# Quantum particle in a random potential: Exact solution and its implications

Mark Ya. Azbel'
*Raymond and Beverly Sackler Faculty of Exact Sciences, School of Physics and Astronomy, Tel Aviv University, Ramat Aviv, Tel Aviv 69978, Israel*
(Received 21 June 1991)

I accurately solve the Schrödinger equation in a magnetic field $B$ in an arbitrary set of two-dimensional point potentials. When $B=0$, they yield a mobility edge. When $B\neq 0$, all states are localized below a certain energy $E_c(B)$. Above $E_c(B)$, they are extended at the Landau energies. At other energies the localization length is a discontinuous function of $B$ at every rational value of $eBd^2/ch$, where $d$ is an average interpotential distance.

## I. LOCALIZATION AND MAGNETIC FIELD

Eigenstates determine the statistical mechanics of a system and are crucial for its transport properties. In a periodic system the accurate Bloch theorem[1] proves that all eigenstates are extended and yield a zero resistance. This theorem dominated the approach to weak disorder[2] until Anderson suggested[3] disorder-induced localization. Soon thereafter all states in one dimension (1D) were found to be localized by even an infinitesimally weak disorder,[4] with the localization length $\xi \propto c_{\rm im}^{-1}$ ($c_{\rm im}$ is the impurity concentration). This implied the zero residual conductivity of an infinite 1D system. Much later a one-parameter scaling theory[5] demonstrated the localization of all 2D states, with $\ln\xi \propto c_{\rm im}^{-1}$.

The magnetic field $B$ is a formidable problem even in the Bloch case. Weak $B$, with the flux through a unit cell small compared to the flux quantum $hc/e$, may be approximately accounted for[2,6] by the quantum Lorentz transformation $\mathbf{q}\rightarrow i^{-1}\nabla - e\mathbf{A}/\hbar c$ ($\mathbf{A}$ is the vector potential, $\mathbf{q}$ is the wave vector) in the Bloch energy $E=E(\mathbf{q})$. The resulting Zil'berman-Harper-Lifshitz equation[2,6] leads to the devil's-staircase spectrum.[7] The quantum Hall effect (QHE) revived interest in the magnetotransport problem and resulted in a study of its invariants.[8] Disordered systems, even besides integer and fractional QHE, yield a very rich behavior, with positive, negative, and oscillatory magnetoresistance.[9] Theoretically[10,11] and experimentally[12] it was demonstrated that a disordered system has one extended state per each disorder broadened Landau level, i.e., per each energy interval $\Delta E_L=\hbar eB/Mc$ ($M$ is a mass, $B$ is a magnetic field). However, both theory and experiment were related either to a strong enough magnetic field[10,12] or to the scaling approach.[11]

In this paper I *accurately* solve a 2D Schrödinger equation in magnetic field for a random set of special "Impurity D-function" (IDF) potentials, introduced in Ref. 13. Then I consider IDF's of random strength situated on a periodic lattice.

I prove that when the energy $E$ is below a certain critical energy $E_c(B)$, then all states are localized. When $E>E_c(B)$, then the Landau energies $E_p=(\hbar eB/Mc)(p+\frac{1}{2})$ ($p$ is an integer) are extended in any $B$, while all other energies are localized. This is consistent with the results of Refs. 10–12, but specifies the mobility energies and is valid down to $B=0$. When $B=0$, extended states exist at all energies above the mobility edge $E=0$. This agrees with Ref. 14, but disagrees with Ref. 5. The residual resistance in $B=0$ depends on the relative number of extended states. In a finite system it scales with its size.

The accurate solution in $B\neq 0$ yields an unusual result, which is reminiscent of the devil's-staircase states in periodic lattices.[7] For a fixed $E\neq E_p$, the localization length $\xi$ depends on the decomposition of the number $N_q=eBd^2/ch$ of flux quanta per IDF ($d$ is an average inter-IDF distance) into a continuous fraction. The function $\xi(B)$ is discontinuous at every rational $N_q$. Of course, in the Mott variable-range hopping (VRH) this discontinuity is smeared out by finite temperature. Still, when temperature decreases, VRH magnetoresistance may reveal a progressively finer structure which resembles oscillations or fluctuations in $B$-$cf$ experiments in Ref. 15. A similar "superstructure" may appear also in QHE.

In a general case, I present accurate universal transcendental equations for eigenenergies and eigenstates.

The disagreement of my results in $B=0$ with Ref. 5 may indicate that the scaling theory and resistance strongly depend on the ratio of the impurity size $\rho$ to its Bohr radius $\rho_B$.

IDF's may also be useful in other problems, such as the development of chaos in scattering, persistent currents, QHE, Coulomb blockade, transport and tunneling (including resonance tunneling), etc.

## II. A SINGLE IDF POTENTIAL IN MAGNETIC FIELD

In this section I introduce a single IDF potential and discuss its similarity to and difference from a $\delta$ function. It allows for the explicit analytical solution of the Schrödinger equation. Almost all of its eigenfunctions have a very special property: they do not depend on the IDF strength, but they do depend on its location. The IDF yields only $s$ scattering.

 

Following Ref. 13, introduce

$$U(\mathbf{R})=-(\hbar^2/2M)D(R)\ , \tag{1a}$$

where

$$D(R)=\lim_{\rho\to 0}[2\exp(-R^2/\rho^2)/\rho^2\ln(\rho_0/\rho)]\ . \tag{1b}$$

Here $\rho_0$ determines the IDF strength. The width of a potential well $U$, by Eq. (1b), is $\rho\to 0$. So, its zero-point energy is $\hbar^2/2M\rho^2$. Since $|U|\lesssim\hbar^2/2M\rho^2\ln(\rho_0/\rho)$, the well is infinitely shallow[16] when $\rho\to 0$. Such a well has a bound eigenstate with the finite Bohr radius.[13,17(a)] Its wave function $\Psi$ exponentially decays [17(a)] at $R\sim\rho_0$, i.e., infinitely slowly compared to $U$ (which decays at $\rho$). So, similar to a $\delta$ function, $\rho\to 0$ yields[17(a),13] (for both localized and extended states)

$$D(R)\Psi(\mathbf{R})=D(R)\Psi(0)\ . \tag{2}$$

By Eq. (1b), $D(R)$ is different from a similar definition of a $\delta$ function $\delta(\mathbf{R})$ only in a factor $2\pi/\ln(\rho_0/\rho)$. Its "area" is

$$\int D(R)d\mathbf{R}=2\pi/\ln(\rho_0/\rho)\ . \tag{3a}$$

When $\rho\to 0$, the area $\to 0$. This does not allow for the introduction of simple symbolic formulas, such as $\int\Psi(\mathbf{R})\delta(\mathbf{R})d\mathbf{R}=\Psi(0)$, which are so efficient in the case of a $\delta$ function, and calls for accurate calculations. They lead to nontrivial equations (see later, especially Secs. III and IV), different from "boring" ones for a $\delta$ function. By Eq. (3a),

$$\rho_0=\rho e^{2\pi}\simeq 534\rho\to 0 \tag{3b}$$

leads to $D(r)\to\delta(\mathbf{r})$. Due to the wave-function nonlinearity in the potential, $D(r)$ has a bound eigenstate and scatters, while $\delta(\mathbf{r})$ has no bound (finite-energy) eigenstate and does not scatter. To demonstrate the bound state of a single IDF, consider the energy

$$E\equiv-\hbar^2k^2/2M\ . \tag{4}$$

By Eq. (2), the Schrödinger equation is

$$\Delta\Psi-k^2\Psi=-D(R)\Psi(0)\ . \tag{5}$$

To solve this equation, I assume a small but finite $\rho$ in Eq. (1b) and *then* take the limit $\rho\to 0$. The same approach is used throughout the paper. (This is a necessity related to the discussed difference between $D$- and $\delta$ functions.) The homogeneous equation (with $D\equiv 0$) has no solution which is finite at $R\to\infty$. The inhomogeneous solution of Eq. (5) is related to the Green function[18(a)]

$$\Psi(\mathbf{R})=[\Psi(0)/2\pi]\int K_0(k|\mathbf{R}-\mathbf{R}'|)D_0(R')d\mathbf{R}'\ , \tag{6}$$

where $K_0$ is the Bessel function. Choosing $R=0$ in Eq. (6), one determines the eigenenergy from

$$\int_0^\infty K_0(kR)D(R)dR=1\ . \tag{7}$$

When $\rho\to 0$, then, by Eqs. (7) and (1b) and Ref. 19(a),

$$\rho_B\equiv k^{-1}=(\rho_0/2)\exp(C/2)\simeq 0.67\rho_0\ . \tag{8}$$

Here $C$ is the Euler constant. Now Eq. (6) determines $\Psi(\mathbf{R})$. When $R/\rho\to\infty$, then, apart from normalization,

$$\Psi(\mathbf{R})=K_0(R/\rho_B)\ . \tag{9}$$

It has a very weak logarithmical divergence at $R=0$. If $R\gg\rho_B$,

$$\Psi(R)\sim\exp(-R/\rho_B)\ . \tag{10}$$

So, by Eqs. (4), (8), and (10), $\rho_B$ is the Bohr radius of an IDF. When $R\ll\rho$, then Eq. (6) yields

$$\Psi(0)\simeq\ln(\rho_B/\rho)\ , \tag{11}$$

with the interpolation formula for Eqs. (9) and (11) being

$$\Psi(R)\simeq K_0[(R+\rho)/\rho_B]\ . \tag{12}$$

Equations (4) and (8) demonstrate the difference between $D(R)$ and $\delta(R)$. In the case (3b), when $\rho_0=534\rho\to 0$ and $D(R)\to\delta(R)$, Eqs. (4) and (8) yield $E\to-\infty$, i.e., $\delta(R)$ does not have a finite-energy bound state.

Now consider the scattering by a single IDF, with the positive energy of an incoming wave

$$E\equiv\hbar^2k^2/2M\ . \tag{4'}$$

Then, by Eq. (2), the Schrödinger equation is

$$\Delta\Psi+k^2\Psi=-D(R)\Psi(0)\ . \tag{5'}$$

If the incident wave is $\exp(ikY)$, then, by Eq. (5'),

$$\Psi(\mathbf{R})=\exp(ikY)+\tfrac{1}{4}i\Psi(0)\int H_0^{(1)}(k|\mathbf{R}-\mathbf{R}'|)D(\mathbf{R}')d\mathbf{R}'\ , \tag{6'}$$

where $H_0^{(1)}$ is the Hankel function. When $R=0$, then, accounting for Eq. (6') and Ref. 19(b),

$$\Psi(0)=\ln(\rho_0/\rho)/\ln(-ik\rho_B)\ , \tag{11'}$$

where $\rho_B$ is provided by Eq. (8). If $R\gg\rho$, then, by Eq. (6'),

$$\Psi(\mathbf{R})=\exp(ikY)+i\pi H_0^{(1)}(kR)/2\ln(-ik\rho_B)\ . \tag{9'}$$

When $kR\gg 1$, then Eq. (9') provides the scattering amplitude $\propto 1/\ln(-ik\rho_B)$. By Eq. (3b), when $D(\mathbf{R})\to\delta(\mathbf{R})$, i.e., $\rho_B\to 0$, there is no scattering. This happens due to the wavelength $k^{-1}$ being infinitely larger than the Bohr radius. The scattering also vanishes in the opposite limiting case $k\rho_B\to\infty$ of an infinitely high energy $E$ [see Eq. (4')], or of an infinitely weak IDF ($\rho_B\to\infty$). When $k\rho_B=1$, the scattering is maximal (resonance).

By Eq. (9'), the scattering is extremely nonuniversal with respect to the angular moment $l$: only $l=0$ is scattered, other $l$'s do not notice the IDF. Now consider IDF eigenstates with positive energy $E>0$. When $l\neq 0$, then, by Eq. (5'), $\Psi_l=\exp(il\phi)J_l(kR)$, $l\neq 0$, does not notice the IDF. The eigenstate with $l=0$ may be derived via the Green function of Eq. (5'):

$$\Psi_0(R)=J_0(kR)-\tfrac{1}{4}\Psi_0(0)\int D(R')N_0(k|\mathbf{R}-\mathbf{R}'|)d\mathbf{R}' , \tag{13}$$

where $N$ is the Neumann function.[19(b)] Calculating $\Psi_0(0)$, one obtains

$$\Psi_0(R)=J_0(kR)-N_0(kR)/4\ln(k\rho_B) \quad \text{if } R/\rho\to\infty , \tag{14}$$

$$\Psi_0(R)/\Psi(0)\sim\ln(\rho_B/\rho)/\ln(k\rho_B) .$$

The $l\neq 0$ eigenstate $\Psi_l(R)\to 0$ when $R\to 0$. The degeneracy allows for an extended eigenstate

$$\Psi(\mathbf{R})=\exp(ikY)-J_0(kR) . \tag{15}$$

It also does notice the IDF position: $\Psi(0)=0$, but it is independent of the IDF strength. Thus, this feature is common for almost all IDF eigenstates (the exceptions are $s$ states). It is related to the IDF point nature and it vanishes when $\rho\neq 0$. Consider, e.g., the scattering by the potential (1b) with small, but finite $\rho$. Then the Schrödinger equation

$$\Delta\Psi+k^2\Psi=-D(R)\Psi(R) \tag{16}$$

yields

$$\Psi(\mathbf{R})=\exp(ikY)+\tfrac{1}{4}i\int H_0^{(1)}(k|\mathbf{R}-\mathbf{R}'|)D(\mathbf{R}')\Psi(\mathbf{R}')d\mathbf{R}' . \tag{17}$$

Consider

$$\Psi(\mathbf{R})=\sum_l\Psi^{(l)}(R)\exp(il\phi) . \tag{18}$$

Then, accounting for Ref. 19(c), one obtains

$$\Psi^{(l)}(R)=J_l(kR)+\tfrac{1}{4}i\int H_0^{(1)}(k\sqrt{R^2+R'^2-2RR'\cos\theta})\times D(R')\cos(l\theta)\Psi^{(l)}(R')dR'd\theta . \tag{19}$$

When $l\neq 0$, in the leading approximation $\Psi^{(l)}(R)=J_l(kR)$. Further perturbations in Eq. (19) yield the scattering amplitude $\propto(\rho/\rho_0)^l/\ln(\rho_B k)$ when $\rho\ll\rho_0$.

## III. A SINGLE IDF POTENTIAL IN MAGNETIC FIELD

Consider magnetic field $\mathbf{B}=B\hat{z}$. In the symmetric vector potential gauge $\mathbf{A}=\frac{1}{2}\mathbf{B}\times\mathbf{r}$ the Schrödinger equation, by Eq. (2), reads

$$(\nabla-e\mathbf{B}\times\mathbf{r}/2c\hbar)^2\Psi+2ME\Psi/\hbar^2=-D(R)\Psi(0) . \tag{20}$$

The homogeneous equation (with $D\equiv 0$) yields the Landau eigenenergy[17(b)]

$$E_p=(\hbar eB/Mc)(p+\tfrac{1}{2}) \tag{21}$$

($p$ is an integer) and the homogeneous eigenfunction

$$\Psi_{\text{homog}}(\mathbf{R})=l_p(\pi eBR^2/ch) , \quad l_p(\zeta)=\exp(-\zeta/2)L_p(\zeta) . \tag{22}$$

Here $l_p$ is the Laguerre function and $L_p$ is the Laguerre polynomial.[19(d)] The homogeneous $\Psi_{\text{homog}}$, by Eq. (22), does not satisfy Eq. (20) with $D_0\neq 0$. The inhomogeneous $\Psi(\mathbf{R})$ is related to the Green function[18] and equals

$$\Psi(\mathbf{R})=\Psi(0)\int D(\mathbf{R}')G_a(b|\mathbf{R}-\mathbf{R}'|^2)d\mathbf{R}' , \tag{23}$$

$$G_a(\zeta)=\Gamma(a)\zeta^{1/2}W_{(1/2)-a,0}(\zeta) , \tag{24}$$

$$a=\tfrac{1}{2}-McE/\hbar eB , \quad \mathbf{b}=\hbar e\mathbf{B}/ch . \tag{25}$$

Here $\Gamma$ and $W$ are the gamma[19(e)] and Whitteker[19(f)] functions correspondingly. Choosing $R=0$ in Eq. (23) and accounting for Ref. 19(g), one obtains the equation for the energy

$$C+2\ln(\rho_0/L_B)+\psi(a)=0 , \quad L_B^2=1/2b , \tag{26}$$

where $\psi$ is the Euler $\psi$ function and $L_B$ is the magnetic length. When $\rho_0\gg L_B$ or $\rho_0\ll L_B$, then, by Eq. (26) and Ref. 19(i),

$$a\simeq -p-\tfrac{1}{2}\ln(\rho_0/L_B) . \tag{27}$$

When $\rho_0/L_B\to 0$ or $\rho_0/L_B\to\infty$, then $a\to -p$, i.e., eigenenergies are independent of the IDF strength, as in the case of $B=0$. When $R\gg\rho$, then, by Eq. (23), apart from normalization,

$$\Psi(\mathbf{R})=G_a(bR^2) . \tag{27a}$$

When $bR^2\gg|a|$, i.e., $R\gg L_H^2(ME/\hbar^2)^{1/2}$, then

$$\Psi(R)\sim(bR^2)^{a-1}\exp(-bR^2/2) . \tag{27b}$$

## IV. GENERAL CASE SOLUTION

Consider an arbitrary set of IDF's in magnetic field $\mathbf{B}=B\hat{z}$. Denote the ordinal number of an IDF by $s$, its strength by $\rho_s$ [i.e., $\rho_s$ replaces $\rho_0$ in Eq. (1b)], its position by $\mathbf{R}_s$, and its potential by $U_s=-(\hbar^2/2M)D_s(|\mathbf{R}-\mathbf{R}_s|)$. Similar to Eq. (2), $D_s(|R-\mathbf{R}_s|)\Psi(\mathbf{R})=D_s(|\mathbf{R}-\mathbf{R}_s|)\Psi(\mathbf{R}_s)$. So, in the symmetric vector-potential gauge, the Schrödinger equation reads

$$(\nabla-e\mathbf{B}\times\mathbf{r}/2c\hbar)^2\Psi+2ME\Psi/\hbar^2=-\sum_s D_s(\mathbf{R}-\mathbf{R}_s)\Psi_s , \quad \Psi_s\equiv\Psi(\mathbf{R}_s) . \tag{28}$$

Suppose $E\neq E_p=(\hbar eB/Mc)(p+\frac{1}{2})$, where $E_p$ is the Landau energy. Then Eq. (28) has no homogeneous solutions, and the inhomogeneous $\Psi(\mathbf{R})$ is related, by the Green function, to $\Psi(\mathbf{R}_s)$:

$$\Psi(\mathbf{R})=\sum_s\Psi_s\exp(i\mathbf{b}\cdot\mathbf{R}_s\times\mathbf{R})\times\int D_s(\mathbf{R}')G_a(b|\mathbf{R}-\mathbf{R}_s-\mathbf{R}'|^2)d\mathbf{R}' . \tag{29}$$

Choosing $\mathbf{R}=\mathbf{R}_\sigma$, where $\mathbf{R}_\sigma$ is the location of the $\sigma$th IDF, I obtain the equation for the eigenvalues of $E$ and $\Psi_s$. When $\rho\to 0$, it reads (cf. Ref. 13),

$$\varepsilon_\sigma\Psi_\sigma=\sum_s{}'\Psi_s\exp(i\mathbf{b}\cdot\mathbf{R}_s\times\mathbf{R}_\sigma)G_a(b|\mathbf{R}_\sigma-\mathbf{R}_s|^2) , \tag{30}$$

$$\varepsilon_\sigma=\eta_\sigma+\psi(a)\ ,\quad \eta_\sigma=C+\ln(2b\rho_\sigma^2)\ . \tag{31}$$

The prime in Eq. (30) denotes that the diagonal term $s=\sigma$ is missing in the sum; $a$ is from Eq. (25). This equation allows one to easily study a small number of IDF's. Consider, e.g., an instructive example of three identical IDF's with $\rho_s=\rho_0$, situated at the sites of an equilateral triangle with side $d$. The determinant of three linear equations (30) must be zero. This yields $a$ and thus, by Eq. (25), the energy

$$\begin{aligned} &E=(0.5-a)\hbar eB/Mc\ ,\\ &-\psi(a)=C+\ln(2\alpha\rho_0^2/d^2)+\tilde{\varepsilon}(\alpha)/G_a(\alpha)\ ,\\ &\alpha=bd^2 \end{aligned} \tag{32}$$

where $\tilde{\varepsilon}(\alpha)$ satisfies the equation

$$\tilde{\varepsilon}^3-3\tilde{\varepsilon}+2\cos(\alpha\sqrt{3}/2)=0\ . \tag{33}$$

When $E\to E_p$, then $a=-p+\gamma$, $\gamma\to 0$, and, by Eqs. (21) and (31) and Ref. 19(i), $\varepsilon_0\to -1/\gamma$, $G_a(\zeta)\to f_p(\zeta)/\gamma$. So, by Eq. (32),

$$\tilde{\varepsilon}=1/l_p(\alpha) \tag{34}$$

is independent of the IDF strength. By Eqs. (33) and (34) the equation

$$2l_p^3(\alpha)\cos(\alpha\sqrt{3}/2)-3l_p^2(\alpha)+1=0 \tag{34a}$$

determines the values of $\alpha=\alpha(p)$ when this happens. One readily verifies that IDF strength-independent eigenstates exist even when IDF strengths are different. Then they also have the Landau energies.

Back to a general case. When $\mathbf{R}\neq\mathbf{R}_s$ (i.e., $|\mathbf{R}-\mathbf{R}_s|/\rho\to\infty$ for all $s$), then, by Eq. (29), the wave function, apart from normalization, is

$$\begin{aligned} \Psi(\mathbf{R})&=\sum_s \Psi_s \exp(i\mathbf{b}\cdot\mathbf{R}_s\times\mathbf{R})G_a(b|\mathbf{R}-\mathbf{R}_s|^2)\\ &\equiv\pi\Psi^*(\mathbf{R})/\sin(\pi a)\ . \end{aligned} \tag{35a}$$

[I introduce $\Psi^*$ for further convenience—see Eq. (35c).] By Refs. 20 and 19(f), the Green function $G_a(b)$ may be presented through the Laguerre functions $l_\nu(R)$:

$$G_a(\zeta)=\sum_{\nu=0}^{\infty} l_\nu(\zeta)/(\nu+a)\ . \tag{35b}$$

So, $\Psi^*$ from Eq. (35a) equals

$$\begin{aligned} \Psi^*(\mathbf{R})&\equiv\Psi(\mathbf{R})\sin(\pi a)/\pi\\ &=\sum_{\nu=0}^{\infty}\Psi^\nu(\mathbf{R})\sin(\pi a)/\pi(\nu+a)\ , \end{aligned} \tag{35c}$$

where

$$\Psi^\nu(\mathbf{R})=\sum_s \Psi_s\Psi^{s\nu}(\mathbf{R}-\mathbf{R}_s)\ , \tag{36}$$

$$\Psi^{s\nu}(\mathbf{R}-\mathbf{R}_s)=\exp(i\mathbf{b}\cdot\mathbf{R}_s\times\mathbf{R})l_\nu(b|R-\mathbf{R}_s|^2)\ . \tag{37}$$

Equation (35) explains the introduction of $\Psi^*(\mathbf{R})$, with the normalization different from $\Psi(\mathbf{R})$. When $a\to -p$, then $\Psi(\mathbf{R})$ diverges, while

$$\Psi^*(\mathbf{R})\to\Psi^p(\mathbf{R})\ . \tag{37a}$$

However, in a rather general case (see Sec. V) $\Psi^p(\mathbf{R})\equiv 0$ if $\Psi_s$ yields Eq. (30) with $a+p=\gamma\to 0$ (i.e., with $E\to E_p$). Then it is more convenient to use $\Psi(\mathbf{R})$, which, by Eq. (35) and the l'Hopitale formula, equals

$$\Psi(\mathbf{R})=\left[\frac{d\Psi^\nu}{d\gamma}\right]_{\gamma=0}=\sum_s\Psi^{s\nu}(\mathbf{R}-\mathbf{R}_s)\left[\frac{d\Psi_s}{d\gamma}\right]_{\gamma=0}\ . \tag{37b}$$

By Eq. (36) $\Psi^\nu$ is the Wannier-type presentation of the homogeneous solution to Eq. (20) with $E=E_\nu$, and $\Psi^{s\nu}(\mathbf{R}-\mathbf{R}_s)$ is the eigenfunction (22) centered at $\mathbf{R}_s$ rather than at $\mathbf{R}=0$, with the phase accounted for. Equation (30), which determines $E$ and $\Psi_s=\Psi(\mathbf{R}_s)$, and Eq. (35), which determines $\Psi$ when $\mathbf{R}\neq\mathbf{R}_s$, are universal, general, and accurate for IDF's. If $E=E_p$, then the homogeneous solution $\Psi^p$ must be orthogonal to the right-hand side of Eq. (28), i.e., $\sum_s|\Psi^p(\mathbf{R}_s)|^2=0$. So, by Eq. (36),

$$\Psi^p(\mathbf{R}^s)=\sum_s\Psi_s\Psi^{sp}(\mathbf{R}-\mathbf{R}_s)=0 \tag{38}$$

for all $s$.

For simplicity, from now on, unless stated otherwise, consider IDF's on a square lattice $\mathbf{R}_s=\mathbf{r}_s d$, $\mathbf{r}_s=mx+ny\equiv\mathbf{r}_{mn}$, $\mathbf{r}_\sigma=m_0x+n_0y$. Introduce

$$\Psi_{mn}=\exp(i\alpha mn)\Psi^L_{mn}\ ,\quad \alpha=bd^2 \tag{39}$$

("$L$" denotes the Landau gauge). Then, by Eq. (30),

$$\begin{aligned} \varepsilon_{m_0n_0}\Psi^L_{m_0n_0}&=\sum_{m,n}\Psi^L_{mn}\exp[i\alpha(m-m_0)(n+n_0)]\\ &\quad\times G_{m-m_0,n-n_0}, \end{aligned} \tag{40a}$$

$$\varepsilon_{mn}=\psi(a)+C+\ln(2\alpha\rho_{mn}^2/d^2)\ ,$$

$$G_{mn}=G_a(\alpha|m^2+n^2|)\ \text{ if } m^2+n^2\neq 0\ ,\quad G_{00}=0\ . \tag{40b}$$

Equation (40a) is universal, general, and accurate. It has only diagonal disorder, related to the IDF strength $\rho_{mn}$. Nondiagonal elements in Eq. (40a) depend on two parameters: the number of flux quanta $\alpha/\pi$ per unit cell and the dimensionless energy $a$ from Eq. (25). Consider special cases. Suppose a total IDF potential is periodic along $x$ and is arbitrary along $y$, i.e., $\varepsilon_{mn}=\varepsilon_n$. Then

$$\Psi^L_{mn}=\exp(imq_1)\Psi^L_n \tag{41a}$$

and Eq. (40a) accurately reduces to a 1D equation for $\Psi^L_n$, which provides a test stone for 2D studies:

$$\varepsilon_{n_0}\Psi^L_{n_0}=\sum_n\Psi^L_n\sum_m\cos\{m[\alpha(n+n_0)+q_1]\}G_{m,n-n_0}\ . \tag{41b}$$

In periodic total IDF potential with $\varepsilon_{mn}=\varepsilon$, choose $\alpha=\pi Q_1/Q$, (where $Q_1$ and $Q$ are integers, $Q_1/Q$ is the number of magnetic flux quanta per unit cell) and $n=NQ+\mu$, $n_0=N_0Q+\lambda$, $0\le\mu,\lambda\le Q-1$. Introduce

$$\Psi^L_n=\Psi^L_{NQ+\mu}=\exp(iq_2N)\tilde{\Psi}_\mu\ . \tag{42a}$$

Since $\exp[2i\alpha n_0(m-m_0)]$ is periodic in $n_0\to n_0+Q$, Eq.

(41) reduces then to $Q$ linear algebraic equations for $\tilde{\Psi}_\mu$:

$$\varepsilon\tilde{\Psi}_\lambda=\sum_{\mu=0}^{Q-1}\tilde{G}^t_{\lambda\mu}\tilde{\Psi}_\mu\ ,\quad G^t_{\lambda\mu}=\sum_{m,N=-\infty}^{\infty}\tilde{G}_{mN\lambda\mu}$$
$$\tilde{G}_{mN\lambda\mu}=(-1)^{mNQ_1}\cos\{m[\alpha(\mu+\lambda)+q_1]\}$$
$$\times\exp(iq_2N)G_{m,NQ+\mu-\lambda}\ , \tag{42b}$$

where the superscript "$t$" denotes the total. Equation (42b) is the exact IDF version of the Zil'berman-Harper-Lifshitz equation.[2,6] Different from the latter, it already has proper quantum numbers and depends on magnetic field via $\alpha$ and $a$. More important, while Eqs. (40a) and (41b) depend on a parameter $\alpha$, it explicitly depends on the (integer) nominator $Q_1$ and denominator $Q$ of $\alpha/\pi=Q_1/Q$. For instance, the number of linear equations in Eq. (42b) is 2 if $\alpha/\pi=0.5$ and is $10^6$ if $\alpha/\pi=0.500\,001$. It is natural that such a change in the number of equations significantly changes the eigenenergy. Similar to Ref. 7, this leads to the devil's-staircase-type spectrum, which is related to the decomposition of $\alpha/\pi$ into the continuous fraction and which has a singularity at every rational value of $\alpha/\pi$. All of the above equations are accurate. Now consider an approximate case. The change $\mathbf{s}\to\mathbf{s}+\boldsymbol{\sigma}$ in Eq. (30) on a square lattice (where $\mathbf{r}_s=\mathbf{s}$) leads to $\Psi_{s+\sigma}=\exp(sd/d\sigma)\Psi_\sigma$. When $\alpha\ll1$, the coefficients in Eq. (30) change slowly. If $\varepsilon_\sigma$ also slowly changes with $\sigma$ together with $\rho_\sigma$ [see Eq. (40a)], then in the leading approximation one replaces $id/d\sigma$ by the dimensionless generalized wave vector $\mathbf{q}_g$, with $\mathbf{q}=\mathbf{q}_g+\boldsymbol{\alpha}\cdot\mathbf{r}_s\times\mathbf{r}_\sigma$ being the kinematic wave vector. Then one obtains the accurate classical dispersion in magnetic field:

$$\varepsilon=\sum_{m,n}\cos(\mathbf{q}\cdot\mathbf{r}_{mn})G_{mn}\ . \tag{43}$$

Different from Refs. 2 and 6, it depends on magnetic field via $\alpha$ and $a$ from Eqs. (40b) and (25). Now consider $\alpha=\pi Q_0+\alpha_1$, where $|\alpha_1|\ll1$, $Q_0\geq1$. In the leading approximation ($\alpha_1\to0$) one may replace $\exp[i\alpha(m-m_0)(n+n_0)]$ in Eq. (40a) by $\exp[i\pi Q_0(n-n_0)]$. If $\varepsilon_{m_0n_0}$ slowly changes, then $\Psi^L_{mn}=\exp(i\mathbf{q}\cdot\mathbf{r}_{mn})$ yields, by Eq. (40a),

$$\varepsilon=\sum_{m,n}(-1)^{mnQ_0}\cos(\mathbf{q}\cdot\mathbf{r}_{mn})G_{mn} \tag{44}$$

which, by Eq. (43), includes the case of $Q_0=0$. So, Eq. (44) is valid for any $\alpha/\pi$ close to an integer. It is accurate for $\varepsilon=$const and $\alpha=\pi Q_0\neq0$. When $\pi Q_0\gg1,|a|$, then

$$\varepsilon\simeq(\cos q_1+\cos q_2)\Gamma(a)(\pi Q_0)^{-a}\exp(-\pi Q_0/2)\ . \tag{45}$$

This is the tight-binding case. The leading approximation in $\exp(-\pi Q_0/2)$ corresponds to a single IDF with $\varepsilon=0$ [cf. Eq. (26)]. The next approximation yields the effective mass[2] $m^*=(\frac{1}{2}\pi)\partial S/\partial E$ [$S$ is the area of $E(\mathbf{q})=E$] of a Bloch electron, which exponentially increases $\propto\exp(\pi Q_0/2)$ with magnetic field. When $-a\gg1$, then[19(i)]

$$a\simeq-p+(-1)^p(p^{-3/2}/\pi\ln p)(\pi eQ_0/p)^p$$
$$\times\exp(-\pi Q_0/2)(\cos q_1+\cos q_2)\ , \tag{46}$$

where $p$ is an integer.

## V. LANDAU LEVELS AND DEVIL'S-STAIRCASE LOCALIZATION

When $E\to E_p$, i.e., when, by Eq. (25), $a=-p+\gamma$, $\gamma\to0$, then, by Eqs. (24) and (31) and Ref. 19(i), $G_a(\zeta)\to l_p(\zeta)/\gamma$, $\varepsilon_0\to-1/\gamma$. So, Eq. (40a) reduces to

$$\sum_{m,n=-\infty}^{\infty}\Psi^L_{mn}\exp[i\alpha(m-m_0)(n+n_0)]l_p(\alpha r^2_{m-m_0,n-n_0})$$
$$=0\ . \tag{47}$$

By Eq. (35c), when $a\to-p$, then[21] $\Psi^*(\mathbf{r})\to\Psi^p(\mathbf{r})$. Since Eq. (47) is clearly ignorant of randomness, follow the reasoning of the periodic case (42b) and obtain *extended* solutions for $\gamma\to0$ [cf. Ref. 17(b) for a free particle]. Choose $\alpha/\pi=Q_1/Q$, $n=NQ+\mu$, $n_0=N_0Q+\lambda$, $0\leq\lambda$, $\mu\leq Q-1$, and

$$\Psi^L_{mn}=\Psi^L_{m,NQ+\mu}=\exp(iq_1m+iq_2N)c_\mu\ . \tag{48}$$

Then, by Eq. (47),

$$\sum_{\mu=0}^{Q-1}S^t_{\lambda\mu}c_\mu=0,\quad S^t_{\lambda\mu}=\sum_{m,N=-\infty}^{\infty}S_{mN\lambda\mu}=\overline{S^t_{\mu\lambda}}\ , \tag{49}$$

$$S_{mN\mu\lambda}=(-1)^{mNQ_1}\exp\{im[\alpha(\mu+\lambda)+q_1]+iq_2N\}$$
$$\times l_p(\alpha r^2_{m,NQ+\mu-\lambda})$$
$$=\overline{S_{-m,NQ+\mu-\lambda}}\ , \tag{50}$$

where $r^2_{mn}=m^2+n^2$ [cf. Eqs. (49), (50), and (42b)]. A bar denotes the complex conjugation. The Landau energy states (with $a\to-p$) are extended when the $Q\times Q$ determinant of the matrix $S^t$ from Eq. (49) is zero:

$$D(\mathbf{q},p,Q_1,Q)\equiv\mathrm{Det}S^t=0\ . \tag{51}$$

When $\alpha=0$, i.e., $B=0$, then, using the Green function of Eq. (5′), Eq. (47) is replaced by $\sum_{m,n}\Psi_{mn}J_0(kdr_{m-m_0,n-n_0})=0$. Its Fourier transformation[22]

$$\sum_{m,n}J_0(kdr_{mn})\cos(\mathbf{q}\cdot\mathbf{r}_{mn})\exp(-\alpha r^2_{mn}/2)=0$$

for all $\mathbf{q}$ in virtue of the Poisson summation formula and Refs. 19(c) and 19 (k). So, all states (for $E>0$) are extended.

The Landau energy ($a=-p$) states are extended if $D=0$ in Eq. (51). Consider localization in their vicinity ($|\gamma|\ll1,\gamma=a+p$) for $\mathbf{q}$ which yields $D(\mathbf{q},p,Q_1,Q)=0$. Introduce in Eq. (40a)

$$\varepsilon_{mn}=\varepsilon_{m,NQ+\mu}=\varepsilon^{*}_{mN\mu}-1/\gamma\ ,\quad G_{mn}=G^{*}_{mn}+l_p(\alpha r^2_{mn})/\gamma\ \text{ if } m\neq n\ ,\quad G^{*}_{00}=0\ , \tag{52}$$

$$\Psi^{L}_{mn}=\Psi^{L}_{m,NQ+\mu}=\exp(iq_1m+iq_2N)\tilde{\Psi}_{mN\mu}\ ,\quad 0\le\mu\le Q-1\ . \tag{53}$$

When $\gamma\to 0$, then $G^{*}_{mn}$ and $\varepsilon^{*}_{mn}$ are calculated according to Ref. 19(i), and

$$\varepsilon^{*}_{mn}\equiv\varepsilon^{*}_{mN\mu}=\psi(1+p)+C+\ln(2\alpha\rho^2_{m,NQ+\mu}/d^2)\ . \tag{54}$$

By Eqs. (40a), (52), and (53),

$$\sum_{\mu}S^{t}_{\lambda\mu}\tilde{\Psi}_{m_0N_0\mu}=\gamma\left[\varepsilon^{*}_{m_0N_0}\tilde{\Psi}_{m_0N_0\lambda}-\sum_{m,N,\mu}S^{*}_{mN\lambda\mu}\tilde{\Psi}_{m+m_0,N+N_0,\mu}\right]-\sum_{m,N,\mu}S_{mN\lambda\mu}(\tilde{\Psi}_{m+m_0,N-N_0,\mu}-\tilde{\Psi}_{m_0N_0\mu})$$
$$\equiv F_{m_0N_0\lambda}\ . \tag{55}$$

Here $S$ is presented by Eqs. (49), (50), and

$$S^{*}_{mN\lambda\mu}=G^{*}_{m,NQ+\mu-\lambda}(-1)^{mNQ_1}\times\exp\{im[\alpha(\lambda+\mu)+q_1]+q_2N\}=\overline{S^{*}_{-m,-N,\mu\lambda}}\ . \tag{56}$$

The "homogeneous" Eq. (55) with $F_{m_0N_0\lambda}=0$, by Eq. (49), has a solution $\tilde{\Psi}_{mN\mu}=c_\mu$. Therefore, the inhomogeneous Eq. (55) with $F_{m_0N_0\lambda}\neq 0$ has a solution if

$$\sum_{\lambda}c_\lambda F_{m_0N_0\lambda}=0\ . \tag{57}$$

Consider

$$\tilde{\Psi}_{mN\mu}=\chi_{mN}c_\mu \tag{58a}$$

and assume that $\chi_{mN}$ [which was constant in Eq. (48)] slowly changes with $m,N$. Then with the accuracy $\gamma$ one may replace $\tilde{\Psi}_{mN\mu}$ by $\chi_{m_0N_0}c_\mu$ in the figure brackets in Eq. (55), and rewrite Eq. (57) as

$$\gamma g_{m_0N_0}\chi_{m_0N_0}=\sum_{m,n}(\chi_{m+m_0,N+N_0}-\chi_{m_0N_0})S^{T}_{mN}\ . \tag{58b}$$

Here

$$S^{T}_{mN}=\sum_{m,N}c_\lambda c_\mu S_{mN\lambda\mu}=\overline{S^{T}_{-m,-N}}\ , \tag{59}$$

$$g_{m_0N_0}=\sum_{\lambda}|c_\lambda|^2\varepsilon^{*}_{m_0N_0\lambda}-\sum_{m,N,\lambda,\mu}c_\lambda c_\mu S^{*}_{mN\lambda\mu}=\overline{g_{m_0N_0}}\ .$$

"$T$" denotes total. Expanding in Eq. (58b) $\chi_{m_0+m,N_0+N}-\chi_{m_0N_0}$ in $m,N$, one obtains the equation for $\chi_{mN}$:

$$\langle m^2\rangle\frac{\partial^2\chi}{\partial m^2}+\langle N^2\rangle\frac{\partial^2\chi}{\partial N^2}+2\langle mN\rangle\frac{\partial^2\chi}{\partial m\,\partial N}-2i\left[\langle mi\rangle\frac{\partial\chi}{\partial m}+\langle Ni\rangle\frac{\partial\chi}{\partial N}\right]=2\gamma g\chi\ , \tag{60}$$

$$\langle f_{mN}\rangle\equiv\sum_{m,N}f_{mN}S^{T}_{mN}\ , \tag{61}$$

where, by Eq. (59), $\langle m^2\rangle$, $\langle N^2\rangle$, $\langle mN\rangle$, $\langle mi\rangle$, and $\langle Ni\rangle$ are real constants (independent of $m,N$). The linear transformation from $m,N$ to $\tilde{m},\tilde{N}$ allows one to eliminate $\partial^2\chi/\partial\tilde{m}\,\partial\tilde{N}$ in Eq. (60). Then the phase shift

$$\chi=\exp(i\kappa_1\tilde{m}+i\kappa_2\tilde{N})\tilde{\chi} \tag{62}$$

with properly chosen (real and constant) $\kappa_1,\kappa_2$ eliminates $\partial\tilde{\chi}/\partial\tilde{m},\partial\tilde{\chi}/\partial\tilde{N}$ and leads to the Schrödinger-type equation

$$\frac{\partial^2\tilde{\chi}}{\partial\tilde{m}^2}+\frac{\partial^2\tilde{\chi}}{\partial\tilde{N}^2}+(\kappa^2+\gamma g)\tilde{\chi}=0\ ,\quad \kappa^2=\kappa_1^2+\kappa_2^2\ , \tag{63}$$

$$g=g(\tilde{m},\tilde{N})\ . \tag{64}$$

Here $\kappa_1,\kappa_2$ and random $g$ depend on $Q_1,Q$. The $\tilde{\chi}$ phase in Eq. (63) must be chosen to provide slowly changing $\chi(m,N)$. The localization length $\xi\to\infty$ when $\gamma\to 0$. In a 1D case (when $\varepsilon_{mn}=\varepsilon_n$) it may be calculated analytically, similar to Ref. 23. When $\kappa\neq 0$ and $g$ is an analytical function, then

$$\ln\xi\approx\kappa\gamma^{-1/2}\ . \tag{63'}$$

When $g$ is not an analytical function and has a singularity, then

$$\ln\xi\propto|\ln\gamma|\ . \tag{63''}$$

The factors in Eqs. (63') and (63'') depend, via $\kappa$ and $g$, on $Q$ and $Q_1$. The crucial point in the above reasoning is the existence of $\mathbf{q}$ which satisfies Eq. (51). Consider the case when Eq. (40a) reduces to Eq. (44). Then $a+p=\gamma\to 0$, by Eq. (44), yields

$$\sum_{m,N=-\infty}^{\infty}(-1)^{mnQ_0}\cos(\mathbf{q}\cdot\mathbf{r}_{mn})l_p(\alpha|m^2+n^2|)=0\ . \tag{65}$$

If $Q_0$ is even, the Poisson summation formula, together with Refs. 19(k) and 19(l) and Eq. (22) for $l_p$, yields

$$\sum_{m,N=-\infty}^{\infty}l_p(|2\pi r_{ls}+\mathbf{q}|^2/\alpha)=0\ . \tag{66}$$

If $p\neq 0$, $\alpha\ll 1/p$, the leading term in Eq. (66) corresponds to $l=s=0$. Then $L_p(q/\alpha)=0$ yields[19(m)] $q\simeq(\pi/2)(n-\frac{1}{4})(\alpha/p)^{1/2}$, where $n$ is an integer. Since one may choose $|q_1|,|q_2|\le\pi$ in Eq. (65), the maximal $n$ is an integer part of $[2(2p/\alpha)^{1/2}+\frac{1}{4}]\gg p$.

When $\alpha\gg 1,p$, the only term in Eq. (65) which is not

exponentially small with $\alpha$ has $m=n=0$. It is $\neq 0$, and thus Eq. (65) has no solutions.[24] Since Eq. (65) has solutions when $\alpha \ll 1/p$ and does not have them when $\alpha \gg p$, there must exist the "critical line(s)" $\alpha_c = \alpha_c(p)$, with only localized states above it and the extended Landau states below it. So, weak fields $\alpha \ll 1$ always yield extended Landau states (this is consistent with the considered case of $\alpha \equiv 0$). Physical intuition and numerical experiments verify that (for a fixed $\alpha$) higher energies $p \gg 1/\alpha \gg 1$ enhance extended Landau states; $\alpha_c$ starts with a finite value, numerically $\simeq 1$.

The real solutions to Eq. (51) vanish at degeneracy points. So, $\alpha_c$ is provided by

$$D = \frac{\partial D}{\partial q_i} = 0 , \tag{67}$$

where $q_i$ is $q_x$ or $q_y$. Since this equation depends on $Q_1, Q$ and since, by definition, $p$ is an integer, $\alpha_c(p)$ will probably form a fractal line for a fixed $p$. Its calculation is planned to be the subject of a separate study.

## VI. THE EXAMPLE

A general analysis may be demonstrated by the example of $\alpha = 2\pi$ (i.e., $Q_1 = 2$, $Q = 1$), $p = 1$, and $q_1 = q_2 = 0$. Then, by Eq. (53), $\lambda = \mu = 0$, $N = n$, and $c_\mu = c_0$ may be chosen to be $c_0 = 1$. By Eq. (50),

$$S_{mn00} = l_1(2\pi r_{mn}^2) = (1 - 2\pi r_{mn}^2)\exp(-\pi r_{mn}^2) \tag{68}$$

and thus $q_1 = q_2 = 0$ satisfies Eq. (49), if

$$\sum_{m,n} l_1(\alpha r_{mn}^2) = 0 . \tag{69}$$

By Eq. (48), $\Psi_{mn}^L = 1$; by Eq. (39),

$$\Psi_{mn} = 1 . \tag{69a}$$

So, by Eqs. (36) and (37a),

$$\Psi^*(\mathbf{r}) = \Psi^1(\mathbf{r}) = \exp(2\pi i x y)\tilde{\Psi}^1(\mathbf{r}) ,$$

$$\tilde{\Psi}^1(\mathbf{r}) = \sum_{m,n=-\infty}^{\infty} \exp[2\pi i(m-x)(n+y)] l_1(2\pi|\mathbf{r}-\mathbf{r}_{mn}|^2)$$

$$= \tilde{\Psi}^1(x+1, y) . \tag{70}$$

Equation (69), by Eq. (70), may be presented as

$$\tilde{\Psi}^1(0,0) = 0 . \tag{71}$$

Finally, by Eqs. (61) and (68), $\langle m \rangle = \langle N \rangle = \langle mN \rangle = 0$, $\langle m^2 \rangle = \langle n^2 \rangle$, and thus Eq. (60) reads

$$\langle r_{mn}^2 \rangle \Delta\chi = 4\gamma g \chi . \tag{72}$$

Now calculate the Fourier components $\tilde{\Psi}_\nu^1(y)$ of $\tilde{\Psi}(\mathbf{r})$ in $x$. By Eq. (70), after a straightforward calculation,

$$\begin{aligned}\tilde{\Psi}_\nu^1(y) &= \sum_{m,n=-\infty}^{\infty} \int_0^1 \exp\{2\pi i[\nu x + (m-x)(n-y)]\} l_1\{2\pi[(x-m)^2 + (y-n)^2]\} dx \\ &= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \cos[2\pi x(y+n-\nu)] l_1\{2\pi[x^2 + (y-n)^2]\} dx \\ &= (2y-\nu)\exp[-2\pi y(y-\nu) - \pi\nu^2] \sum_{n=-\infty}^{\infty} (2n-\nu)\exp[-2\pi n(n-\nu)] \equiv 0 .\end{aligned} \tag{73}$$

Since $\tilde{\Psi}_\nu^1(y) = 0$ for all $\nu$, so $\tilde{\Psi}_\nu^1(x,y) = 0$, and, in particular, Eq. (71) is satisfied. The situation, when by Eq. (69a) $\Psi_{mn} = 1$ while $\Psi^*(\mathbf{r}) \equiv 0$, is similar to the case of Eqs. (10)–(12) when $\rho \to 0$; in both cases the probability density is infinitely (logarithmically in $\rho$) larger at the lattice sites than elsewhere.

This seems to be the case for all $p, \mathbf{q}$ and $\alpha/\pi < n+1$, studied in numerical calculation. In a general case, then[25]

$$\Psi_p(\mathbf{q}, \mathbf{R}) = 0 \quad \text{when } D(\mathbf{q}) = 0 \tag{74}$$

leads to an unusual identity for the Laguerre functions, which calls for a special study. In this case one should determine, by Eq. (37b), $\Psi(\mathbf{r})$ rather than $\Psi^*(\mathbf{r})$ for $|\mathbf{r}-\mathbf{r}_s| \gg \rho$.

## VII. OUTSTANDING PROBLEMS

(1) For simplicity, I studied only random-strength IDF's on a square lattice. However, a random IDF situation may be seen as a disorder on a lattice whose period $\to 0$. Thus, I conjecture that all results are valid in a general case. However, an accurate study is in order.

(2) In the absence of magnetic field *all* states (for any $q_1, q_2$) are extended above the mobility edge. This may suggest a *zero* residual resistance (at zero temperature) of a 2D system with random IDF disorder, and thus an example of a perfect disordered conductor. It also means that a residual resistivity decreases with the size $\rho$ ( *not* the Bohr radius $\rho_B$) of impurities, and also calls for a detailed study.

(3) In magnetic field, when the Fermi energy $E_F = E_p$, Eq. (51) yields $q_1 = q_1(q_2)$. In a finite system $L \times L$ this typically means the fraction of extended states, which is $\propto 1/L$. Since the effective mean free path for an extended state is $L$, then the effective residual 2D conductivity $\sigma_{\text{eff}}$ should be size independent.

(4) One expects that the lower the temperature $T$ is, the finer the details in the discontinuous $\xi(B)$ are manifested. What is the exact dependence of $\xi(B)$ and $\sigma_{\text{eff}}(B,T)$? The

first question may be answered, e.g., by the numerical study of the derived equations; the second demands a finer analysis (see Ref. 13 for the numerical approach to the Mott variable-range hopping in a random IDF system).

(5) To determine the applicability of the scaling theory,[5] one must calculate the mean free path for weak localization for the IDF disorder.

## VIII. SUMMARY

(i) Start with the listing of derived equations. The main equation, which determines the eigenenergy $E$ and eigenvalues of the wave function $\Psi_s$ at the IDF sites (for their arbitrary locations $\mathbf{R}_s$ and strengths $\rho_s$), is Eq. (30), where $a$, $b$, and $G$ are provided by Eqs. (24) and (25), and $\varepsilon_\sigma$ by Eq. (31). The wave function $\Psi(\mathbf{R})$ everywhere, except for the IDF locations, is presented by Eqs. (35a)–(37), where $l_\nu$ is the Laguerre function. When IDF's are situated on a square lattice with period $d$, then Eq. (30) reduces to universal Eqs. (40a) and (40b), with diagonal disorder only. If the energy $E \to E_p = (p+\frac{1}{2})\hbar eB/Mc$ is the Landau level and $\alpha = \pi Q/Q_1$, $Q, Q_1$, and $p$ are integers, then Eq. (30) reduces, by Eq. (48) (extended Landau states), to Eqs. (49) and (50), which are independent of the IDF strength, while $\Psi$ reduces to $\Psi^p(\mathbf{r})$ from Eqs. (36) and (37) or to $\Psi$ from Eq. (37b). If $E \simeq E_p$, i.e., $a+p=\gamma$, $|\gamma| \ll 1$, then Eqs. (53), (58a), (60), and (61) describe localization, with the localization length $\xi$ provided by Eqs. (63′) and (63″) with the factor depending on $Q, Q_1$. If the total IDF potential is periodic along $x$, then the Schrödinger equation reduces to 1D Eqs. (41a) and (41b). When the total IDF potential is periodic in both directions and the number of flux quanta per unit cell is $\alpha/\pi = Q_1/Q$, where $Q_1, Q$ are integers, then, by Eq. (42a), the problem reduces to $Q$ linear algebraic equation (42b). When $\alpha = \pi Q_0 + \alpha_1$, $Q_0$ is an integer, $|\alpha_1| \ll 1$, the dispersion in magnetic field is described by Eq. (44).

(ii) The main features of the presented equations, which are absent in all other models, are the following. The main equation is general, accurate, and universal. On a lattice it has only diagonal disorder. It depends explicitly not only on the (conventionally accounted for) Lorentz phase shift in magnetic field, but also on the exponential shrinkage of the wave function with magnetic field. In the vicinity of the Landau levels (of a free electron in magnetic field) it reduces to $Q$ linear algebraic equations (where $Q$ is the integer denominator in the number of flux quanta $\alpha/\pi = Q_1/Q$ per unit cell), whose coefficients depend on $Q_1, Q$.

In any magnetic field they reduce to a 1D situation if the total IDF potential is periodic along one of the lattice axis, and to $Q$ algebraic equations if it is periodic along both axis (the Bloch case). A comprehensive picture of 2D localization in a random IDF set may be obtained from computer calculations of the presented equations. Even in the Bloch case it would be interesting to find out how the wave-function shrinkage affects the Hofstadter batterfly spectrum.[7]

(iii) Analytical calculations prove that in any 2D system all Landau levels $E = (p+\frac{1}{2})\hbar eB/Mc$ are extended when $p \geq p_c(B)$. When $B \geq B_c$, then $p_c(B)$ depends on $Q, Q_1$, which are related to the number of flux quanta $eBd^2/ch = Q/Q_1$ per IDF. When $B \leq B_c$, then $p_c(B) = 0$. This implies the mobility edge $E = 0$ in a 2D system in the absence of magnetic field. In the vicinity of the Landau levels I determine the behavior of the localization length $\xi(B)$.

## ACKNOWLEDGMENTS

I am grateful to Dr. Mints for discussions and computer simulations, Dr. M. Paris for numerical calculations, Dr. P. M. Platzman for very useful discussions, Dr. P. Erdös and Dr. M. Balkanskii, and to the R. and J. Meyerhoff Chair for Solid State Theory and Thermodynamics for financial support.

[1] F. Bloch, Z. Phys. **52**, 555 (1928).

[2] See, e.g., I. M. Lifshitz, M. Ya. Azbel', and M. I. Kaganov, *Electron Theory of Metals* (Consultant Bureau, New York, 1973).

[3] P. W. Anderson, Phys. Rev. **109**, 1492 (1958).

[4] P. Edrös and R. C. Herndon, Adv. Phys. **31**, 65 (1982).

[5] E. Abrahams, P. W. Anderson, D. C. Licciardello, and T. V. Ramakrishnan, Phys. Rev. Lett. **42**, 673 (1979); P. W. Anderson, D. J. Thouless, E. Abrahams, and D. S. Fisher, Phys. Rev. B **22**, 3519 (1980).

[6] P. G. Harper, Proc. Phys. Soc. A **68**, 879 (1955). See also M. Ya. Azbel', Phys. Rev. B **39**, 6241 (1989), and references therein.

[7] M. Ya. Azbel', Zh. Eksp. Teor. Fiz. **46**, 435 (1964) [Sov. Phys. JETP **19**, 292 (1964)]; D. Hofstadter, Phys. Rev. B **14**, 2239 (1976).

[8] D. J. Thouless, M. P. Nightingale, and M. den Nijs, Phys. Rev. Lett. **49**, 405 (1982); D. J. Thouless, J. Phys. C **17**, L325 (1984); Phys. Rep. **110**, 279 (1984); *Number Theory and Physics,* edited by M. Waldschmidt and P. Moussa (Springer-Verlag, Berlin, 1990), p. 170, and references therein.

[9] See, e.g., B. I. Shklovskii and A. L. Efros, *Electronic Properties of Doped Semiconductors* (Springer, New York, 1984), Vol. 45, p. 210; B. I. Shklovskii and B. Z. Spivak, in *Hopping Transport in Solids,* edited by B. Shklovskii and M. Pollak (Elsevier, New York, 1990).

[10] R. F. Kazarinov and S. Luryi, Phys. Rev. B **25**, 7626 (1982); **27**, 1386 (1983); M. Ya. Azbel' and O. Entin-Wohlman, *ibid.* **32**, 562 (1985), and references therein.

[11] A. M. M. Pruisken, Phys. Rev. Lett. **61**, 1297 (1988), and references therein.

[12] H. P. Wei, D. C. Tsui, M. A. Paalanen, and A. M. M. Pruisken, Phys. Rev. Lett. **61**, 1294 (1988), and references therein.

[13] M. Ya. Azbel', Phys. Rev. B **43**, 2435 (1991); **43**, 6717 (1991); see also Phys. Rev. Lett. **47**, 1015 (1981); **67**, 1787 (1991).

[14] M. Ya. Azbel', Solid State Commun. **47**, 699 (1983), Phys. Rev. B **26**, 4735 (1982).

[15] F. P. Milliken and Z. Ovadyahu, Phys. Rev. Lett. **65**, 911

(1990), and references therein.

[16]In fact, this is the only important point in the definition of $D$: when $\rho \to 0$, then $\rho^2 D(R) \to 0$ for any $R$, and $D(R) \to 0$ when $R/\rho \to \infty$.

[17]L. D. Landau and E. M. Lifshitz, *Quantum Mechanics* (Pergamon, Oxford, 1965): (a) Chap. 6, Sec. 45 and problems; (b) Chap. 15, Sec. 112, and problem 1.

[18]P. M. Morse and M. Feshbach, *Methods in Theoretical Physics* (McGraw-Hill, New York, 1953), Pt. I: (a) Eq. (7.2.15); (b) Eq. (5.3.75).

[19]I. S. Gradsteyn and I. M. Ryzhik, *Tables of Integrals, Series and Products* (Academic, New York, 1980): (a) Eq. (8.447.3); (b) Eqs. (8.441.1, 8.444.1, and 4.331.1); (c) Eq. (8.411.1); (d) Eq. (8.970.2); (e) Eq. (8.310.1); (f) Eqs. (9.220.2, 9.220.4, and 9.210.2); (g) Eq. (9.237.1); (h) Eq. (9.227); (i) Eqs. (8.365.8, 8.334.3, and 8.343.2); (j) Eqs. (6.633.2 and 7.421.1); (k) Eq. (8.451.5); (l) Eq. (8.978).

[20]E. Erdelyi, *Higher Transcendental Functions* (McGraw-Hill, New York, 1953), Vol. I, Eq. (6.12.3).

[21]By Eqs. (47) and (35c)–(37), when $a \to -p$, then $\Psi^*(\mathbf{r}_{mn})=0$, and thus $D(\mathbf{r}-\mathbf{r}_{mn})\Psi^*(\mathbf{r}_{mn})=0$. This is the crux of the difference between $E=E_p$ and $E \to E_p$: in the former case $\Psi(\mathbf{r})=\Psi^p(\mathbf{r})$, and Eq. (38) corresponds to $\Psi(\mathbf{r}_{mn})=0$; in the latter case Eq. (12) implies $\Psi^*(\mathbf{r}_{mn})=0$, while $\Psi_{mn}=\Psi(\mathbf{r}_{mn})$ may be $\neq 0$. On this point see also Sec. V.

[22]The exponent with $\alpha \to 0$ is introduced to provide explicit convergence. When $\alpha=0$, the spectrum of the homogeneous equation is continuous, and one should account for both homogeneous and inhomogeneous solutions to Eq. (28).

[23]M. Ya. Azbel' and P. Soven, Phys. Rev. Lett. **49**, 751 (1982); Phys. Rev. B **27**, 831 (1983); M. Ya. Azbel', *ibid.* **28**, 4106 (1983).

[24]In this case localized states in Eq. (30) correspond to the tight binding, with the leading approximation $\varepsilon_\sigma=0$, $\Psi_s=\delta_{s\sigma}$.

[25]S. A. Gredeskul, Y. Avishai, M. Ya. Azbel' (unpublished).