Noise, coherence, and reversibility in Josephson arrays

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The noise sensitivity of Josephson-junction arrays is greatly enhanced in the presence of a so-called "reversibility" symmetry. We use a simple dynamical picture corresponding to a diffusion process in phase space to calculate the power spectrum of the total array voltage. Breaking the reversibility symmetry leads to a crossover from incoherent to coherent oscillations, reflected by improved scaling of the voltage with respect to array size. In the limit where the in-phase orbit is strongly attracting, the results agree with previous theoretical and experimental work.

I. INTRODUCTION

There is great current interest in the study of Josephson-junction arrays, both as physical systems in their own right,[1–6] and as a class of nonlinear dynamical systems with many degrees of freedom.[7–11] Special attention has focused on current-biased series arrays, which have the important property that the voltage across the entire array can be large, even though each junction generates a relatively small voltage. From the physical point of view, there is substantial leeway in the specific circuit to be studied, in terms of both the type of Josephson-junction element used, and the type of "load circuit" which serves to couple the individual nonlinear oscillators.

Very recently, a remarkable dynamical property of certain Josephson arrays was discovered, which can be expressed formally as a symmetry in the underlying circuit equations.[12,13] In addition to the obvious mathematical interest, this symmetry has profound physical implications: in particular, it rules out the possibility of stable phase locking between array elements. The purpose of the present paper is to focus on the practical ramifications of this observation, and to draw a connection with earlier theoretical and experimental work on Josephson junction arrays.[1]

In particular, this paper presents a calculation of the total voltage output in the simplest dynamical regime (which is also the most important regime from a practical point of view), depending on whether or not the "reversibility" property described in Ref. 12 is present. Our starting point is a model for the phase-space flow which embodies the most essential properties of the dynamics. In this way, it is possible to draw quantitative conclusions about the power spectra without making explicit use of the detailed circuit equations. Our model is based on only three features: (i) the topology of the phase space (the dynamics takes place on an N-torus); (ii) the absence or presence of an in-phase attractor (depending on the load); and (iii) the symmetry of the governing equations (in addition to the reversibility property, the Josephson array equations have a permutation symmetry).

Consider the circuit depicted in Fig. 1. It was shown that for Josephson junctions of negligible capacitance, the array dynamics cannot stably phase lock if the coupling load is purely resistive, though such behavior is possible for inductor-capacitor loads.[12,13] Numerical simulations[13] verify this picture, by demonstrating the concomitant noise sensitivity in the absence of a phase-locked attractor, also called the in-phase state (see Fig. 2). In this same work, a simple model of the deterministic phase-space flow was introduced to estimate the power spectra, which reproduced the gross features of the numerical simulations.[13] However, the analytic results showed a flaw in the proposed model, since it predicted perfectly sharp spectral lines in the in-phase case, which is unreasonable on physical grounds. The purpose of the present paper is to introduce an improved model for the phase-space dynamics which corrects this flaw, yet is still simple enough to be analytically tractable. We will show that only a few fundamental properties of the deterministic flow are needed to get an accurate approximation of the power spectrum.

Of direct relevance is the work carried out several years ago based on the "slowly varying amplitude" approximation by Likharev and co-workers.[1,14] The basic idea there was to launch a direct attack on the governing nonlinear differential equations. By dividing the response of each junction into a low-frequency part and a high-

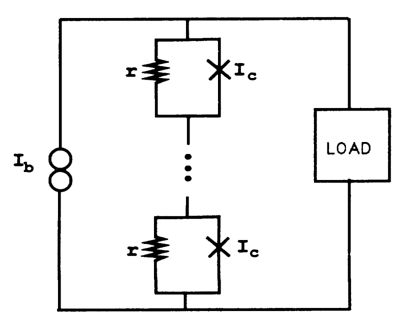

FIG. 1. Lump-circuit schematic of the Josephson-junction-series array; the load may be any combination of passive circuit elements.

frequency part, a quasilinear averaging scheme yielded a set of self-consistency equations, which allowed those authors to compute a variety of things, including the power spectrum for the total voltage across the array when all of the oscillators were mutually entrained. (They could also calculate the range over which nonidentical elements would mutually entrain, which goes well beyond the scope of the calculations presented below.) The basic predictions for the scaling of the linewidth and total power were consistent with experiments on series arrays having up to 99 junctions.[1]

In contrast, the picture presented below is based on general "geometric" properties of the dynamical system rather than the details of the governing differential equations. Nevertheless, there is the opportunity for direct comparison in the limit of a strongly attracting in-phase orbit. In this limit, we recover the earlier findings, giving the same line shape, linewidth, and total integrated power as a function of the noise strength and the array size (i.e., the number of junctions). In addition, we obtain results valid when the reversibility symmetry is present, which gives dramatically different scaling properties.

II. PHASE-SPACE MODEL AND LANGEVIN EQUATIONS

The series array Josephson circuit with inductor-capacitor load obeys the following ordinary differential equations:[4]

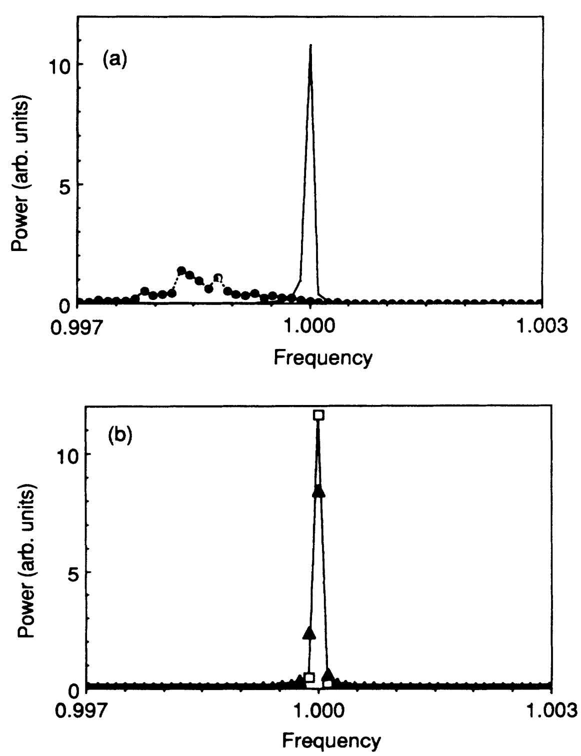

FIG. 2. Results of numerical simulations of the Josephson circuit equations (1) and (2) augmented by independent noise sources for each junction, as reported in Ref. 13. (a) Resistor load; (b) inductor-capacitor load.

$$\frac{\hbar}{2er}\dot{\phi}_k + I_c \sin\phi_k = I_b \ , \tag{1}$$

$$RI = \frac{\hbar}{2e}\sum_k \dot{\phi}_k \ , \tag{2a}$$

$$L\ddot{I} + (1/C)I = \frac{\hbar}{2e}\sum_k \ddot{\phi}_k \ , \tag{2b}$$

where (2a) obtains for a pure resistive load, and (2b) for a combination inductor-capacitor load. Here, ϕ_k is the phase difference of the macroscopic wave function across the kth junction, r is the junction resistance, I_c is the critical current, I_b is the bias current, e is the electron charge, $\hbar$ is Planck's constant divided by 2π, L is the load inductance, C is the load capacitance, R is the load resistance, and the overdot denotes differentiation with respect to time.

The array equations (1) and (2b) exhibit stable in-phase periodic oscillations over a range of parameter space.[3] In this case, each junction oscillates with exactly the same wave form, corresponding to the largest possible voltage oscillations across the entire array. In contrast, the array (1) and (2a) *cannot exhibit stable in-phase periodic oscillations for any parameters,* due to an underlying symmetry. The goal of this paper is to compute the line shape of the total voltage oscillations in the presence of random noise, e.g., due to the junction resistance, for both situations.

One approach to this problem is to augment Eq. (1) by independent random processes, and to attempt the solution of the corresponding Langevin equations. However, we proceed instead by writing down alternative differential equations which nevertheless capture the essential features of the observed phase-space trajectories of the "true" evolution equations, and augment *these* by random noise terms. In particular, our model incorporates both the topology of the phase space (i.e., the variables ϕ_k are phases and so defined mod2π) and the permutation symmetry of the dynamical equations.

Now, except for parameter values I_b very close to I_c, the in-phase solution corresponds to nearly sinusoidal voltage oscillations, so that

$$\phi_k = \omega t + a \sin(\omega t + \alpha) \ . \tag{3}$$

In the presence of noise, the trajectories deviate from this unperturbed solution. An important observation is that for initial conditions off of the in-phase orbit, the variables ϕ_k are nearly periodic, and adjust only on a time scale long compared to $2\pi/\omega$.[9] This may be modeled by a diffusing phase in Eq. (3), so that

$$\dot{\phi}_k = \omega + a\omega \cos[\omega t + \alpha_k(t)] \ . \tag{4}$$

The next step is to define more precisely the diffusion process governing the $\alpha_k(t)$. In particular, if the in-phase orbit is stable, then the line $\alpha_1 = \alpha_2 = \cdots = \alpha_N$ must be attracting. Moreover, any periodic orbit must be neutrally stable along the orbit. Finally, the original equations have permutation symmetry owing to the global coupling. A model incorporating all of these features is

$$\dot{\alpha}_k = -\lambda\alpha_k + \frac{\lambda}{N}\sum_j \alpha_j + \xi_k(t) \ , \tag{5}$$

where the parameter λ is related to the degenerate Floquet exponent of the attracting in-phase orbit, and each phase is subject to independent white-noise sources

$$\langle \xi_k(t) \rangle = 0, \quad \langle \xi_j(t)\xi_k(s) \rangle = \kappa\delta_{jk}\delta(t-s) . \tag{6}$$

Equations (5) are appropriate for an attracting in-phase orbit; however, they can also describe the free diffusion over the entire N-torus if one takes $\lambda=0$. This limiting case is therefore a model when there are no attractors at all: this is precisely the situation for the pure resistive array Eqs. (1) and (2a).

Our plan is as follows. Based on the dynamics (5), we first determine the total array voltage $V(t)$, given by

$$V(t) = \frac{\hbar}{2e}\sum_j \dot{\phi}_j = N\omega + \sum_j a\omega\cos[\omega t + \alpha_k(t)] . \tag{7}$$

Then, we will compute the correlation function of $V(t)$ and the resulting power spectrum. Our main concern is how the line shape depends on the array size N and the noise strength κ.

III. THE FOKKER-PLANCK EQUATION

The N coupled Langevin equations (5) can be recast as a Fokker-Planck equation for the probability density $P(\alpha_1, \ldots, \alpha_N, t)$. It is convenient to first transform the Langevin system so as to decouple the N variables. Let

$$\psi_k = \frac{1}{\sqrt{k(k+1)}}\left[\sum_{i=1}^{k}\alpha_i - k\alpha_{k+1}\right], \quad k=1,\ldots,N-1 ,$$
$$\psi_N = \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\alpha_i . \tag{8}$$

(A simpler transformation to center-of-mass and relative coordinates which has been used to great effect elsewhere,[3] is unhelpful here, since it leads to correlated noise sources and so a relatively complicated Fokker-Planck equation.)

In terms of this orthogonal transformation, the Langevin system (5) becomes

$$\dot{\psi}_k = -\lambda\alpha_k + \Xi_k, \quad k=1,\ldots,N-1$$
$$\dot{\psi}_N = \Xi_N(1/\sqrt{N})\sum_{j=1}^{N}\xi_j , \tag{9a}$$

where

$$\Xi_k = \frac{1}{\sqrt{k(k+1)}}\left[\sum_{j=1}^{k}\xi_j - k\xi_{k+1}\right]$$

and (9b)

$$\Xi_N = \frac{1}{\sqrt{N}}\sum_{j=1}^{N}\xi_j .$$

In these coordinates, all of the input noise cross correlations vanish:

$$\langle \Xi_k(t) \rangle = 0, \quad \langle \Xi_j(t)\Xi_k(s) \rangle = \kappa\delta_{jk}\delta(t-s) . \tag{10}$$

Clearly, the coordinate ψ_N evolves differently than the others. However, it is mathematically convenient to recast Eqs. (9) in a slightly different, but equivalent, form:

$$\dot{\psi}_k = -\lambda_k\psi_k + \Xi_k, \quad k=1,\ldots,N , \tag{11}$$

where $\lambda_k=\lambda$ for $k=1,\ldots,N-1$, and $\lambda_N=0$. The advantage of this formulation is twofold: First, it allows us to write some of the expressions generated below more symmetrically; and second, our results can be easily extended to another interesting case encountered in certain Josephson arrays, namely when the dynamics is neutrally stable over the entire N-torus.[12,13]

The Fokker-Planck equation corresponding to Eqs. (11) is

$$\partial_t P(\underline{\Psi};t) = \sum_{k=1}^{N}\partial_k(\lambda_k\psi_k P) + \frac{\kappa}{2}\sum_{k=1}^{N}\partial_{kk}^2 P , \tag{12}$$

where $\underline{\Psi}=(\psi_1,\ldots,\Psi_N)$, and ∂_k denotes partial differentiation with respect to ψ_k. This equation is easily solved. For a δ-function initial condition at an arbitrary position, the solution is the product of Gaussians

$$P_C(\underline{\Psi};t) = \prod_{k=1}^{N}(2\pi\sigma_k^2)^{-1/2}\exp\left[\frac{-[\psi_k-\psi_k(0)e^{-\lambda_k t}]^2}{2\sigma_k^2}\right] , \tag{13}$$

where

$$\sigma_k^2 = \frac{\kappa}{2\lambda_k}(1-e^{-2\lambda_k t}), \quad 1,\ldots,N-1 ,$$
$$\sigma_N^2 = \kappa t .$$

and where we have written P_C to emphasize that this is the conditional probability given this specific initial condition. In what follows, we need the joint probability density $P(\underline{\Psi},t;\underline{\Psi}',t')$. We can ensure stationarity by taking the distribution at the earlier time t to be the equilibrium distribution P_{eq} [given by the $t\to\infty$ limit of Eq. (13)], so that

$$P(\underline{\Psi},t;\underline{\Psi}',t') = P_{eq}(\underline{\Psi})\prod_{k=1}^{N}[2\pi\sigma_k^2(\tau)]^{-1/2}\exp\left[\frac{-(\psi_k'-\psi_k e^{-\lambda_k\tau})^2}{2\sigma_k^2(\tau)}\right], \tag{14}$$

where $\tau=t-t'$. In particular, we have

$$P_{eq}(\underline{\Psi}) = \prod_{k=1}^{N}P_k^{eq}(\psi_k) \tag{15}$$

with

$$P_k^{eq}(\psi_k) = (2\pi\sigma^2)^{-1/2}\frac{1}{\sqrt{2\pi\sigma^2}}\exp(-\psi_k^2/2\sigma^2), \tag{16}$$

where $\sigma^2=\kappa/2\lambda$ for $k=1,\ldots,N-1$, while for $k=N$ we must take the limit as $\sigma^2\to\infty$, which correctly recovers the fact that the stationary probability density is uniform along the limit cycle.

IV. CORRELATION FUNCTION FOR THE TOTAL VOLTAGE

From the probability density (14), we can compute the correlation function $C(\tau)=\langle V(t)V(t+\tau)\rangle$ for the total voltage (7), where the angular brackets denote an ensemble average. Subtracting out the dc contribution to $V(t)$, which gives rise to a δ function in the power spectrum at zero frequency, we have

$$C(t,\tau)=\frac{1}{2}(a\omega)^2\sum_{m,n}\langle\cos(\alpha_m-\omega\tau-\alpha_n')+\cos(2\omega t+\alpha_m+\omega\tau+\alpha_n')\rangle\ , \quad (17)$$

where the primed and unprimed variables are evaluated at times $t+\tau$ and t, respectively. The second term oscillates rapidly, and disappears upon averaging over one period. In the remaining term, the slow drift of the phases α_m gives rise to the linewidth observed in the power spectrum.

It is convenient to introduce complex notation, so that the desired correlation function is

$$C(\tau)=\frac{1}{4}(a\omega)^2\sum_{m,n}e^{-i\omega t}\langle e^{i\alpha_m}e^{-i\alpha_n'}\rangle+\text{c.c.}\ , \quad (18)$$

where "c.c." denotes the complex conjugate. The required averages are more simply performed in terms of the trasnformed variables ψ_k. From Eq. (8), we have the linear transformation

$$\alpha_k=\sum_j c_{kj}\psi_j\ , \quad (19)$$

where

$$\begin{aligned}&c_{kN}=N^{-1/2}\ ,\\&c_{kj}=[j(j+1)]^{-1/2},\ \text{for } j=k,\ldots,N-1\ ,\\&c_{k,k-1}=-[(k-1)/k]^{1/2}\ ,\\&c_{kj}=0,\ \text{for } j=1,\ldots,k-2\ .\end{aligned} \quad (20)$$

It follows that

$$\langle e^{i\alpha_m}e^{-i\alpha_n'}\rangle=\left\langle\prod_{k=1}^{N}e^{ic_{mk}\psi_k}e^{-ic_{nk}\psi_k'}\right\rangle\ . \quad (21)$$

Since the joint probability density is simply the product of terms, one for each k, we have [using Eqs. (14) and (15)]

$$\langle e^{i\alpha_m}e^{-i\alpha_n'}\rangle=\prod_{k=1}^{N}\int_{-\infty}^{\infty}d\psi_k\int_{-\infty}^{\infty}d\psi_k' e^{ic_{mk}\psi_k}e^{ic_{nk}\psi_k'}P_k^{\text{eq}}(\psi_k)[2\pi\sigma_k^2(\tau)]^{-1/2}\exp\left[\frac{-(\psi_k'-\psi_k e^{-\lambda_k\tau})^2}{2\sigma_k^2(\tau)}\right]\ . \quad (22)$$

The Gaussian integrals are readily evaluated. [Some care must be taken with the $k=N$ factor; recall that $P_N^{\text{eq}}(\psi_N)$ is the $\lambda\to 0$ limit of Eq. (16).] Performing first the integration over the primed coordinates yields

$$\langle e^{i\alpha_m}e^{-i\alpha_n'}\rangle=\prod_{k=1}^{N}\exp[-\tfrac{1}{2}\sigma_k^2(\tau)c_{nk}^2]\int_{-\infty}^{\infty}d\psi_k e^{ic_{mk}\psi_k}P_k^{\text{eq}}(\psi_k)e^{-ic_{nk}\psi_k e^{-\lambda_k\tau}}\ . \quad (23)$$

For $k\neq N$, the integral over ψ_k has the value [see Eq. (16)]

$$\exp[-\tfrac{1}{2}\sigma^2(c_{mk}-c_{nk}e^{-\lambda_k\tau})^2],\quad k=1,\ldots,N-1\ , \quad (24a)$$

while for $k=N$ we have instead (recall that $\lambda_N=0$)

$$\lim_{L\to\infty}\frac{1}{2L}\int_{-L}^{L}d\psi_N e^{ic_{mN}\psi_N}e^{-ic_{nN}\psi_N}=1\ , \quad (24b)$$

where we have used the fact that, from Eqs. (20), $c_{mN}=1/\sqrt{N}$ for all m. Moreover, since $\sigma_1=\sigma_2=\cdots=\sigma_{N-1}$ and $\lambda_1=\lambda_2=\cdots=\lambda_{N-1}$, we have

$$\langle e^{i\alpha_m}e^{-i\alpha_n'}\rangle=\exp\left[-\frac{1}{2N}\sigma_N^2(\tau)\right]\exp\left[-\frac{1}{2}\sigma_1^2(\tau)\sum_{k=1}^{N-1}C_{nk}^2-\frac{1}{2}\sigma^2\sum_{k=1}^{N-1}(c_{mk}-c_{nk}e^{-\lambda_1\tau})^2\right]\ . \quad (25)$$

Using Eqs. (20), the sums are readily performed, with the result

$$\langle e^{i\alpha_m}e^{-i\alpha_n'}\rangle=\exp\left[\frac{-1}{2N}\sigma_N^2-\frac{N-1}{2N}\sigma_1^2-\frac{N-1}{2N}\sigma^2(1+e^{-2\lambda_1\tau})+\sigma^2 e^{-\lambda_1\tau}\left[\delta_{m,n}-\frac{1}{N}\right]\right]\ . \quad (26)$$

Substitution of this expression into the correlation function Eq. (18) yields, upon performing the double sum,

$$C(\tau)=\frac{1}{2}(a\omega)^2\cos(\omega\tau)\exp\left[\frac{-1}{2N}\sigma_N^2-\frac{N-1}{2N}\sigma_1^2-\frac{N-1}{2N}\sigma^2(1+e^{-2\lambda_1\tau})\right]$$
$$\times\left[N\exp\left[\frac{N-1}{N}\sigma^2e^{-\lambda_1\tau}\right]+N(N-1)\exp\left[\frac{-1}{N}\sigma^2e^{-\lambda_1\tau}\right]\right]. \tag{27}$$

Finally, reintroducing the explicit expressions for σ_1^2, σ_N^2, and σ^2, this becomes

$$C(\tau)=\frac{1}{2}(a\omega)^2\exp\left[-\frac{N-1}{N}\frac{\kappa}{2\lambda}\right]\cos(\omega\tau)\exp\left[-\frac{\kappa\tau}{2N}\right]\left[N\exp\left[\frac{N-1}{N}\frac{\kappa}{2\lambda}e^{-\lambda\tau}\right]+N(N-1)\exp\left[-\frac{1}{N}\frac{\kappa}{2\lambda}e^{-\lambda\tau}\right]\right]. \tag{28}$$

The form of Eq. (28) is suggestive. First, we see that the output is the sum of two terms: Apparently, the one proportional to N represents the contribution from incoherent superposition, while the term proportional to $N(N-1)$ represents coherent superposition. Second, the correlation function has an exponentially decaying prefactor, which contributes an additional width to the resulting spectral line centered at frequency ω.

V. POWER SPECTRUM: COMPARISON OF TWO LOADS

The power spectrum $S(\Omega)$ for the total voltage is related to Eq. (28) via

$$S(\Omega)=(2/\pi)\int_0^\infty C(\tau)\cos(\Omega\tau)d\tau .$$

Now, since $C(\tau)$ is the product of a pure cosine with a (somewhat complicated) decaying function, we see that $S(\Omega)$ has the form of a line centered at $\Omega=\omega$, with the line shape given by the transform of the decaying factor. Equation (28) is valid for any choice of parameters; however, we are especially interested in two limiting cases, both of which correspond to small noise intensities, so that $\kappa \ll 1$.

The first case is when the in-phase state is a *bona fide* attractor. Physically, this can happen as long as the junctions have substantial capacitance [so that there is also a term in Eq. (1) proportional to the second derivative of ϕ_k], or if the circuit load is not purely resistive, as in Eq. (2b). In this event, the relaxation rate λ is (at least) of order unity, and we can expand the exponentials appearing in the square bracket of Eq. (28), with the result

$$C(\tau)=\frac{(a\omega)^2}{2}\exp\left[-\frac{N-1\kappa}{N2\lambda}\right]\cos(\omega\tau)\exp\left[-\frac{\kappa\tau}{2N}\right]\left[N^2+\tfrac{1}{2}(N-1)\left[\frac{\kappa}{2\lambda}\right]^2e^{-2\lambda\tau}+O(\kappa^3)\right]. \tag{29}$$

(We have not bothered to expand the first exponential prefactor, which is just an overall constant very nearly equal to unity.) The corresponding power spectrum is

$$S(\Omega)=\frac{1}{2\pi}(a\omega)^2\exp\left[-\frac{N-1}{N}\frac{\kappa}{2\lambda}\right]\left[\frac{N^2\gamma_1}{\gamma_1^2+(\Omega-\omega)^2}+\frac{N-1}{2}\left[\frac{\kappa}{2\lambda}\right]^2\frac{\gamma_2}{\gamma_2^2+(\Omega-\omega)^2}\right]+(\omega\to-\omega)\ , \tag{30}$$

where $\gamma_1=\kappa/(2N)$ and $\gamma_2=\kappa/(2N)+2\lambda$. The dominant contribution to the output thus has the two hallmark features expected for coherent phase locking: overall growth as N^2 and a linewidth which narrows inversely with N. Dynamically, the magnitude is set by the (very slow) diffusion along the limit cycle. This result differs from an earlier calculation based on an oversimplified model which led to a spectral line with zero linewidth.[13] Equation (30) gives the more palatable result of a finite linewidth, which is nevertheless seen to be quite small in the weak noise limit. Moreover, in addition to the dominant sharp line, there is a small broadband "skirt" whose width is determined by the relaxation rate perpendicular to the limit cycle, and whose relative magnitude becomes increasingly negligible for larger N. (Its absolute magnitude grows linearly with N, and so represents a manifestly incoherent contribution to the output.) Finally, the term proportional to N^2 with $(\omega\to-\omega)$ is negligible, since it corresponds to the tail of a line sharply peaked at the negative frequency $\Omega=-\omega$.

These scaling results are in agreement with calculations based on the slowly varying amplitude approximation and also experiments on Josephson arrays having up to 99 elements.[1] In the present context, we see that this corresponds to the limit of a strongly attracting limit cycle, which is the most desirable circumstance for all practical applications.

The second case of interest corresponds to no attracting in-phase limit cycle, but instead a neutrally stable N-torus in phase space. While this seems like a situation so special as to be of no practical interest, the Josephson array has an underlying dynamical symmetry which forces this behavior if the junction capacitance is negligible [as in Eq. (1)] and the circuit load is a resistor, conditions which on physical grounds are certainly realizable.[15] Indeed, the effects of random noise will serve to extend the range of validity of the purely diffusive behavior even in the presence of weakly attracting orbits. This case corresponds to $\lambda=0$ in the Langevin system (5); we can still use Eq. (28) if we take the limit as $\lambda\to 0$. Taking this limit yields the simple result

$$C(\tau)=\tfrac{1}{2}(a\omega)^2\cos(\omega\tau)Ne^{-\kappa\tau/2} . \quad (31)$$

This result makes good sense: It is just the incoherent sum of N independent and identical phase diffusion processes. The power spectrum is then

$$S(\Omega)=N\frac{1}{2\pi}(a\omega)^2\frac{\kappa/2}{(\kappa/2)^2+(\Omega-\omega)^2}+(\omega\to-\omega) . \quad (32)$$

For the small noise regime we have been considering, the Lorentian is very narrow, so that the $(\omega\to-\omega)$ term is negligible at positive frequencies. Note that the linewidth is now independent of array size N, while the total power grows only linearly with N.

As a final point, we note that the numerical simulations reproduced in Fig. 2, involved time series that were hundreds of oscillation periods, which is far too short for the statistics to reach the stationary state. There, the initial conditions were always taken on the in-phase orbit, and while the resulting power spectra clearly show the severe noise sensitivity of the LC-load circuit when compared with the corresponding R-load circuit, the amount of time required for those simulations to reach the stationary limit considered in the present work would have been enormous. On the other hand, for a real Josephson-junction system, the characteristic oscillation frequency is anywhere from 1 to 500 GHz, so that laboratory experiments are likely to recover the stationary limit even for time series lasting "only" a few seconds. Consequently, such experiments can be quantitatively compared with the above results, in order to determine the validity of the underlying dynamical model given by the Langevin system Eqs. (5).

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[15]This is an example where the notions of "genericity" and "structural stability" from dynamical systems theory must be carefully defined relative to a significantly restricted class of vector fields, owing to structure inherent in the physics.