilon/2}\\
&\times\left[\frac{\Gamma(2+\epsilon/2)\Gamma(\epsilon/2)}{\Gamma(2)\Gamma(\epsilon)}+\frac{\Gamma(2+\epsilon/2)\Gamma(2-\epsilon)\Gamma(-\epsilon/2)}{\Gamma(2)\Gamma(2-3\epsilon/2)\Gamma(\epsilon/2)}\left[1+\frac{q^2}{t}\right]^{-\epsilon/2}+O(\epsilon t/q^2)\right]\\
=&I(q,t)[I(0,t)-I_{sp}\times O((q/\kappa)^{-\epsilon})] .
\end{aligned}
\tag{A14}
$$

The above result shows that the diagram D_8 is represented with the product of diagrams in $\Gamma^{(0,2)}$ and $C(q)$. Therefore, we are able to conclude

$$F(q,t,g,\kappa)=1+O(g^2) . \tag{A15}$$

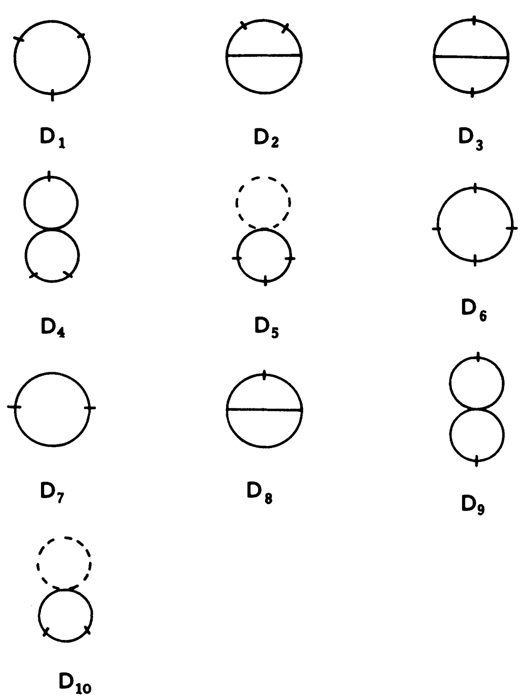

FIG. 1. Diagrams for the contribution of the vertex functions referred to in the text. (See Appendix A.) Solid lines represent the Gaussian propagator $G(k,t)=1/[t+k^2+\alpha^2(k_z^2/k^2)]$ and dashed lines represent the propagator $G(k,t)G(k,t=0)$. ($\alpha=0$ for the short-range interaction case and $\alpha\neq 0$ for the dipolar case.)

APPENDIX B: THE OPERATOR-PRODUCT EXPANSION COEFFICIENT $C(q)$ AND $F(q,t,g,\kappa)$ IN THE UNIAXIAL DIPOLAR SYSTEM

As noted in the Introduction, previous work has indicated that the critical properties of gadolinium sufficiently close to T_c (i.e., within approximately 0.5 K) are those of a uniaxial ($M=1$) dipolar system. As described by Brézin and Zinn-Justin, the appropriate bare Hamiltonian for such a system is

$$
\begin{aligned}
-\mathcal{H}=&\int d\mathbf{q}(r_0+q^2)+\alpha_0\frac{q_z^2}{q^2}\varphi(\mathbf{q})\varphi(-\mathbf{q})\\
&+u\int d\mathbf{q}_1 d\mathbf{q}_2 d\mathbf{q}_3\varphi(\mathbf{q}_1)\varphi(\mathbf{q}_2)\varphi(\mathbf{q}_3)\\
&\times\varphi(-\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3) .
\end{aligned}
\tag{B1}
$$

The diagrams required for the graphical expansions of vertex functions in the uniaxial dipolar case have the same structure as those required for the short-range interaction case; however, the propagator is that discussed by Brézin and Zinn-Justin.[19] For example, the bubble integral $I(q,t)$ of Appendix A is replaced by

$$I(q_b,q_z,t)=\int d^dkG(k_b+q_b,\ k_z+q_z,\ t)G(k_b,k_z,t) . \tag{B2}$$

The integration over k_z can be done by contour integration

$$J_z=\int dk_z \frac{1}{(A^2+k_z^2)[B^2+(k_z+q_z)^2]}$$
$$=\frac{\pi(A+B)}{AB[q_z^2+(A+B)^2]}\ , \qquad \text{(B3)}$$

where $A=[(t+k_b^2)k_b^2/\alpha^2]^{1/2}$ and

$$B=\{[t+(k_b+q_b)^2](k_b+q_b)^2/\alpha^2\}^{1/2}\ .$$

In this discussion, we set $d=3-\epsilon$ and take the $\epsilon \to 0$ limit at the end. As $t\to 0$, $I(q_b,q_z,t)$ is approximately

$$I(q_b,q_z,t)=\frac{\pi}{2\alpha}\int d^{d-1}k_b \frac{k_b^2+(q_b/2)^2}{(\alpha q_z/2)^2+[k_b^2+(q_b/2)^2]^2}$$
$$=S'_{d-1}\frac{\pi}{2\alpha}\Gamma(\epsilon/2)\Gamma(1-\epsilon/2)(\alpha q_z)^{-\epsilon/2}\left[\left[\frac{q_b^2}{4\alpha q_z}\right]^2+1\right]^{1/2-\epsilon/4}\sin\left[\frac{d-1}{2}[\pi-\tan^{-1}(4\alpha q_z/q_b^2)]\right]$$
$$+S'_{d-1}\frac{\pi}{2\alpha}\Gamma(-\epsilon/2)\Gamma(1+\epsilon/2)(\alpha q_z)^{-\epsilon/2}\frac{q_b^2}{4\alpha q_z}\left[\left[\frac{q_b^2}{4\alpha q_z}\right]^2+1\right]^{-\epsilon/4}\sin\left[\frac{-\epsilon}{2}[\pi-\tan^{-1}(4\alpha q_z/q_b^2)]\right]\ . \qquad \text{(B4)}$$

For special cases, $I(q_b,q_z,t)$ becomes (i) $q_b \gg \alpha q_z$,

$$I\approx\frac{\pi}{4\alpha}\Gamma(1-\epsilon/2)\Gamma(\epsilon/2)\left[\frac{q_b^2}{4}\right]^{-\epsilon/2}\ , \qquad \text{(B5)}$$

(ii) $q_b \ll \alpha q_z$,

$$I\approx\frac{\pi}{8\alpha}\Gamma(1-\epsilon/4)\Gamma(\epsilon/4)\left[\frac{\alpha q_z}{2}\right]^{-\epsilon/2}\ . \qquad \text{(B6)}$$

The other integrals may be evaluated in a similar manner (Fujiki[23]). In particular, the integrals D_2 and D_3 appearing in Appendix A are now replaced by

$$D_2=\int d^dk[G(k_b,k_z,t)]^3 I(q_b+k_b,\, q_z+k_z,\, t)\ .$$

It is seen that the integration over k is dominated by the small-k behavior at $t\approx 0$. Therefore, $q+k$ can be replaced by q and we conclude

$$D_2\approx D_7 I(q_b,q_z,0)\ . \qquad \text{(B7)}$$

On the other hand, the integral D_3 is

$$D_3=\int d^dk[G(k_b,k_z,t)]^2 I_2(q_b+k_b,\, q_z+k_z,\, t)\ , \qquad \text{(B8)}$$

where

$$I_2(q_b,q_z,t)=\int d^dk[G(k_b,k_z,t)]^2 G(q_b+k_b,\, q_z+k_z,\, t)\ . \qquad \text{(B9)}$$

This is evaluated by contour integration. Finally, for the special case $q_b \gg \alpha q_z$, for simplicity,

$$D_3 \propto \int d^dk[G(k_b,k_z,t)]^2[(q_b+k_b)^2/2]^{-1-\epsilon/2}\ . \qquad \text{(B10)}$$

Similar arguments applied to D_2 yield

$$D_3\approx O(D_2 t/(q_b)^2)\ . \qquad \text{(B11)}$$

The integral D_8 of Appendix A is replaced by

$$D_8=\int d^dk[G(k_b,k_x,t)]^2 I(q_b+k_b,\, q_z+k_z,\, t)\ . \qquad \text{(B12)}$$

Following the consideration of D_8 in the short-range interaction case, and noting that the integral is still dominated by small k for $t\approx 0$, the leading contribution becomes

$$D_8\approx D_7 I(q_b,q_z,t)\ . \qquad \text{(B13)}$$

We conclude that the operator-product expansion for $\Gamma^{(2,2)}$ and $\Gamma^{(2,1)}$ follows that described in Appendix A.

In particular,

$$C(q)=S'_{d-1}\frac{\alpha}{\pi}\frac{M+2}{3M}\{g-g^2[I(q_b,q_z,0)-I_{\rm sp}]\}+O(g^3) \qquad \text{(B14)}$$

and

$$F(q,t,g,\kappa)=1+O(g^2)\ . \qquad \text{(B15)}$$

The final explicit forms for $C(q)$ and $F(q)$ in cases of physical interest are given in the text.

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