Exact solution of the master equation for ultrasmall normal tunnel junctions

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An exact solution for the stochastic master equation describing the semiclassical Coulomb-blockade theory is obtained for an arbitrary shunt resistance at zero temperature. This solution is used to systematically evaluate various quantities characterizing the dynamics of single-electron-tunneling oscillations. Our solution provides analytic expressions of the voltage spectra and sum rules.

I. INTRODUCTION

Small-capacitance normal tunnel junctions have attracted a great deal of attention since it was predicted (for reviews, see Refs. 1–3) and experimentally corroborated[4–8] that the charging effect of a single electron leads to a family of phenomena at low temperatures, where the charging energy $e^2/2C$ exceeds the background thermal energy k_BT. Single-electron-tunneling (SET) oscillations are particularly fascinating. They are the voltage oscillations with frequency I_{dc}/e that are generated in a small normal tunnel junction when a charge transfer across the tunnel barrier in *discrete* units of the electronic charge e is supplemented by the *continuous* injection of current I_{dc} from an external circuit.

Within the semiclassical approximation, the dynamics of small-capacitance normal tunnel junctions can be described by a stochastic master equation. Most articles have reported numerical simulations rather than analytical solutions of the master equation. The numerical approach successfully yields the *static* properties of SET oscillations, such as current-voltage (I-V) characteristics.[9–11] However, it is not suitable for investigating their *dynamic* properties, such as voltage spectra, because SET oscillations have an extremely long-time correlation that severely restricts the accuracy of computer simulation. An alternative time-domain approach to small tunnel junctions[12–14] gives complete information about two consecutive tunneling events and hence the analytic expression of I-V characteristics for arbitrary bias conditions, but this approach has not been extended to study voltage spectra.

The purpose of this paper is to develop an analytic approach to small tunnel junctions by *exactly* solving the stochastic master equation. This approach not only affirms previously obtained results but also gives an analytic expression of the voltage spectra of SET oscillations under arbitrary bias conditions. This analytic expression allows a deeper understanding of SET oscillations, and clears up ambiguities about the accuracy of computer simulation. Special emphasis is put on sum rules and the linewidth problem of SET oscillations.

This paper is organized as follows. Section II describes a semiclassical model for small-capacitance normal tunnel junctions and the corresponding stochastic master equation. Section III obtains the exact solution of the master equation under arbitrary bias conditions, and discusses two important cases: constant-current operation and the stationary case. In particular, it is proved that the linewidth of SET oscillations vanishes for constant-current operation. Section IV uses the exact solution to calculate the charge (or voltage) correlation function and the power spectra for both current-driven and shunted operations. The results are used to discuss the sum rules and linewidth of SET oscillations. Up to Sec. IV, the discussion is only treated in the frequency domain. However, another analytic method reported in Refs. 12–14 describes the physics in the time domain. Therefore, Sec. V connects the two descriptions by showing how the key distributions in the time-domain description can be constructed from the solution of the master equation. Section VI shows how our method can be used to calculate the current-voltage characteristics. Section VII uses the solution of the master equation for a detailed discussion of the linewidth problem. Section VIII summarizes the main results. Some complicated algebraic manipulations are relegated to appendices to avoid digressing from the main subject.

II. FORMULATION OF THE PROBLEM

We briefly describe here our semiclassical model and assumptions. We consider a simple circuit [see Fig. 1(a)] consisting of a normal tunnel junction, with electrostatic capacitance C and tunnel resistance R_T, and a resistance $R_{\rm series}$ in series with a voltage source V. All other circuit elements, such as stray capacitance and inductance, are ignored for simplicity. This configuration is equivalent to a conventional one,[9] in which a normal tunnel junction and a shunt resistor $R_{\rm shunt}$ are connected parallel to a current source [Fig. 1(b)]; the two circuits are connected by the relation $V = IR_{\rm shunt}$. In this paper we use the former circuit and denote the resistance $R_{\rm series}$

as source resistance R_S. Constant-current operation can be achieved by taking the limits $V \to \infty$ and $R_S \to \infty$ with V/R_S fixed at I_{dc}. We shall refer to cases of finite V and R_S as shunted.

Both source resistance R_S and tunnel resistance R_T are assumed to be much larger than the quantum unit of resistance $R_Q = h/e^2$, so that the effects of the electromagnetic environment[15–17] and the quantum fluctuations of electric charge[18,19] can be disregarded. The condition $R_T \gg R_Q$ means that an electron is almost always localized on either side of the tunnel barrier. The barrier traversal time[20] and the thermal equilibration time inside the electrodes are also assumed to be negligible compared to circuit time constants and the average time between adjacent tunneling events. These assumptions allow us to make the Markov approximation and describe the dynamics of small tunnel junctions by the stochastic master equation,[9,11]

$$\frac{\partial}{\partial t}P(Q,t) = \frac{1}{CR_S}\frac{\partial}{\partial Q}\left(Ck_BT\frac{\partial P(Q,t)}{\partial Q} - (CV-Q)P(Q,t)\right) + r(Q+e)P(Q+e,t) + l(Q-e)P(Q-e,t) - [r(Q)+l(Q)]P(Q,t), \quad (2.1)$$

where $P(Q,t)$ is the probability density of charge Q at time t, and the forward and backward tunneling rates, $r(Q)$ and $l(Q)$, are given by

$$r(Q) = \frac{1}{eR_TC}\frac{Q-\frac{e}{2}}{1-\exp\left[-\frac{e}{Ck_BT}\left(Q-\frac{e}{2}\right)\right]}, \qquad l(Q) = r(-Q). \quad (2.2)$$

The point here is that the right-hand side of Eq. (2.1) consists of two parts, corresponding to the *continuous* charge-up process and the *discrete* tunneling process in units of e, both of which are essential for SET oscillations. Such coexistence of continuous and discrete processes in a single equation makes Eq. (2.1) unique but intractable.

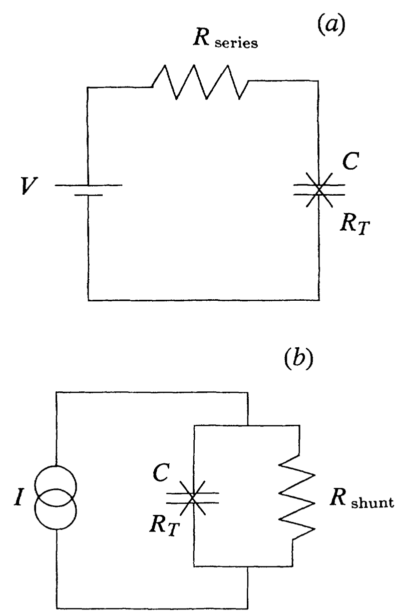

FIG. 1. (a) Schematic illustrations of the circuit treated in this paper and (b) the equivalent circuit.

Henceforth, we restrict our discussion to zero temperature, where the master equation (2.1) reduces to

$$\frac{\partial}{\partial t}P(Q,t) = -\frac{CV-Q}{CR_S}\frac{\partial}{\partial Q}P(Q,t) + \frac{1}{CR_S}P(Q,t) + r(Q+e)P(Q+e,t) - r(Q)P(Q,t), \quad (2.3)$$

with the forward tunneling rate,

$$r(Q) = \begin{cases} 0, & Q \le e/2 \\ \dfrac{Q-e/2}{eR_TC}, & Q \ge e/2. \end{cases} \quad (2.4)$$

Because tunneling is forbidden for $|Q| < e/2$, it is sufficient to consider only $CV > e/2$.

III. EXACT SOLUTION OF THE MASTER EQUATION

A. General solution

This section formulates and solves an initial-value problem of the stochastic master equation (2.3). We seek a solution of Eq. (2.3) with the following initial conditions:

$$P(Q,t=0) = \delta(Q-Q_0) \text{ and } P(Q,t<0) = 0. \quad (3.1)$$

For clarity, we denote the solution of this initial-value problem as $P(Q_0,Q;t)$, which is the retarded Green's function of the master equation.

The procedure for obtaining $P(Q_0,Q;t)$ is outlined as follows. We first solve the master equation (2.3) without the tunneling term $r(Q+e)P(Q+e,t)$. Using this solution, we transform the full master equation into an integral equation. Finally, the integral equation is solved for $CV < 3e/2$, which is sufficient for investigating SET oscillations.

Let us first solve the master equation without the tunneling term

$$\frac{\partial}{\partial t}P_0(Q_0,Q;t) = -\frac{CV-Q}{CR_S}\frac{\partial}{\partial Q}P_0(Q_0,Q;t) + \frac{1}{CR_S}P_0(Q_0,Q;t) - r(Q)P_0(Q_0,Q;t), \quad (3.2)$$

with the initial conditions

$$P_0(Q_0,Q;t=0) = \delta(Q-Q_0) \text{ and } P_0(Q_0,Q;t<0) = 0. \quad (3.3)$$

Since the probability that no tunneling event occurs while the charge increases from Q_1 to Q_2 is given by[13]

$$P_0(Q_1,Q_2) = \begin{cases} 1, & Q_1 < Q_2 < e/2 \\ \exp\left(-\int_A^{Q_2} dq\frac{r(q)}{i(q)}\right), & Q_2 > e/2, \end{cases} \quad (3.4)$$

where $A \equiv \max\{Q_1, e/2\}$ and

$$i(q) = \frac{CV-q}{CR_S}, \quad (3.5)$$

we obtain the solution of Eq. (3.2) as

$$P_0(Q_0,Q;t) = \Theta(t)P_0(Q_0,Q)\delta(Q-CV-(Q_0-CV)\exp(-t/CR_S)), \quad (3.6)$$

where $\Theta(t)$ is the unit-step function. The δ function in Eq. (3.6) ensures that the time for the junction to charge from Q_0 to Q is equal to t.

Using the fact that $P_0(Q_0,Q;t)$ is the retarded Green's function of Eq. (3.2), we can transform Eq. (2.3) into an integral equation,

$$P(Q_1,Q_2;t) = P_0(Q_1,Q_2;t) + \int_0^t dt' \int_{e/2}^{CV} dQ P(Q_1,Q;t')r(Q)P_0(Q-e,Q_2;t-t'). \quad (3.7)$$

This integral equation is equivalent to the initial-value problem of the stochastic master equation (2.3) with the initial conditions (3.1). We solve this equation in the frequency domain. The Fourier transform of Eq. (3.7) yields

$$P(Q_1,Q_2;\omega) = P_0(Q_1,Q_2;\omega) + \int_{e/2}^{CV} dQ\, P(Q_1,Q;\omega)r(Q) \times P_0(Q-e,Q_2;\omega), \quad (3.8)$$

where $P(Q_1,Q_2;\omega)$ and $P_0(Q_1,Q_2;\omega)$ are defined by

$$P(Q_1,Q_2;\omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega t}P(Q_1,Q_2;t), \quad (3.9)$$

$$P_0(Q_1,Q_2;\omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega t}P_0(Q_1,Q_2;t). \quad (3.10)$$

The Fourier transform of Eq. (3.6), on the other hand, is given by

$$P_0(Q_1,Q_2;\omega) = CR_S\Theta(Q_2-Q_1)P_{01}(Q_1;\omega)P_{02}(Q_2;\omega), \quad (3.11)$$

where

$$P_{01}(Q_1;\omega) = \begin{cases} (CV-Q_1)^{i\omega CR_S}, & Q_1 < e/2, \\ (CV-Q_1)^{i\omega CR_S}\left(\frac{CV-e/2}{CV-Q_1}\right)^{(1/e)(R_S/R_T)(CV-e/2)} \exp\left[\frac{1}{e}\frac{R_S}{R_T}\left(\frac{e}{2}-Q_1\right)\right], & Q_1 \geq e/2, \end{cases} \quad (3.12)$$

and

$$P_{02}(Q_2;\omega) = \begin{cases} (CV-Q_2)^{-1-i\omega CR_S}, & Q_2 < e/2 \\ (CV-Q_2)^{-1-i\omega CR_S}\left(\frac{CV-Q_2}{CV-e/2}\right)^{(1/e)(R_S/R_T)(CV-e/2)} \exp\left[\frac{1}{e}\frac{R_S}{R_T}\left(Q_2-\frac{e}{2}\right)\right], & Q_2 \geq e/2. \end{cases} \quad (3.13)$$

Using Eq. (3.11), Eq. (3.8) can be expanded by iteration to yield

$$\begin{aligned}P(Q_1,Q_2;\omega) &= P_0(Q_1,Q_2;\omega) + \int_{e/2}^{CV} dq_1 P_0(Q_1,q_1;\omega)r(q_1)P_0(q_1-e,Q_2;\omega)\\
&\quad + \int_{e/2}^{CV} dq_1 \int_{e/2}^{CV} dq_2 P_0(Q_1,q_1;\omega)r(q_1)P_0(q_1-e,q_2;\omega)r(q_2)P_0(q_2-e,Q_2;\omega)+\cdots\\
&= CR_S P_{01}(Q_1;\omega)P_{02}(Q_2;\omega)\\
&\quad\times\Bigg(\Theta(Q_2-Q_1)+CR_S\int_{e/2}^{CV} dq_1\Theta(q_1-Q_1)\Theta(Q_2-q_1+e)P_{02}(q_1;\omega)r(q_1)P_{01}(q_1-e;\omega)\\
&\qquad +(CR_S)^2\int_{e/2}^{CV} dq_1\int_{e/2}^{CV} dq_2\Theta(q_1-Q_1)\Theta(q_2-q_1+e)\Theta(Q_2-q_2+e)\\
&\qquad\qquad\times P_{02}(q_1;\omega)r(q_1)P_{01}(q_1-e;\omega)\\
&\qquad\qquad\times P_{02}(q_2;\omega)r(q_2)P_{01}(q_2-e;\omega)+\cdots\Bigg).\end{aligned} \tag{3.14}$$

This series expansion can be summed exactly when voltage V is less than $3e/2C$, because in this case, the step function in the integrand, $\Theta(q_{i+1}-q_i+e)$, always gives unity, so the multiple integrals can be reduced to products of single integrals:

$$\begin{aligned}P(Q_1,Q_2;\omega) &= CR_S P_{01}(Q_1;\omega)P_{02}(Q_2;\omega)\\
&\quad\times\Bigg\{\Theta(Q_2-Q_1)+\Theta(Q_2-Q_1+e)CR_S\int_A^B dq_1 P_{02}(q_1;\omega)r(q_1)P_{01}(q_1-e;\omega)\\
&\qquad +(CR_S)^2\int_A^{CV} dq_1 P_{02}(q_1;\omega)r(q_1)P_{01}(q_1-e;\omega)\\
&\qquad\qquad\times\int_{e/2}^{B} dq_2 P_{02}(q_2;\omega)r(q_2)P_{01}(q_2-e;\omega)\\
&\qquad\qquad\qquad\times\Bigg[1+CR_S\int_{e/2}^{CV} dq P_{02}(q;\omega)r(q)P_{01}(q-e;\omega)\\
&\qquad\qquad\qquad\quad+\left(CR_S\int_{e/2}^{CV} dq P_{02}(q;\omega)r(q)P_{01}(q-e;\omega)\right)^2+\cdots\Bigg]\Bigg\},\\
&= CR_S P_{01}(Q_1;\omega)P_{02}(Q_2;\omega)\\
&\quad\times\Bigg[\Theta(Q_2-Q_1)+CR_S\Theta(Q_2-Q_1+e)\int_A^B dq P_{02}(q;\omega)r(q)P_{01}(q-e;\omega)\\
&\qquad +(CR_S)^2\int_A^{CV} dq_1 P_{02}(q_1,\omega)r(q_1)P_{01}(q_1-e;\omega)\\
&\qquad\qquad\times\int_{e/2}^{B} dq_2 P_{02}(q_2;\omega)r(q_2)P_{01}(q_2-e;\omega)\\
&\qquad\qquad\times\left(1-\int_{e/2}^{CV} dq P_{02}(q;\omega)r(q)P_{01}(q-e;\omega)\right)^{-1}\Bigg],\end{aligned} \tag{3.15}$$

where $A\equiv\max\{Q_1,e/2\}$ and $B\equiv\min\{Q_2+e,CV\}$. Introducing the quantity

$$I(q_1,q_2;\omega)=CR_S\int_{q_1}^{q_2} dq P_{02}(q;\omega)r(q)P_{01}(q-e;\omega), \tag{3.16}$$

we finally obtain

$$P(Q_1,Q_2;\omega)=CR_S P_{01}(Q_1;\omega)P_{02}(Q_2;\omega)\left(\Theta(Q_2-Q_1)+\Theta(Q_2-Q_1+e)I(A,B;\omega)+\frac{I(A,CV;\omega)I(e/2,B;\omega)}{1-I(e/2,CV;\omega)}\right). \tag{3.17}$$

This is an exact solution of the stochastic master equation (2.3) for $CV<3e/2$ with initial conditions (3.1). That is, it is the retarded Green's function of the master equation. The quantity $I(q_1,q_2;\omega)$ can be evaluated from Eq. (3.16) using Eqs. (2.4), (3.12), and (3.13).

Thus we have demonstrated that the master equation (2.3) can be solved exactly for $CV<3e/2$. This solution is nonetheless applicable to $CV>3e/2$ as long as the probability that the charge on the junction capacitor exceeds $3e/2$ is negligible. Mathematically, this condition can be expressed as

$$\exp\left(-\int_{e/2}^{3e/2} dq \frac{r(q)}{i(q)}\right) = \left(\frac{CV-3e/2}{CV-e/2}\right)^{(1/e)(R_S/R_T)(CV-e/2)} \times \exp\left(\frac{R_S}{R_T}\right) \ll 1. \quad (3.18)$$

For constant-current operation, Eq. (3.18) reduces to

$$\exp\left(-\frac{e}{2R_TCI_{\rm dc}}\right) \ll 1, \quad (3.19)$$

which is satisfied when the driving current $I_{\rm dc}$ is much smaller than e/R_TC. For example, the left-hand side of Eq. (3.19) gives 6.7×10^{-3} for $I_{\rm dc} = 0.1e/R_TC$. Because SET oscillations can only be observed at low currents, this condition is usually satisfied in the cases that we are interested in. The solution (3.17) is therefore useful for investigating the dynamics of SET oscillations.

Some comments on the key quantity $I(q_1,q_2;\omega)$ are in order here. Substituting Eqs. (3.12) and (3.13) into Eq. (3.16) yields

$$I(q_1,q_2;\omega) = \int_{q_1}^{q_2} dq \exp[i\omega\tau(q-e,q)]P\left(\frac{e}{2},q\right), \quad (3.20)$$

where the quantity

$$\tau(Q_1,Q_2) \equiv CR_S \ln\frac{CV-Q_1}{CV-Q_2} \quad (3.21)$$

gives the time needed for charging from Q_1 to Q_2 without tunneling events, and

$$P\left(\frac{e}{2},q\right) = \frac{r(q)}{i(q)}\exp\left(-\int_{e/2}^{q} dq' \frac{r(q')}{i(q')}\right) \quad (3.22)$$

is the probability density that the first tunneling event occurs at charge q, given that the initial charge was less than $e/2$.[13]

Noting that the quantity

$$\overline{\tau} \equiv \int_{e/2}^{CV} dq\, \tau(q-e,q)P\left(\frac{e}{2},q\right) \quad (3.23)$$

gives the average time required to charge the junction by e, which is also equal to the average interval between consecutive tunneling events, we find that $I(e/2,CV;\omega)$ is the moment-generating function for that time:

$$\overline{\tau^n} = \frac{\partial^n}{\partial(i\omega)^n} I(e/2,CV;\omega)\Big|_{\omega=0}. \quad (3.24)$$

Now let us examine properties of the solution (3.17) in some detail by reducing it in two important cases: constant-current operation and the stationary case. The reduced expressions will be used in later discussions.

B. Solution for constant-current operation

For constant-current operation, the retarded Green's function (3.17) of the master equation (2.3) can be greatly simplified, because the quantity $\tau(q-e,q)$ in $I(q_1,q_2;\omega)$ becomes $e/I_{\rm dc}$ regardless of ω. In the limits $V\to\infty$ and $R_S\to\infty$ with $V/R_S = I_{\rm dc}$, Eq. (3.16) reduces to

$$I(q_1,q_2;\omega) = e^{i\omega e/I_{\rm dc}}\left[\exp\left(-\frac{(q_1-e/2)^2}{2eR_TCI_{\rm dc}}\right) - \exp\left(-\frac{(q_2-e/2)^2}{2eR_TCI_{\rm dc}}\right)\right]. \quad (3.25)$$

Substituting Eq. (3.25) into Eq. (3.17) yields, to the first order in $\exp(-e/2R_TCI_{\rm dc})$,

$$P(Q_1,Q_2;\omega) = \begin{cases} \frac{1}{I_{\rm dc}} e^{i\omega(Q_2-Q_1)/I_{\rm dc}}\left\{\Theta(Q_2-Q_1) + \frac{e^{i\omega e/I_{\rm dc}}}{1-e^{i\omega e/I_{\rm dc}}}\left[1-\exp\left(-\frac{(Q_2+e/2)^2}{2eR_TCI_{\rm dc}}\right)\right]\right\} & \text{for } -e/2 < Q_1,Q_2 < e/2 \\ \frac{1}{I_{\rm dc}} e^{i\omega(Q_2-Q_1)/I_{\rm dc}}\frac{1}{1-e^{i\omega e/I_{\rm dc}}}\exp\left(-\frac{(Q_2-e/2)^2}{2eR_TCI_{\rm dc}}\right) & \text{for } -e/2 < Q_1 < e/2 < Q_2 < 3e/2 \\ \frac{1}{I_{\rm dc}} e^{i\omega(Q_2-Q_1)/I_{\rm dc}}\left\{\Theta(Q_2+e-Q_1)e^{i\omega e/I_{\rm dc}}\left[1-\exp\left(-\frac{(Q_2+e/2)^2-(Q_1-e/2)^2}{2eR_TCI_{\rm dc}}\right)\right] + \frac{e^{2i\omega e/I_{\rm dc}}}{1-e^{i\omega e/I_{\rm dc}}}\left[1-\exp\left(-\frac{(Q_2+e/2)^2}{2eR_TCI_{\rm dc}}\right)\right]\right\} & \text{for } -e/2 < Q_2 < e/2 < Q_1 < 3e/2 \\ \frac{1}{I_{\rm dc}} e^{i\omega(Q_2-Q_1)/I_{\rm dc}}\left[\Theta(Q_2-Q_1)\exp\left(-\frac{(Q_2-e/2)^2-(Q_1-e/2)^2}{2eR_TCI_{\rm dc}}\right) + \frac{e^{i\omega e/I_{\rm dc}}}{1-e^{i\omega e/I_{\rm dc}}}\exp\left(-\frac{(Q_2-e/2)^2}{2eR_TCI_{\rm dc}}\right)\right] & \text{for } e/2 < Q_1,Q_2 < 3e/2. \end{cases} \quad (3.26)$$

From Eq. (3.26) we find that $P(Q_1, Q_2; \omega)$ has poles on the real ω axis at $\omega = 2\pi n I_{dc}/e$ ($n = 1, 2, \ldots$), which indicates that the SET peaks in the power spectra have no linewidth. Although Eq. (3.26) itself is correct only to the first order in $\exp(-e/2R_TCI_{dc})$, this conclusion holds true as long as the junction is driven by a constant-current source. A more rigorous proof is given in Appendix A.

C. Stationary solution

From the solution of the master equation, the stationary solution or the charge distribution can be calculated as

$$P(Q) = \overline{P(Q,t)} = \lim_{T\to\infty} \frac{1}{T}\int_0^T dt P(Q_1, Q; t) = -i \lim_{\omega\to 0} \omega P(Q_1, Q; \omega), \tag{3.27}$$

where the bar over $P(Q,t)$ denotes the time average. Substituting Eqs. (3.17) and (3.16) into Eq. (3.27), we obtain the charge distribution for CV less than $3e/2$ as

$$P(Q) = \begin{cases} \frac{1}{\overline{\tau}}\frac{CR_S}{CV-Q}\left\{1-\left(\frac{CV-Q-e}{CV-e/2}\right)^{(1/e)(R_S/R_T)(CV-e/2)}\exp\left[\frac{1}{e}\frac{R_S}{R_T}\left(Q+\frac{e}{2}\right)\right]\right\}, & -e/2 < Q < CV-e \\ \frac{1}{\overline{\tau}}\frac{CR_S}{CV-Q}, \quad CV-e < Q < e/2 \\ \frac{1}{\overline{\tau}}\frac{CR_S}{CV-Q}\left(\frac{CV-Q}{CV-e/2}\right)^{(1/e)(R_S/R_T)(CV-e/2)}\exp\left[\frac{1}{e}\frac{R_S}{R_T}\left(Q-\frac{e}{2}\right)\right], & e/2 < Q < CV, \end{cases} \tag{3.28}$$

where $\overline{\tau}$ is the average dwell time given by Eq. (3.23). This result is identical to that obtained in Ref. 13 using the probability-density-function approach. In particular, for constant-current operation, the charge distribution is obtained from Eq. (3.26) as

$$P(Q) = \begin{cases} \frac{1}{e}\left[1-\exp\left(-\frac{(Q+e/2)^2}{2eR_TCI_{dc}}\right)\right], \\ \qquad -e/2 < Q < e/2 \\ \frac{1}{e}\exp\left(-\frac{(Q-e/2)^2}{2eR_TCI_{dc}}\right), \quad Q > e/2, \end{cases} \tag{3.29}$$

which, of course, holds only for $I_{dc} \ll e/R_TC$.

The average and mean square of charge can be calculated from Eq. (3.29):

$$\overline{Q(t)} = \sqrt{\frac{\pi}{2}eR_TCI_{dc}}\ , \tag{3.30}$$

$$\overline{Q(t)^2} = \frac{e^2}{12} + 2eR_TCI_{dc}. \tag{3.31}$$

These results will be used later in discussing power spectra and sum rules.

IV. POWER SPECTRA OF SET OSCILLATIONS

This section evaluates various quantities characterizing SET oscillations, such as the charge (or voltage) correlation function and the power spectra. The results are used in discussing the sum rules and linewidth of SET oscillations.

A. General properties of the charge correlation function

We begin by discussing general features of the charge correlation function defined by

$$S_Q(\tau) = \Theta(\tau)[\overline{Q(t)Q(t+\tau)} - \overline{Q(t)}^2]. \tag{4.1}$$

Our primary concern is the analytic behavior of its Fourier transform,

$$S_Q(\omega) = \int_{-\infty}^{\infty} d\tau\, e^{i\omega\tau} S_Q(\tau). \tag{4.2}$$

This quantity can be expressed in terms of $P(Q)$ and $P(Q_1, Q_2; \omega)$ as

$$S_Q(\omega) = \tilde{S}_Q(\omega + i\delta), \tag{4.3}$$

where δ denotes an infinitesimal positive number and

$$\tilde{S}_Q(\omega) = \int_{-e/2}^{CV} dQ_1 \int_{-e/2}^{CV} dQ_2\, Q_1 Q_2 P(Q_1) P(Q_1, Q_2; \omega) - \frac{i}{\omega}\left(\int_{-e/2}^{CV} dQ Q P(Q)\right)^2. \tag{4.4}$$

Other general nth-order correlation functions can also be similarly expressed in terms of $P(Q)$ and $P(Q_1, Q_2; \omega)$. Because the correlation function defined by Eq. (4.1) is the retarded one, $S_Q(\omega)$ is analytic in the upper half of the complex ω plane. By definition, the following sum rule holds:

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} S_Q(\omega) = \tfrac{1}{2}[\overline{Q(t)^2} - \overline{Q(t)}^2]$$

$$= \frac{1}{2}\left[\int_{-e/2}^{CV} dQ\, Q^2 P(Q) - \left(\int_{-e/2}^{CV} dQ\, Q P(Q)\right)^2\right]. \quad (4.5)$$

This sum rule and the analyticity in the upper-half plane determine the asymptotic behavior of $S_Q(\omega)$:

$$S_Q(\omega) \sim \frac{i}{\omega}[\overline{Q(t)^2} - \overline{Q(t)}^2] \text{ for } |\omega| \to \infty\,; \text{ Im}\omega \geq 0. \quad (4.6)$$

We shall refer to the real and imaginary parts of $S_Q(\omega)$ as $S_Q^R(\omega)$ and $S_Q^I(\omega)$. It follows then from the analyticity and Eq. (4.6) that these pairs obey the Kramers-Kronig relations,

$$S_Q^R(\omega) = \frac{1}{\pi}\text{P}\int_{-\infty}^{\infty} d\omega' \frac{S_Q^I(\omega')}{\omega' - \omega}\,, \quad (4.7)$$

$$S_Q^I(\omega) = \frac{1}{\pi}\text{P}\int_{-\infty}^{\infty} d\omega' \frac{S_Q^R(\omega')}{\omega - \omega'}\,, \quad (4.8)$$

so the correlation function can be represented as

$$S_Q(\omega) = \frac{i}{\pi}\int_{-\infty}^{\infty} d\omega' \frac{S_Q^R(\omega')}{\omega - \omega' + i\delta}. \quad (4.9)$$

Equation (4.9) shows that $S_Q^R(\omega)$ can be regarded as the power spectrum of the charge fluctuations or SET oscillations.[21] It can be easily proved that $S_Q^R(\omega)$ has a finite value rather than a δ-function peak at $\omega = 0$.

B. Power spectra and sum rules

This section examines analytic properties of the power spectra and discusses the related sum rules for constant-current operation. $\tilde{S}_Q(\omega)$ can be calculated by substituting Eqs. (3.26) and (3.29) into (4.4). Lengthy but straightforward calculation yields

$$\begin{aligned}\tilde{S}_Q(\omega) = {}& \frac{i}{\omega}\left[\frac{e^2}{12} + 2eR_TCI_{\text{dc}}\left(1 - \frac{\pi}{4}\right)\right] + \frac{eI_{\text{dc}}}{2\omega^2}\frac{1 + e^{i\omega e/I_{\text{dc}}}}{1 - e^{i\omega e/I_{\text{dc}}}} - \frac{iI_{\text{dc}}^2}{\omega^3} \\ &+ \frac{e}{I_{\text{dc}}}\frac{e^{i\omega e/I_{\text{dc}}}}{1 - e^{i\omega e/I_{\text{dc}}}}\int_0^e dQ_1 \exp\left(i\frac{\omega Q_1}{I_{\text{dc}}} - \frac{Q_1^2}{2eR_TCI_{\text{dc}}}\right)\int_0^e dQ_2 \exp\left(-i\frac{\omega Q_2}{I_{\text{dc}}} - \frac{Q_2^2}{2eR_TCI_{\text{dc}}}\right) \\ &- \frac{2e}{\omega}\frac{e^{i\omega e/I_{\text{dc}}}}{1 - e^{i\omega e/I_{\text{dc}}}}\int_0^e dQ \sin\frac{\omega Q}{I_{\text{dc}}}\exp\left(-\frac{Q^2}{2eR_TCI_{\text{dc}}}\right).\end{aligned} \quad (4.10)$$

The real part of $S_Q(\omega)$ is given from Eqs. (4.3) and (4.10) as

$$S_Q^R(\omega) = S_Q^{\text{ped}}(\omega) + \sum_n{}' \frac{2\pi}{R_TC}\delta(\omega - 2\pi nI_{\text{dc}}/e)S_Q^{\text{SET}}(\omega), \quad (4.11)$$

where $\sum'$ denotes the summation over all integers except $n = 0$,

$$S_Q^{\text{ped}}(\omega) = \frac{e}{\omega}f(\omega) - \frac{e}{2I_{\text{dc}}}\{[f(\omega)]^2 + [g(\omega)]^2\}, \quad (4.12)$$

and

$$S_Q^{\text{SET}}(\omega) = \frac{R_TC}{2}\left[\left(\frac{I_{\text{dc}}}{\omega} - f(\omega)\right)^2 + [g(\omega)]^2\right]. \quad (4.13)$$

We have introduced here the functions $f(\omega)$ and $g(\omega)$, which are defined by

$$f(\omega) = \int_0^e dQ \sin\frac{\omega Q}{I_{\text{dc}}}\exp\left(-\frac{Q^2}{2eR_TCI_{\text{dc}}}\right), \quad (4.14)$$

$$g(\omega) = \int_0^e dQ \cos\frac{\omega Q}{I_{\text{dc}}}\exp\left(-\frac{Q^2}{2eR_TCI_{\text{dc}}}\right). \quad (4.15)$$

Equation (4.11) shows that $S_Q^R(\omega)$ consists of δ-function peaks $S_Q^{\text{SET}}(\omega)$ at $\omega = 2\pi nI_{\text{dc}}/e$ and what is called the noise pedestal $S_Q^{\text{ped}}(\omega)$.

The contribution of the pedestal to the power spectrum $S_Q^R(\omega)/\pi$ is given by[22]

$$\int_{-\infty}^{\infty}\frac{d\omega}{\pi}S_Q^{\text{ped}}(\omega) = e\overline{Q(t)}\left(1 - \frac{1}{\sqrt{2}}\right) = \left(1 - \frac{1}{\sqrt{2}}\right)\sqrt{\frac{\pi}{2}e^3R_TCI_{\text{dc}}}. \quad (4.16)$$

Equation (4.16) shows that the contribution of the pedestal is proportional to the average charge of the junction. This is a characteristic feature of a Poisson random-point process, so it confirms the statement that the pedestal is generated due to the statistical randomness of individual tunneling events.[9] The value of $S_Q^{\text{ped}}(\omega)$ at $\omega = 0$ is given by[22]

$$S_Q^{\text{ped}}(0) = e^2R_TC\left(1 - \frac{\pi}{4}\right), \quad (4.17)$$

while its asymptotic behavior for $\omega \gg \sqrt{I_{\text{dc}}/eR_TC}$ is given (see Appendix B) by

$$S_Q^{\rm ped}(\omega) = \frac{eI_{\rm dc}}{2\omega^2}\left[1-\left(\frac{I_{\rm dc}}{\omega^2 eR_TC}\right)^2 -6\left(\frac{I_{\rm dc}}{\omega^2 eR_TC}\right)^3 - \cdots\right]. \quad (4.18)$$

The contribution of the δ-function peaks is, on the other hand, given by

$$\int_{-\infty}^{\infty}\frac{d\omega}{\pi}\sum_n{}' \frac{2\pi}{R_TC}\delta(\omega - 2\pi n I_{\rm dc}/e)S_Q^{\rm SET}(\omega)$$
$$= \overline{Q(t)^2} - \overline{Q(t)}^2 - e\overline{Q(t)}\left(1-\frac{1}{\sqrt{2}}\right)$$
$$= \frac{e^2}{12} + 2eR_TCI_{\rm dc}\left(1-\frac{\pi}{4}\right) - \left(1-\frac{1}{\sqrt{2}}\right)\sqrt{\frac{\pi}{2}e^3R_TCI_{\rm dc}}. \quad (4.19)$$

The right-hand side of Eq. (4.19) depends on $\overline{Q(t)^2}$, which reflects the fact that SET oscillations are generated due to correlation of tunneling events.

$S_Q^{\rm ped}(\omega)$ and $S_Q^{\rm SET}(\omega)$ are shown in Fig. 2 for typical values of the current, $I_{\rm dc}R_TC/e = 0.01$ and 0.05. It can be seen that, as the driving current increases, the pedestal tends to overcome the contribution of the SET peaks. This means that SET oscillations are less obvious for larger currents. In addition, as ω increases, the SET peaks decrease rapidly as $S_Q^{\rm SET}(\omega) \sim 1/\omega^6$.

For shunted operation, the power spectra of SET oscillations can be calculated numerically using Eqs. (3.17) and (4.4). The results are shown in Fig. 3. The figures clearly show that SET peaks appear as the ratio R_S/R_T increases; as this ratio further increases, the background noise corresponding to the pedestal decreases and the SET peaks grow and become sharper.

C. Linewidth of SET oscillations for shunted operation

The linewidth of the SET peaks can be estimated by the relation

$$\Gamma_S = -{\rm Im}\,\omega_n, \quad (4.20)$$

where ω_n is a pole of $P(Q_1,Q_2;\omega)$, determined by the equation

$$I(e/2, CV; \omega_n) = 1. \quad (4.21)$$

To first order in R_T/R_S, we obtain

$$\omega_n = \frac{2\pi n}{CR_S\ln\dfrac{CV+e/2}{CV-e/2}} - i(4-\pi)\frac{R_T}{R_S}\frac{2\left(\dfrac{e}{2CV}\right)^3\left(1-\dfrac{e}{2CV}\right)}{\left[1-\left(\dfrac{e}{2CV}\right)^2\right]^2}\frac{(2\pi n)^2}{CR_S\left(\ln\dfrac{CV+e/2}{CV-e/2}\right)^3}, \quad (4.22)$$

where n is an integer. Thus the linewidth of SET oscillations due to a small source resistance R_S is given by

$$\Gamma_S = \pi^2 n^2(4-\pi)\frac{R_T}{R_S}\frac{1}{CR_S}. \quad (4.23)$$

A more detailed discussion on the poles of $P(Q_1,Q_2;\omega)$ for a shunted junction is given in Appendix C.

V. TIME-INTERVAL DISTRIBUTIONS

So far, the discussion is only treated in the frequency domain. However, Refs. 12–14 use time-domain description. In this section we will show how the key distributions in the time-domain description can be constructed from the solution of the master equation. We discuss the probability distributions of time intervals between tunneling events. In a small tunnel junction at most one electron tunnels at one time, and the tunneling process is Markovian. Such tunneling events can be most directly characterized by second-order correlation functions, and we introduce two such time-interval distributions $P_{s,11}(\tau)$ and $P_{c,11}(\tau)$. The former (latter) distribution gives the probability density that the next (another) tunneling event occurs at time τ after the first event.[13,23] Thus $P_{s,11}(\tau)$ describes the correlation between two successive tunneling events, while $P_{c,11}(\tau)$ describes the correlation between any two tunneling events separated by a time interval of τ.

Using $P_0(Q_1,Q_2;t)$ and $P(Q_1,Q_2;t)$, $P_{s,11}(\tau)$ and $P_{c,11}(\tau)$ can be written as

$$P_{s,11}(\tau) = \int dQ_1\int dQ_2 P_i(Q_1)P_0(Q_1,Q_2;\tau)r(Q_2), \quad (5.1)$$

$$P_{c,11}(\tau) = \int dQ_1\int dQ_2 P_i(Q_1)P(Q_1,Q_2;\tau)r(Q_2), \quad (5.2)$$

where $P_i(Q_1)$ is the charge distribution immediately after a tunneling event and is given, for $e/2 < CV < 3e/2$, by[13]

$$P_i(Q_1) = P\left(\frac{e}{2}, Q_1+e\right), \quad (5.3)$$

where $P(e/2,Q)$ is given by Eq. (3.22). In the limit of constant-current operation, the analytic forms of their Fourier transforms are given by

$$P_{s,11}(\omega) = \frac{e^{i\omega e/I_{dc}}}{(eR_TCI_{dc})^2} \int_0^e dQ_1\, Q_1 \exp\left(i\frac{\omega Q_1}{I_{dc}} - \frac{Q_1^2}{2eR_TCI_{dc}}\right) \int_0^e dQ_2\, Q_2 \exp\left(-i\frac{\omega Q_2}{I_{dc}} - \frac{Q_2^2}{2eR_TCI_{dc}}\right) \tag{5.4}$$

and

$$P_{c,11}(\omega) = \frac{1}{(eR_TCI_{dc})^2} \frac{e^{i\omega e/I_{dc}}}{1 - e^{i\omega e/I_{dc}}} \int_0^e dQ_1\, Q_1 \exp\left(i\frac{\omega Q_1}{I_{dc}} - \frac{Q_1^2}{2eR_TCI_{dc}}\right) \int_0^e dQ_2\, Q_2 \exp\left(-i\frac{\omega Q_2}{I_{dc}} - \frac{Q_2^2}{2eR_TCI_{dc}}\right). \tag{5.5}$$

We see from Eqs. (5.4) and (5.5) that the difference between $P_{c,11}(\omega)$ and $P_{s,11}(\omega)$ is only a factor $1/(1 - e^{i\omega e/I_{dc}})$; that is, $P_{c,11}(\omega)$ is nothing but $P_{s,11}(\omega)$ repeating with period e/I_{dc}. This can easily be understood by noting that the junction charge changes with time as depicted in Fig. 5(a) and that the probability of charging the tunnel juntion above $3e/2$ is assumed to be negligible. Thus every charging process starts from the Coulomb blocked region, and thus the memory of earlier tunneling events does not affect later tunneling events. This relation between $P_{s,11}(\omega)$ and $P_{c,11}(\omega)$ is specific to constant-current operation for low currents and does not hold for larger currents or for shunted operation.

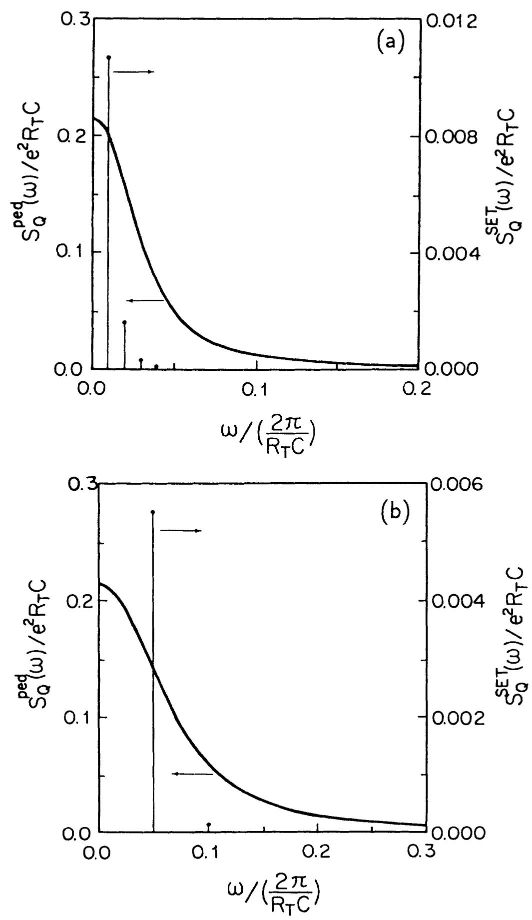

FIG. 2. Real part of the charge correlation function for constant-current operation with (a) $I_{dc}R_TC/e = 0.01$ and (b) $I_{dc}R_TC/e = 0.05$.

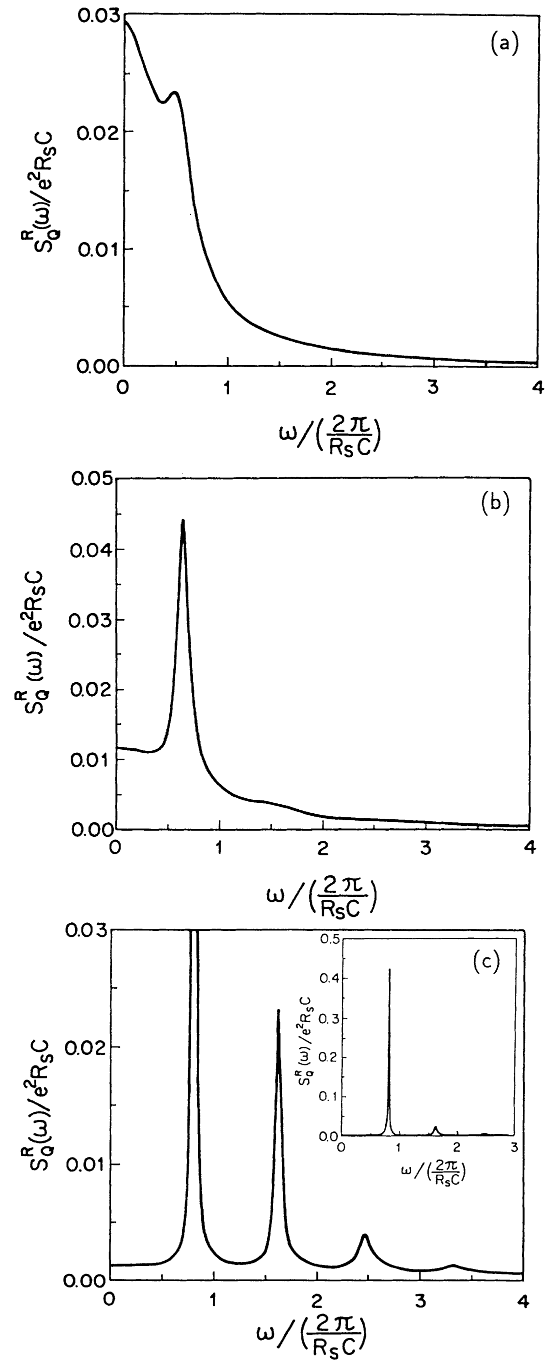

FIG. 3. Real part of the charge correlation function for shunted operation at $CV = e$: (a) $R_S/R_T = 3$, (b) $R_S/R_T = 10$, and (c) $R_S/R_T = 100$, where the inset shows the same figure in a different scale.

VI. CURRENT-VOLTAGE CHARACTERISTICS

The previous sections discussed dynamic properties of SET oscillations. In this section we consider their static properties, i.e., current-voltage characteristic. The time-averaged voltage across the tunnel junction is given by

$$\overline{V} = \frac{1}{C}\int_{-e/2}^{CV} dQ\, Q P(Q). \tag{6.1}$$

Substituting Eq. (3.28) into Eq. (6.1) we obtain Kirchhoff's second law,[13]

$$\overline{V} = V - R_S \overline{I}, \tag{6.2}$$

where the quantity

$$\overline{I} \equiv \frac{e}{\overline{\tau}} \tag{6.3}$$

is the average current through the tunnel junction. It is interesting to note that a classical circuit relation is recovered for a circuit involving a quantum device (i.e., tunnel junction) if we take an ensemble average. This is a tunneling version of Ehrenfest's theorem. Equation (6.2) shows that the current-voltage (I-V) characteristic can be calculated by using Eqs. (3.22) and (3.23) to evaluate the average time interval between tunneling events. The results are shown in Fig. 4. This figure, which clearly demonstrates the crossover from constant-voltage operation ($R_S/R_T \ll 1$) to constant-current operation ($R_S/R_T \gg 1$), agrees well with the previous results.[1] In the limit of constant-current operation, the I-V characteristic has horizontal (insulating) and parabolic (conductive) branches, while the I-V curve is single valued and has an offset $e/2C$ in the limit of constant-voltage operation.

VII. LINEWIDTH PROBLEM REVISITED

Section IV B showed that the linewidth of SET oscillations vanishes for constant-current operation and that the statistical randomness of tunneling leads to the background noise, called noise pedestal. This result has been known for some time,[9] but it has not seemed so obvious because in optical physics statistical fluctuations usually lead to a finite linewidth.[24] In this section we use the exact solution of the master equation to reconsider the physical origin of the vanishing linewidth.

The vanishing linewidth of SET oscillations—or, equivalently, the infinitely long-time correlation of SET events—for constant-current operation originates from the combination of the *discrete* transfer of charge across the barrier with its *linear* supplement from the external circuit. It thus follows that the values that the charge can take at each moment are restricted to a *discrete* set,[3]

$$Q(t) = Q(0) + I_{dc}t - ne, \tag{7.1}$$

where n is the number of tunneling events [see Fig. 5(a)]. Therefore the statistical properties of charge at $t = 0$, $Q(0)$, precisely carry over to those of charge at time t, $Q(t)$, and "memory" of charge at $t = 0$ is preserved. This is why the randomness of individual tunneling events does not lead to deterioration of correlation. In shunted operation, on the other hand, *nonlinearity* of the charging process allows the charge to take values from a *continuous* set after several tunneling events [Fig. 5(b)], and eventually the memory of the charge at $t = 0$ will be completely lost.

The reason the randomness of individual tunneling events does not deteriorate the long-time correlation of SET oscillations can be put in another way. Provided that tunneling events have occurred at $Q = Q_1, Q_2, \ldots, Q_n$ while the charge changes from $Q = Q_0$ to Q, the time taken for the whole process is given, from Eq. (3.21), by

$$\tau = CR_S\left(\ln\frac{CV - Q_0}{CV - Q_1} + \ln\frac{CV - Q_1 + e}{CV - Q_2} + \cdots + \ln\frac{CV - Q_n + e}{CV - Q}\right). \tag{7.2}$$

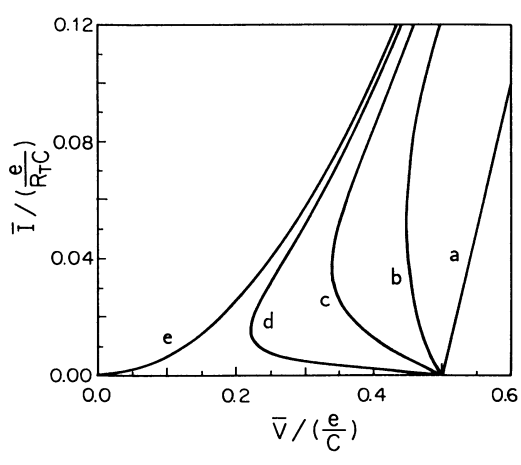

FIG. 4. Current-voltage characteristics for (a) $R_S/R_T = 0$, (b) $R_S/R_T = 3$, (c) $R_S/R_T = 10$, (d) $R_S/R_T = 50$, and (e) $R_S/R_T \to \infty$.

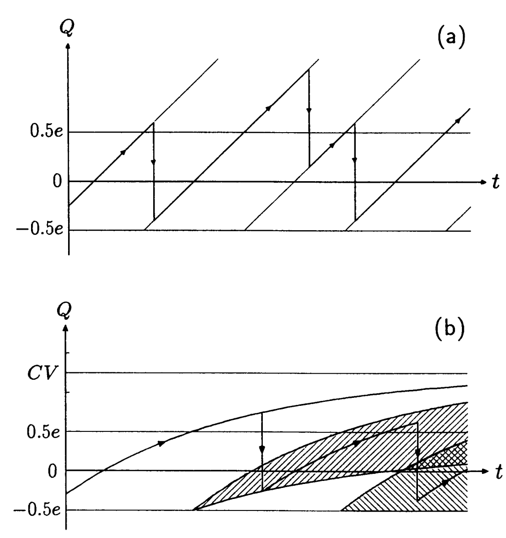

FIG. 5. Regions in the t-Q plane that the junction charge is allowed to occupy: (a) constant-current operation and (b) shunted operation.

In the limit $V \to \infty$ and $R_S \to \infty$ with $V/R_S = I_{\rm dc}$, Eq. (7.2) reduces to

$$\tau = \frac{1}{I_{\rm dc}}[(Q_1 - Q_0) + (Q_2 - Q_1 + e) + \cdots + (Q - Q_n + e)] = \frac{Q - Q_0 + ne}{I_{\rm dc}}. \qquad (7.3)$$

This shows that the randomness of intermediate tunneling events at $Q = Q_1, Q_2, \ldots, Q_n$ does not affect the time τ and hence does not destroy the long-time correlation.

These qualitative explanations for the absence of the linewidth of SET oscillations can be refined by using our analytic solution of the stochastic master equation. In constant-current operation, the time evolution of the charge averaged over the ensemble, in which the initial value of the charge Q_0 is held fixed, is given by

$$\langle Q(t)\rangle = Q_0 + I_{\rm dc}t - e\left[\frac{I_{\rm dc}t + Q_0}{e} + \frac{1}{2}\right] + e\sum_{k=0}^{\infty} kp(k,t), \qquad (7.4)$$

where [ ] is Gauss's symbol, and $p(k,t)$ is a distribution function satisfying the normalization condition,

$$\sum_{k=0}^{\infty} p(k,t) = 1. \qquad (7.5)$$

We have assumed that Q_0 is less than $e/2$. The first three terms of the right-hand side (rhs) of Eq. (7.4) represent completely regular SET oscillations, whereas the last term includes the stochastic nature of the tunneling. The separation of $\langle Q(t)\rangle$ into regular and random parts leads to two distinct components in the power spectrum, namely, a set of δ-function peaks and the pedestal. The explicit form of the last term can be obtained from the analytic solution (3.17) of the master equation. The Fourier transform of $\langle Q(t)\rangle$ is given by

$$\langle Q(\omega)\rangle = \int dQ\, Q P(Q_0, Q; \omega) = \frac{iQ_0}{\omega} - \frac{I_{\rm dc}}{\omega^2} - \frac{ie}{\omega}\frac{\exp[i\omega(e/2 - Q_0)/I_{\rm dc}]}{1 - \exp(i\omega e/I_{\rm dc})} + \frac{e}{I_{\rm dc}}\frac{\exp[i\omega(e/2 - Q_0)/I_{\rm dc}]}{1 - \exp(i\omega e/I_{\rm dc})}\int_0^e dQ \exp\left(i\frac{\omega Q}{I_{\rm dc}} - \frac{Q^2}{2eR_TCI_{\rm dc}}\right). \qquad (7.6)$$

It can be easily verified that the first three terms of the rhs of Eq. (7.6), respectively, correspond to the Fourier transforms of the first three terms (i.e., regular parts) of Eq. (7.4). The last term in Eq. (7.6), on the other hand, includes the random nature of tunneling events; it is a product of a periodically oscillating factor $e^{i\omega(e/2-Q_0)/I_{\rm dc}}/(1 - e^{i\omega e/I_{\rm dc}})$ and an integral arising from the randomness of individual tunneling events. Hence the last term in Eq. (7.4) is given by a convolution of a periodic function with frequency $I_{\rm dc}/e$ and the Fourier transform of the integral,

$$e\exp\left(-\frac{I_{\rm dc}t^2}{2eR_TC}\right) \quad \text{for } 0 \le t \le e/I_{\rm dc}. \qquad (7.7)$$

As noted in Sec. V, the periodicity of the random part is characteristic of low-current operation and is not directly related to the vanishing linewidth. In fact, this periodicity will be lost as the probability that the charge exceeds $3e/2$ increases. Nevertheless, the linewidth of the SET peaks will remain zero as long as the values that the charge can take are restricted to a discrete set.

VIII. CONCLUSIONS

In this paper we have exactly solved the stochastic master equation for small-capacitance normal tunnel junctions under arbitrary bias conditions at zero temperature. We have used this solution to calculate important quantities characterizing SET oscillations both for constant-current and for shunted operation. Analytic expressions of the power spectra and sum rules of SET oscillations are explicitly calculated for constant-current operation. Although thermal noise is not treated in this paper, this effect can be included, for sufficiently low temperatures, by replacing the δ function in $P_0(Q_1, Q_2; t)$ with a Gaussian packet.

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APPENDIX A: POLES OF $P(Q_1, Q_2; \omega)$ FOR CONSTANT-CURRENT OPERATION

In Sec. III B we have explicitly calculated the retarded Green's function $P(Q_1, Q_2; \omega)$ and shown that it has poles on the real ω axis. However, this calculation is correct only to the first order in $\exp(-e/2R_TCI_{\rm dc}) \ll 1$, and thus one may wonder if SET oscillations really have infinitely long time correlation. To answer this question, we rigorously prove here that $P(Q_1, Q_2; \omega)$ has poles on the real ω axis in constant-current operation.

In the limit $V \to \infty$ and $R_S \to \infty$ with $V/R_S = I_{\rm dc}$, the series expansion of $P(Q_1, Q_2; \omega)$, Eq. (3.15), becomes

$$P(Q_1,Q_2;\omega)=\tilde{P}_0(Q_1,Q_2;\omega)\bigg(\Theta(Q_2-Q_1)+e^{i\omega e/I_{dc}}\int_A^{Q_2+e}dq_1\tilde{P}(q_1)+e^{2i\omega e/I_{dc}}\int_{e/2}^{Q_2+e}dq_2\tilde{P}(q_2)\int_A^{q_2+e}dq_1\tilde{P}(q_1)$$
$$+e^{3i\omega e/I_{dc}}\int_{e/2}^{Q_2+e}dq_3\tilde{P}(q_3)\int_{e/2}^{q_3+e}dq_2\tilde{P}(q_2)\int_A^{q_2+e}dq_1\tilde{P}(q_1)+\cdots\bigg), \tag{A1}$$

where $A\equiv\max\{Q_1,e/2\}$,

$$\tilde{P}_0(Q_1,Q_2;\omega)=\begin{cases}\frac{1}{I_{dc}}e^{i\omega(Q_2-Q_1)/I_{dc}}, & Q_2<e/2\\ \frac{1}{I_{dc}}e^{i\omega(Q_2-Q_1)/I_{dc}}\exp\left(-\frac{(Q_2-A)(Q_2+A-e)}{2eR_TCI_{dc}}\right), & Q_2\geq e/2,\end{cases} \tag{A2}$$

and

$$\tilde{P}(q)=\begin{cases}\frac{q-e/2}{eR_TCI_{dc}}\exp\left(-\frac{(q-e/2)^2}{2eR_TCI_{dc}}\right), & e/2<q<3e/2\\ \frac{q-e/2}{eR_TCI_{dc}}\exp\left(-\frac{q-e}{R_TCI_{dc}}\right), & q\geq 3e/2.\end{cases} \tag{A3}$$

Equation (A1) shows that $P(Q_1,Q_2;\omega)/\tilde{P}_0(Q_1,Q_2;\omega)$ is a function of ω with period $2\pi I_{dc}/e$. On the other hand, it can be easily shown from Eq. (3.27) that, in general, $P(Q_1,Q_2;\omega)$ has a pole at $\omega=0$. These two observations lead to the conclusion that $P(Q_1,Q_2;\omega)$ has poles at $\omega=2\pi nI_{dc}/e$ ($n=0,\pm1,\pm2,\ldots$). Hence the SET peaks in the power spectra $S_Q^R(\omega)$ have no linewidth.

APPENDIX B: ASYMPTOTIC FORM AND SUM RULES FOR $S_Q^{\rm PED}(\omega)$ AND $S_Q^{\rm SET}(\omega)$

In this appendix we briefly derive Eqs. (4.16), (4.17), (4.18), and (4.19). We consider the case of low bias currents where the condition $\exp(-e/2R_TCI_{dc})\ll 1$ is satisfied. Thus $f(\omega)$ and $g(\omega)$ are approximately written as

$$f(\omega)=\int_0^\infty dQ\sin\frac{\omega Q}{I_{dc}}\exp\left(-\frac{Q^2}{2eR_TCI_{dc}}\right)$$
$$=\sqrt{2eR_TCI_{dc}}\int_0^T dt\exp(t^2-T^2), \tag{B1}$$

$$g(\omega)=\int_0^\infty dQ\cos\frac{\omega Q}{I_{dc}}\exp\left(-\frac{Q^2}{2eR_TCI_{dc}}\right)$$
$$=\sqrt{\frac{\pi eR_TCI_{dc}}{2}}\exp\left(-\frac{eR_TC\omega^2}{2I_{dc}}\right), \tag{B2}$$

where

$$T\equiv\omega\sqrt{\frac{eR_TC}{2I_{dc}}}. \tag{B3}$$

From Eq. (4.12) we have

$$\int_{-\infty}^{\infty}\frac{d\omega}{\pi}S_Q^{\rm ped}(\omega)=e\int_0^e dQ\exp\left(-\frac{Q^2}{2eR_TCI_{dc}}\right)\int_{-\infty}^{\infty}\frac{d\omega}{\pi}\frac{1}{\omega}\sin\frac{\omega Q}{I_{dc}}$$
$$-\frac{e}{2I_{dc}}\int_0^e dQ_1\int_0^e dQ_2\exp\left(-\frac{Q_1^2+Q_2^2}{2eR_TCI_{dc}}\right)\int_{-\infty}^{\infty}\frac{d\omega}{\pi}\exp\left(-i\frac{\omega(Q_1-Q_2)}{I_{dc}}\right)$$
$$=e\int_0^e dQ\exp\left(-\frac{Q^2}{2eR_TCI_{dc}}\right)-e\int_0^e dQ\exp\left(-\frac{Q^2}{eR_TCI_{dc}}\right)$$
$$=e\overline{Q(t)}\left(1-\frac{1}{\sqrt{2}}\right). \tag{B4}$$

Combining Eqs. (4.5) and (B4), we obtain Eq. (4.19).

$S_Q^{\rm ped}(\omega=0)$ can also be calculated from Eq. (4.12) as

$$S_Q^{\rm ped}(0)=\frac{e}{I_{dc}}\int_0^e dQ\,Q\exp\left(-\frac{Q^2}{2eR_TCI_{dc}}\right)-\frac{e}{2I_{dc}}\left[\int_0^e dQ\exp\left(-\frac{Q^2}{2eR_TCI_{dc}}\right)\right]^2$$
$$=e^2R_TC\left[1-\exp\left(-\frac{e}{2R_TCI_{dc}}\right)-\frac{\pi}{4}\right], \tag{B5}$$

which is equal to Eq. (4.17) to the zeroth order in $\exp(-e/2R_TCI_{dc})$. On the other hand, using the asymptotic expansion,

$$e^{-x^2}\int_0^x dt\, e^{t^2} = \frac{1}{2x} + \frac{1}{2^2x^3} + \frac{1\times 3}{2^3x^5} + \frac{1\times 3\times 5}{2^4x^7} + \cdots \quad (x \gg 1), \tag{B6}$$

and Eq. (B1), the asymptotic behavior of $S_Q^{\rm ped}(\omega)$ for $\omega \gg \sqrt{I_{\rm dc}/eR_TC}$ is obtained as

$$\begin{aligned} S_Q^{\rm ped}(\omega) &= \frac{eI_{\rm dc}}{\omega^2}\left(1 + \frac{I_{\rm dc}}{\omega^2 eR_TC} + \frac{3I_{\rm dc}^2}{\omega^4e^2R_T^2C^2} + \cdots\right) - \frac{eI_{\rm dc}}{2\omega^2}\left(1 + \frac{I_{\rm dc}}{\omega^2 eR_TC} + \frac{3I_{\rm dc}^2}{\omega^4e^2R_T^2C^2} + \cdots\right)^2 \\ &= \frac{eI_{\rm dc}}{2\omega^2}\left[1 - \left(\frac{I_{\rm dc}}{\omega^2eR_TC}\right)^2 - 6\left(\frac{I_{\rm dc}}{\omega^2eR_TC}\right)^3 - \cdots\right]. \end{aligned} \tag{B7}$$

APPENDIX C: POLES OF $P(Q_1, Q_2; \omega)$ FOR SHUNTED OPERATION

From Eqs. (3.20), (3.21), (3.22), and (4.21), the equation determining the poles of the Green's function $P(Q_1, Q_2; \omega)$ is given by

$$I(e/2, CV; \omega) = \int_{e/2}^{CV} dQ\left(\frac{CV - Q + e}{CV - Q}\right)^{i\omega CR_S}\frac{r(Q)}{i(Q)}\exp\left(-\int_{e/2}^{Q}dQ'\frac{r(Q')}{i(Q')}\right) = 1. \tag{C1}$$

The problem here is to solve this equation for nearly constant-current operation; we solve Eq. (C1) up to the lowest order with respect to R_T/R_S.

Introducing the quantities $p \equiv e/2CV$ and $q \equiv (Q - e/2)/CV$, we have

$$\left(\frac{CV - Q + e}{CV - Q}\right)^{i\omega CR_S} = \left(\frac{CV + e/2}{CV - e/2}\right)^{i\omega CR_S}\left(1 + i\omega CR_S\frac{2pq}{1-p^2} + \frac{2pq^2}{(1-p^2)^2}\left[i\omega CR_S - p(\omega CR_S)^2\right] + O(q^3)\right). \tag{C2}$$

Such an expansion is justified because moments of q rapidly converge as the ratio R_S/R_T becomes large:[14]

$$\int_{e/2}^{CV} dQ\frac{Q - e/2}{CV}\frac{r(Q)}{i(Q)}\exp\left(-\int_{e/2}^{Q}dQ'\frac{r(Q')}{i(Q')}\right) = \left(\pi p(1-p)\frac{R_T}{R_S}\right)^{1/2}, \tag{C3}$$

$$\int_{e/2}^{CV} dQ\left(\frac{Q - e/2}{CV}\right)^2\frac{r(Q)}{i(Q)}\exp\left(-\int_{e/2}^{Q}dQ'\frac{r(Q')}{i(Q')}\right) = 4p(1-p)\frac{R_T}{R_S}. \tag{C4}$$

Combining Eqs. (C1), (C2), (C3), and (C4), we obtain

$$\begin{aligned} I(e/2, CV; \omega) = \left(\frac{CV + e/2}{CV - e/2}\right)^{i\omega CR_S} &\left[1 + i\omega CR_S\frac{2p}{1-p^2}\left(\pi p(1-p)\frac{R_T}{R_S}\right)^{1/2}\right. \\ &\left. + \frac{8p^2(1-p)}{(1-p^2)^2}\frac{R_T}{R_S}\left[i\omega CR_S - p(\omega CR_S)^2\right] + O((R_T/R_S)^{3/2})\right]. \end{aligned} \tag{C5}$$

Hence the logarithm of Eq. (C1) reads

$$\begin{aligned} 2\pi i n &= i\omega CR_S \ln\frac{CV + e/2}{CV - e/2} + \ln\left[1 + i\omega CR_S\frac{2p}{1-p^2}\left(\pi p(1-p)\frac{R_T}{R_S}\right)^{1/2}\right. \\ &\qquad \left. + \frac{8p^2(1-p)}{(1-p^2)^2}\frac{R_T}{R_S}\left[i\omega CR_S - p(\omega CR_S)^2\right] + O((R_T/R_S)^{3/2})\right] \\ &= i\omega CR_S\left[\ln\frac{CV + e/2}{CV - e/2} + \frac{2p}{1-p^2}\left(\pi p(1-p)\frac{R_T}{R_S}\right)^{1/2} + \frac{R_T}{R_S}\frac{8p^2(1-p)}{(1-p^2)^2}\right] \\ &\quad - i(\omega CR_S)^2(4-\pi)\frac{R_T}{R_S}\frac{2p^3(1-p)}{(1-p^2)^2} + O((R_T/R_S)^{3/2}), \end{aligned} \tag{C6}$$

where n is an integer. We finally obtain

$$\omega_n = \frac{2\pi n}{CR_S\left[\ln\frac{CV+e/2}{CV-e/2}+\frac{2p}{1-p^2}\left(\pi p(1-p)\frac{R_T}{R_S}\right)^{1/2}+\frac{R_T}{R_S}\frac{8p^2(1-p)}{(1-p^2)^2}\right]}$$
$$-i(4-\pi)\frac{R_T}{R_S}\frac{2p^3(1-p)}{(1-p^2)^2}\frac{(2\pi n)^2}{CR_S\left(\ln\frac{CV+e/2}{CV-e/2}\right)^3}+O((R_T/R_S)^{3/2}). \tag{C7}$$

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