Algebraic scaling of material strength

W. A. Curtin and H. Scher
BP Research, 4440 Warrensville Center Road, Cleveland, Ohio 44128
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The strength of a brittle material is determined by the extreme tail of its crack-size distribution. The Weibull ansatz of algebraic distributions has been widely used but is not derived from any physical mechanism, and recent results on randomly depleted networks exhibiting exponential distributions have called this ansatz into question. In this paper a simple model for the formation and subsequent time-dependent growth of cracklike defects in brittle materials, presumed to occur during processing, is introduced to study possible crack-size distributions. The key aspect of the model is that the crack growth rate has a nonlinear dependence on the local stress at the crack tip, and hence on the crack size. The model predicts evolving crack-size distributions showing a wide range of behaviors: For sufficient nonlinearity the crack distribution rapidly becomes nearly algebraic (Weibull) in form and then evolves with a time-dependent power law, or Weibull modulus; in the absence of any nonlinearity the model exactly reproduces the exponential distribution of the one-dimensional random depletion problem. Both algebraic and exponential crack distributions can thus be considered as manifestations of an underlying nonlinear crack growth process. Moreover, the ubiquity of Weibull-like strength distributions observed in brittle materials, with a wide range of Weibull moduli, may be due to the physically expected nonlinearity of damage formation in real materials.

I. INTRODUCTION

Materials such as ceramics at low temperature are brittle in nature; that is, they are linearly elastic up to a stress at which failure occurs by the propagation of a single crack across the entire sample. As a consequence of this absence of any nonlinear or plastic behavior, the strength of any single sample is determined by the weakest preexisting defect in that sample since the weakest defect becomes unstable to catastrophic propagation at the lowest macroscopic applied stress. The strengths measured in a batch of nominally identical samples are thus distributed, since the weakest defect will vary from sample to sample, and so the "strength" of a brittle material is a statistical quantity. Being dependent on the weakest defect, the statistics of brittle failure is extreme; i.e., only the extreme tail of the defect distribution determines the measured strength. This extreme behavior also implies that the material strength is an *extrinsic* quantity: The larger the sample volume, the more likely a weaker defect will be encountered somewhere in that volume, and hence the strength decreases with increasing sample volume.

The quantity at the heart of the statistical distribution of brittle material strength is the defect strength distribution $\bar{N}(\sigma)$. $\bar{N}(\sigma)\Delta\sigma\Delta V$ is the number of defects of strength $(\sigma,\sigma+\Delta\sigma)$ in a volume ΔV. The strength distribution $\bar{N}(\sigma)$ is assumed to arise from an underlying crack size distribution $N(c)$. If the defects are all assumed to be penny cracks oriented perpendicular to the applied stress then the relation between defect strength σ and defect size c is simply $\sigma=K_{Ic}/\pi c^{1/2}$, where K_{Ic} is the material toughness. Given $N(c)$, and hence $\bar{N}(\sigma)$, the probability of survival of the volume ΔV at stress σ is

$$P_s(\Delta V,\sigma)=1-\Delta V\int_0^\sigma d\sigma'\bar{N}(\sigma') . \quad (1)$$

For a component divisible into N subvolumes ΔV_i, the failure of the component corresponds to the failure of any one subvolume and the survival probability is the product $\prod_i P_s(\Delta V_i,\sigma)$. For uniform stress σ, the survival probability of the entire component can be written as

$$P_s(\sigma,V)=\exp\left[-V\int_0^\sigma d\sigma'\bar{N}(\sigma')\right] . \quad (2)$$

This simple form exhibits the extrinsic dependence of survival on the volume V and the sensitivity of failure to the tail region $\bar{N}(\sigma)\sim 1/V$ of the defect distribution for large V.

The true form of $\bar{N}(\sigma)$ for any material is unknown *a priori*. Decades ago Weibull[1] introduced an empirical form for $\bar{N}(\sigma)$ by assuming it to be algebraic in σ,

$$\bar{N}(\sigma)=\frac{m}{\sigma_0 V_0}(\sigma/\sigma_0)^{m-1} , \quad \sigma>0 \quad (3)$$

with m and σ_0 to be used as fitting parameters for strength data obtained on samples of volume V_0. This assumption leads to the now well-known Weibull failure probability $P_f(\sigma,V)=1-P_s(\sigma,V)$, or

$$P_f(\sigma,V)=1-\exp\left[-\frac{V}{V_0}(\sigma/\sigma_0)^m\right] . \quad (4)$$

This form has been used almost exclusively and quite successfully as a characterization of brittle failure for years[2] despite a complete lack of any justification of the algebraic form of $\bar{N}(\sigma)$, or equivalently $N(c)$.

Recent work[3] on randomly depleted elastic spring network models has shown that, for an admittedly idealized but tractable model, the $N(c)$ distribution is expected to be exponential and far better fits to the failure probability form

$$P(\sigma,V)=1-\exp(-ae^{-k\sigma^{-\mu}}) , \quad 1<\mu<2 , \quad (5)$$

which rises from an exponential $N(c)$, have been ob-

tained for this system (a, k are constants). The exponential cracklike defect distribution $N(c)$ is generated by randomly removing elastic spring elements from the network until some fraction p of the springs have been removed. Well below the percolation threshold p^* at which the network elastic modulus goes to zero, $p \ll p^*$, the removed spring elements are organized in isolated defect clusters and the distribution of cluster sizes is known to scale exponentially. With the approximate link between the square root of the cluster (i.e., crack) size and the stress σ required to propagate a cluster, it follows that the strength distribution $N(\sigma)$ should also be exponential in this case. The above work raises, once again, the question of why the Weibull form works so well in such a variety of materials,[2] or nearly equivalently, how algebraic crack-size distributions can be generated in physical systems. One may also ask if the exponential distribution is somehow more fundamental than the algebraic form. In the absence of any fundamental justification, measured Weibull distributions with large power laws ($m \geq 10$) have been called "anomalous" and "unphysical." In this paper, we introduce a simple model for the formation and subsequent time-dependent growth of cracklike defects which is presumed to take place during material processing and which leads to a variety of crack-size distributions $N(c)$. Of most importance here is that the growth of a crack to the next increment of size by the evolution of a new defect at the crack tip is assumed to depend on the local stress at the crack tip and hence have a nonlinear dependence on the crack size. For sufficient nonlinearity, the crack-size distribution evolves into a nearly algebraic (Weibull) form. The Weibull modulus m varies with "time" but the distribution remains largely algebraic as opposed to exponential and the Weibull modulus measured in post-processing strength tests is then simply that $m(t)$ prevailing at the end of processing. When this nonlinear enhancement of the crack growth rate is suppressed, and linking of intersecting defects is accounted for, the model becomes identical to the random-depletion model and an exponential crack-size distribution arises.

The main point of this work is *not,* however, to prove that Weibull distributions are "better" than exponential distributions, or that only algebraic and exponential distributions are possible. Rather, our intent is to show that, using a physically plausible model for crack formation, a wide range of $N(c)$ are possible, varying from exponential in one limit of parameter space to algebraic at another limit. A secondary result is that the algebraic, or Weibull form, does persist over a wide range and that distributions with large m values are not "anomalous" or "unphysical."

II. A SIMPLE MODEL FOR CRACK-SIZE DISTRIBUTIONS

Our model of defect nucleation and growth is as follows. The model material is described by a network of N_T potential defect sites, for instance, grain-facet-sized boundaries. Defect formation is assumed to be a nucleation event, with the probability per unit time of nucleating a defect at site i being a monotonically increasing function $r(\sigma_i(t))$ of the local stress on site i at time t. Under a global applied stress σ_{app}, the local stress enhancement $\sigma_i/\sigma_{\text{app}}$ can differ from unity by the occurrence of *load transfer* from nearby defects.

With the above nucleation rate, the evolution of the system in time is qualitatively as follows. At the shortest times, single isolated defects are nucleated at rate r_0 [$r_0 \equiv r(\sigma_{\text{app}})$]. The enhanced stress in the vicinity of these defects increase the nucleation rate at nearby sites. There are relatively few of these higher-rate sites, however, and so nucleation at one of these sites to form a size two defect can take much longer than the time $(r_0 N_T)^{-1}$ to form another isolated defect. Ultimately, the enhanced rate does lead to the development of defect clusters. As the clusters grow larger in size by addition of further defect sites, load transfer increases and so the clusters grow faster The relative rates of nucleation of new isolated clusters and growth of small clusters and large clusters is governed by both the load transfer and functional form of $r(\sigma)$. In general, there is a competition between growth at the many lower-rate sites and growth at the fewer higher-rate sites that operates during the evolution of the crack distribution.

In general, the evolution of the entire system is difficult to discuss analytically because of the spatial variations of both load transfer and defect-defect interactions. We simplify this picture by focusing only on the load transfer to the z neighbors at the tip of an existing crack, where the stress is highest. We neglect the load transfer and/or load shielding at sites away from crack tip sites. With this simplification, each crack becomes essentially independent of all other cracks (except when crack linking is included, see below). Furthermore, all sites in the material are either (i) part of an existing crack, (ii) at the tip of an existing crack, or (iii) isolated and under only the applied load σ_{app}. Evolution of the entire system is then easily described by the evolution of the crack-size distribution $N_c(t)$, where $N_c(t)$ is the number of cracks of size c existing at time t. Since cracks of size c grow to size $c+1$ at a rate $r(\sigma_c)$, where σ_c is the enhanced stress at the tip of the size crack, we can write down "kinetic" equations for the nucleation and growth of all the cracks as

$$\frac{dN_c}{dt} = zr(\sigma_{c-1})N_{c-1} - zr(\sigma_c)N_c \ , \quad c \geq 1 \ ,$$

or in dimensionless units

$$\frac{dN_c}{d\tau} = \alpha_{c-1}N_{c-1} - \alpha_c N_c \ , \quad c \geq 1 \ , \tag{6}$$

where $\tau \equiv \text{tr}_0$ and $\alpha_c \equiv zr(\sigma_c)/r_0$. The first term on the rhs of Eq. (6) accounts for cracks of size $c-1$ growing to size c while the second term describes cracks of size c growing to size $c+1$ and thus depleting N_c. The evolution of "size" $c=0$ cracks, i.e, the remaining unfailed, nontip sites, is given by

$$\frac{dN_0}{d\tau} = -(1+z)N_T + \sum_{c \geq 1}(z+c)N_c \tag{7}$$

and the initial conditions for (6) and (7) are $N_0(0)=N_T$, $N_c(0)=0$.

It is straightforward to numerically integrate Eqs. (6) and (7) after specifying the functional form of $r(\sigma_c)$ and the relation of σ_c to crack size c. We first, however, will exhibit some general features of N_c by an analytic solution. We introduce the Laplace transform $\tilde{N}_c(s)$ of $N_c(\tau)$ into Eq. (6) to obtain

$$(\alpha_c+s)\tilde{N}_c(s)=\alpha_{c-1}\tilde{N}_{c-1}(s) , \tag{8}$$

a simple linear recursion relation which can be solved by iteration to yield

$$\tilde{N}_c(s)=\tilde{N}_0(s)\prod_{i=1}^{c}\left[\frac{\alpha_{i-1}}{\alpha_i+s}\right] . \tag{9}$$

As time evolves, the nucleation sites $N_0(\tau)$ are monotonically decreasing and, for a finite-size system, this leads to very interesting behavior for the crack-size distribution $N_c(\tau)$. In the present paper we are restricting ourselves to a low density of defects, i.e., $p\equiv\sum_c cN_c/N_T \ll 1$. We emphasize the crack distributions at low-p values because real materials are not highly defective and the larger p need not be consistent with our general neglect of crack-crack interactions.

In the low-p range we can simplify the solution to Eq. (7) by neglecting the depletion of N_0 sites so that

$$N_0(\tau)\simeq N_T . \tag{10}$$

Inserting the Laplace transform of Eq. (10), $\tilde{N}_0(s)=N_T/s$, into Eq. (9), we obtain $N_c(\tau)$ as

$$N_c(\tau)=\frac{\alpha_0 N_T}{\alpha_c}\left[1-\sum_{i=1}^{c} f_i(c)e^{-\alpha_i\tau}\right] \tag{11}$$

with

$$f_i(c)\equiv\prod_{j=1}^{c}{}'\frac{\alpha_j}{(\alpha_j-\alpha_i)} , \tag{12}$$

the prime indicating that the $j=i$ term is excluded in the product.

With the above general solution in hand, we first consider some simplified but illustrative cases of the behavior of $N_c(\tau)$. First, for a single growth rate α, independent of crack size, it is seen from Eq. (9) that

$$N_c(\tau)=N_T e^{-\alpha\tau}\sum_{m=c}^{\infty}\frac{(\alpha\tau)^m}{m!} . \tag{13}$$

The sum in Eq. (13) can be approximately replaced by the first term when

$$\frac{\alpha\tau}{c+1}<\frac{1}{2} , \tag{14}$$

and one finds

$$N_c(\tau)\simeq N_t e^{-\alpha\tau}\frac{(\alpha\tau)^c}{c!} , \tag{15}$$

which decays faster than exponential with crack size c for all times τ. Thus, an algebraic dependence of the crack-size distribution is precluded when the growth rates are independent of c.

To see how $N_c(\tau)$ depends on α_c, e.g., how great a "spread" in the growth rates α_c is required to move away from the regime of exponential dependence of $N_c(\tau)$ on c, we consider the second simple case—the short-time limit where each exponential term in Eq. (11) is expanded in a power series. The result to leading order is

$$N_c(\tau)\simeq N_T\left[\prod_{i=1}^{c}\alpha_{i-1}\right]\frac{\tau^c}{c!} , \tag{16}$$

when

$$\frac{\alpha_c\tau}{c+1}<\frac{1}{2} . \tag{17}$$

The same solution shown in Eq. (16) can be obtained by dropping the depletion term $-\alpha_c N_c$ in Eq. (6) and so the short-time limit corresponds to a net growth rate at each crack size c limited only by the supply at the smaller crack size $c-1$. From Eq. (16), we see that, to obtain slower than exponential c dependence of $N_c(\tau)$ in the short-time limit where Eq. (17) holds, requires that α_c/c be an increasing function of c. If α_c/c is not increasing with c, then Eq. (16) is valid for *all* crack sizes for times $\alpha_1\tau/2<1$, i.e., all crack sizes are in the short-time limit, and $N_c(\tau)$ is at least exponential. If α_c/c is increasing with c, then $N_c(\tau)$ is subexponential in the short-time limit and, moreover, there is a range of crack size $c>c^*$, where c^* is defined by

$$\frac{\alpha_{c^*}\tau}{c^*+1}=\frac{1}{2} , \tag{18}$$

where $N_c(\tau)$ is not described by the early time solution Eq. (16).

In the limit where $c^*\sim 1$, the time-dependent term in Eq. (11) is much less than unity and

$$N_c(\tau)\sim\alpha_0 N_T/\alpha_c \tag{19}$$

for all crack sizes. In this same limit, however, one can show that the cumulant $R_c(\tau)$ of all cracks larger than c,

$$R_c(\tau)=\sum_{c'=c}^{\infty}N_{c'}(\tau) , \tag{20}$$

becomes independent of crack size c; this condition indicates the presence of an unbound crack, i.e., a crack larger than the linear size of the system. This interesting "breakdown" behavior derived form the present model is planned to be discussed in a future publication.

Between the extremes of the solutions Eqs. (16) and (19), there is a range of τ or defect density p over which $N_c(\tau)$ exhibits "intermediate" c dependence. In this range of τ or p, one can rationalize the appearance of algebraic c dependence for $c>c^*$ (which will be confirmed below by numerical evaluation) as follows. For $c>c^*$, the last exponential term in Eq. (11) for $N_c(t)$ is negligible, $e^{-\alpha_c\tau}\ll 1$; hence, the main c dependence in $N_c(t)$ is contained in the coefficients $f_i(c)$ given by Eq. (12). Now the difference between $N_{c-1}(\tau)$ and $N_c(\tau)$ is only that for $N_c(\tau)$ each $f_i(c)$ is increased by a factor

$\alpha_c/(\alpha_c-\alpha_i)$ relative to the value for $N_{c-1}(\tau)$. Thus, the sum in Eq. (11) increases slowly with increasing c, with the largest changes in $f_i(c)$, at $i=c$, weighted by the smallest exponential terms. The net change on going from $N_{c-1}(\tau)$ to $N_c(\tau)$ is thus expected to be a slowly varying function of c.

III. RESULTS

We now choose explicit forms of α_c to show, in detail, the behavior we have been discussing. In general, since the stress at the tip of a crack scales like $c^{1/2}$, we wish to preserve this scaling in the σ_c. We will investigate several choices, but the primary parametrization will be

$$\sigma_c=\left[1+\frac{1}{z}\right]c^{1/2}, \quad c\geq 1 \tag{21}$$

with unit applied stress $\sigma_{\text{app}}=1$ assumed. The $(1+1/z)$ prefactor sets the scale by assuming that the load transferred by a $c=1$ crack is equally distributed among its z neighbors.

The forms chosen for $r(\sigma)$ are physically motivated. A power law is selected because of the observed power-law dependences of damage processes, such as cavitation and slow crack growth, which occur under high temperature or environmental mechanical testing of many ceramics. The same damage process occurring under those conditions may, in fact, be operating during fabrication and lead to the crack distributions in the as-processed material. In addition, at an atomic level, the power-law form for the rate dependence on σ_c can be derived from the stress dependence of the activation energy $U(\sigma)$ for bond rupture based on a Morse potential.[4] A linear approximation to $U(\sigma)$ leads to an exponential rate law. Hence, we use

$$r(\sigma_c)=\sigma_c^n \tag{22}$$

and

$$r(\sigma_c)=e^{\lambda(\sigma_c-1)}, \tag{23}$$

both relative to the rate r_0 at the unit applied stress $\sigma_{\text{app}}=1$.

Although Eqs. (6) and (7) describe evolution in time, the variable to hold fixed for meaningful comparisons between different forms of α_c is p. If, e.g., the applied stress is time dependent, this influences the time evolution through a changing time scale but does not change the evolution of $N_c(\tau)$ as a function of the fraction p of sites broken [for the growth law Eq. (22)]. Enhancing the rates α_c moves the comparisons at the same value of p to earlier times.

Figures 1(a) and 1(b) show the crack-size distribution obtaining from Eqs. (6) and (7) using Eqs. (21) and (22) and $n=0,2,3,4,5$ at $p=0.01$ and $z=2$. Figure 1(a) shows $\ln[N_c(\tau)/N_T]$ versus c, for which exponential distributions plot as straight lines, and Fig. 1(b) shows $\ln[N_c(\tau)/N_T]$ versus $\ln c$, for which algebraic distributions plot as straight lines. There is a clear crossover from exponential or superexponential behavior for $n\leq 2$ to essentially algebraic behavior for $n>3$. The $n\leq 2$ case corresponds to a nonincreasing α_c/c with c; hence, Eq. (17) is satisfied for all c and the superexponential behavior is expected. The behavior for $n>2$ is consistent with our analysis: for $c<c^*$, the $N_c(\tau)$ is subexponential and for $c>c^*$, algebraic scaling can be observed. The algebraic, or Weibull behavior, persists over several decades in crack size. Pursuing the evolution of the Weibull-like behavior further, Fig. 2 shows $\ln[N_c(\tau)/N_T]$ versus $\ln c$ for $n=4$, $z=2$, and increasing values of the defect density p. At low p, $p=0.001$, the size distribution falls off faster than algebraic for the small cracks $c<10$ shown. However, at $p=0.0025$, the tail of the distribution exhibits algebraic behavior, $N_c\sim c^{-\rho}$ with $\rho\simeq 19$ for $c\geq 9$. For $p=0.005$, algebraic behavior with $\rho\simeq 12$ for $c\geq 7$ is observed. One clearly notes here the crossover to algebraic dependence for $c>c^*$ (where c^* is marked explicitly in the figure). For larger p the algebraic scaling extends to even smaller crack sizes, with some slight positive curvature suggesting a slower-than-algebraic falloff in $N_c(\tau)$,

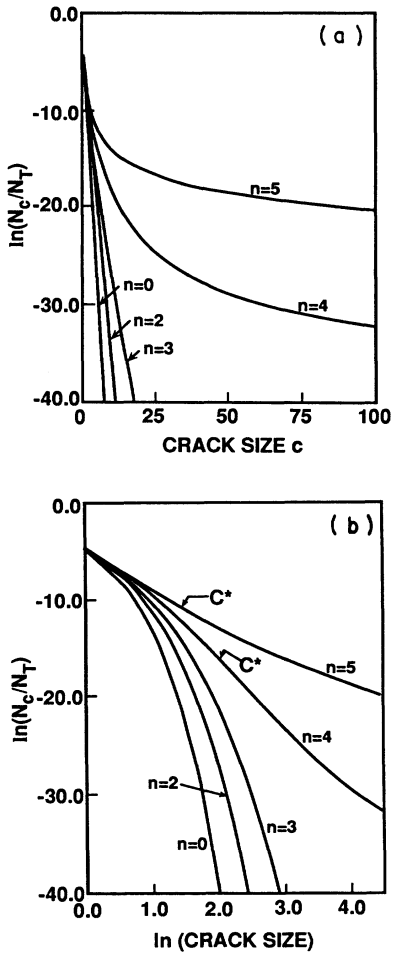

FIG. 1. Crack-size distributions $N_c(\tau)$ for power-law rate enhancement $r(\sigma_c)=\sigma_c^n$, for $n=0,2,3,4,5$ with stress enhancement $\sigma_c=1.5c^{1/2}$ and fraction of broken elements $p=0.01$. (a) $\ln[N_c(\tau)/N_T]$ vs c; exponentials plot as straight lines. (b) $\ln[N_c(t)/N_T]$ vs $\ln(c)$; power laws plot as straight lines. Note crossover from exponential to algebraic scaling with increasing n.

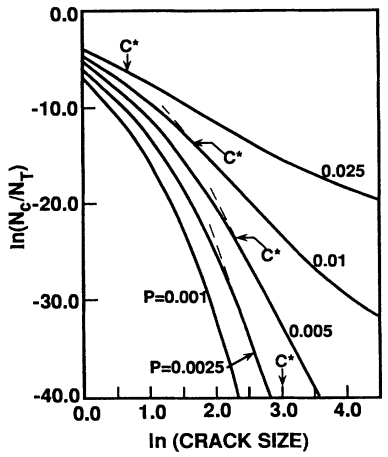

FIG. 2. Crack-size distribution $\ln[N_c(t)/N_T]$ vs $\ln(c)$ at various break fractions p for $r(\sigma_c)=\sigma_c^4$, $\sigma_c=1.5c^{1/2}$. Note the appearance of near-algebraic scaling $N_c\sim c^{-\rho}$ for sizes c larger than the short-time–short-crack to long-time–long-crack crossover crack size c^* (see text). The large negative power laws ρ imply large measured Weibull moduli $m=2(\rho-1)$.

while the approximate value of ρ decreases steadily toward the limiting value of $\rho=2$ (for $n=4$). For sufficiently large n, then, we find $N_c(\tau)$ to be algebraic above a time- or p-dependent crack size which we associate with the c^* given by Eq. (18), and c^* decreases with time until the entire distribution exhibits near-algebraic scaling.

The general behavior observed using Eq. (19) for $n=4$ and $z=2$ persists for other load sharing rules. Figure 3 shows crack-size distributions obtained for $n=4$, and various p using the load transfer rules

$$\sigma_c=\left[1+\frac{1}{z}\right]c^{1/2}\ ,\quad z=4\ , \tag{24a}$$

$$\sigma_c=\left[0.78+\frac{0.58}{c}\right]c^{1/2}\ ,\quad z=4\ , \tag{24b}$$

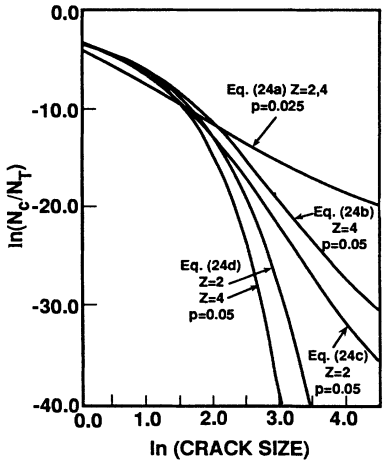

FIG. 3. Crack-size distributions $\ln[N_c(t)/N_T]$ vs $\ln(c)$ for a variety of stress enhancement laws σ_c (see text) with $r(\sigma_c)=\sigma_c^4$.

$$\sigma_c=\sum_{j=1}^{c}\left[\frac{2j+2}{2j+1}\right]\ ,\quad z=2\ , \tag{24c}$$

$$\sigma_c=1+\frac{1}{z}c^{1/2}\ ,\quad z=2,4\ . \tag{24d}$$

Rule (24a) is identical to Eq. (20) except $z=4$ is used, allowing for two sites at each crack tip to be potential growth sites and thus allowing for noncollinear cracks, as might occur in two-dimensional (2D) systems. Rule (24b) has been empirically derived for the tip stresses in a 2D triangular spring network, for which $z=4$.[5] Rule (24c) has been derived by Hedgepeth to describe load transfer around linear breaks in fiber bundles.[6] Rule (24d) is simply a slight modification of Eq. (20). As evident in Fig. 3, Eqs. (24a)–(24c) all yield primarily algebraic behavior for crack sizes $10\le c\le 60$ at these p. Although there is some curvature to these data, the behavior is certainly more algebraic than exponential. Equation (24d), on the other hand, has systematic curvature and is not algebraic for $n=4$, $p=0.05$. However, a crossover to behavior similar to that obtained from Eqs. (24a)–(24c) occurs for $n=5$, $p=0.05$. The precise power n at which algebraiclike behavior sets in thus does depend on the nature of the load transfer σ_c, with larger n required for smaller load transfer functions σ_c at any particular value of p or τ.

The appearance of algebraic N_c is independent of the form of the growth law (i.e., power law versus exponential), as confirmed by the results obtained using Eq. (23), as shown in Fig. 4. Figure 4 shows N_c for $p=0.05$, using Eq. (23) and $z=2$, for a range of λ values. At low λ, N_c is roughly exponential but for larger λ, N_c becomes nearly algebraic, $c^{-\rho}$, with ρ in the range 6–12. The tendency toward subexponential distributions, and the appearance of algebraiclike distributions, is thus independent of the detailed growth rate law and instead depends only on having a sufficient spread in the α_c.

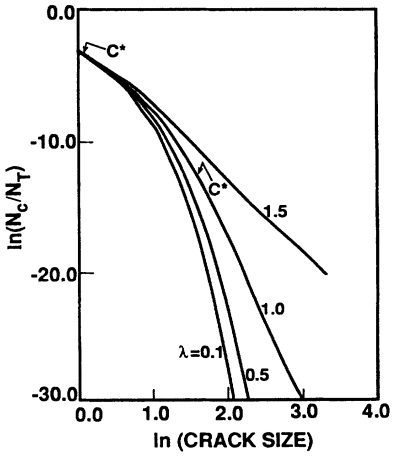

FIG. 4. Crack-size distributions $\ln[N_c(t)/N_T]$ vs $\ln(c)$ for exponential rate law $r(\sigma_c)=\exp[\lambda(\sigma_c-1)]$ with $\sigma_c=1.5c^{1/2}$ and various λ at $p=0.05$. Algebraic scaling again appears over a range of λ values.

IV. DISCUSSION AND SUMMARY

The crack nucleation and growth equations, with crack growth rates α_c depending nonlinearly on the local stress and hence the crack size, have been shown capable of generating a wide spectrum of crack-size distributions. For small nonlinearity (α_c/c decreasing with c), the distributions are more exponential in nature. For stronger nonlinearity (α_c/c increasing with c), algebraic behavior appears, with a wide spectrum of Weibull moduli depending on the rate law $r(\sigma)$, time, and the detailed stress enhancements σ_c. In addition, the Weibull moduli m as determined from strength tests using Eq. (4) are related to the algebraic powers ρ in $N_c \sim c^{-\rho}$ by $m=2(\rho-1)$ and thus m values anywhere from $m=2$ to 35 can be plausibly obtained from this physical model.

A tremendous advantage of the nucleation and growth equations for $N_c(\tau)$ over explicit simulations on finite lattice networks is that the calculations are trivial to perform over decades in crack size (we typically calculate out to $c=100$) with all the probabilities retained to within computer limits ($\ln[N_c(\tau)/N_T] \geq -87$). In the finite lattice networks, the lattice size is limited to perhaps $N_T=10\,000$ sites. Thus, the real extreme tail of the crack distribution where algebraic behavior may be lurking for various growth laws cannot be probed, in practice, by simulations, yet is absolutely necessary to make contact with the failure distributions measured in real macroscopic systems, for which N_T is many orders of magnitude larger.

A disadvantage of the nucleation and growth equations as formulated is that crack-crack interactions are not taken into account. Although we are considering dilute crack concentrations so that interactions are small on average, we are concerned with the extremes of the distributions and interactions can modify the distributions, especially in the tail. We can account exactly, however, for the crudest, but perhaps most important, crack interaction effect of collinear crack linking when there is one site at each crack tip (see the Appendix). Crack linking occurs when two cracks of sizes c and c' are separated by a single uncracked site. Stress concentration at the intervening site, denoted $\sigma_{c,c'}$, is considerably enhanced, scaling more like $(c+c')/2$ than with any square root. Even in the absence of stress-enhanced nucleation, the formation of a $c+c'+1$ crack and concomitant annihilation of a c and a c' crack is an important process governing the evolution of the full crack-size distribution.

Figure 5 shows the effects of collinear crack linking on the crack-size distribution using $\sigma_{c,c'}=1+(c+c')/2$ for the rate law of Eq. (22) with $n=0,2,3,4$ and $\sigma_c=1.5c^{1/2}$ [Eq. (20), $z=2$] at $p=0.025$. Linking has virtually no effect on the distribution at $n=4$, making larger cracks slightly more likely but without modifying the algebraic exponent. This insensitivity to linking arises because the large growth rate at existing crack tips leads to the formation of a dilute collection of longer cracks as compared to $n=0$ at the same total fraction of breaks p. The dilute cracks are less likely to interact (link) and hence linking becomes negligible. For sufficient enhancements σ_c^n, the simple nucleation and growth terms in Eq. (6)

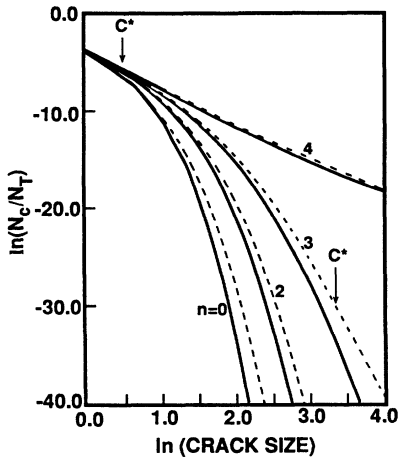

FIG. 5. Crack-size distributions $\ln[N_c(t)/N_T]$ vs $\ln(c)$ without crack linking (——) and with crack linking (— — —), for $r(\sigma_c)=\sigma_c^n$, $\sigma_c=1.5c^{1/2}$, $p=0.025$. With linking, the $n=0$ distribution becomes exactly exponential, the $n=2$ distribution is slightly subexponential, the $n=3$ distribution becomes algebraic in its tail, and the $n=4$ distribution is nearly unchanged.

dominate the evolution of the N_c. Linking is also able to drive the $n=3$ distribution from subalgebraic to algebraic in the tail. This general tendency of linking to "stretch-out" subalgebraic distributions is observed for a wide range of σ_c, n, and p. In contrast to the $n=4$ results, for $n=0$ there is a considerable enhancement of the tail of the crack distribution. In fact, the inclusion of linking at $n=0$ changes the distribution from superexponential to *exactly* exponential. Since $n=0$ implies no enhancement of the nucleation rate at the crack tip, this limit corresponds precisely to the random-depletion model, which has an exponential crack-size distribution. Thus, by including the crack linking of collinear cracks, we have reproduced the one-dimensional random-depletion model, for which the crack-size distribution is trivially $N_c(p)=(1-p)^2e^{c(\ln p)}$. Although $n=0$ is exponential with linking, $n=2$ is only very slightly subexponential with linking and so our conclusions on algebraic behavior appearing for only increasing α_c/c are essentially preserved. Investigation of additional linking effects for a wide range of n, load rules, and p indicates that the appearance of algebraiclike behavior is relatively insensitive to the linking effects. This insensitivity implies that the simple nucleation and growth Eqs. (6) and (7) contain the key elements of the time-dependent crack growth.

The strengths of real ceramic materials are controlled by the distribution of defects in them. The defect distributions $N_c(\tau)$ are, in turn, significantly influenced by processing details, usually involving exposure to high temperatures and nonzero stresses. If the generation of growth of defects is random, i.e., the growth rates are independent of local stresses, crack size, etc., then $N_c(\tau)$ shows exponential c dependence, as established both by simulation[3] and the present model (with linking). However, if the growth rates are dependent on the stress environment and if this dependence is sufficiently coupled to the defect size, as is typical of crack growth in brittle ma-

terials, than the present model for the $N_c(\tau)$ admits a wide range of c dependence. In particular, the occurrence of an algebraic tail is quite generally predicted for defect growth rates α_c which increase faster than linear in crack size c. The precise value of the measured Weibull modulus m corresponding to these algebraic $N_c(\tau)$ is, of course, dependent on the nonlinearity of α_c, the defect density p, and the range of sample volume V tested [see Eqs. (2) and (4)]. However, the resulting range of Weibull moduli in strength distributions in the low-defect density regime spans from at least 3 to 30, dispelling the notion that observed Weibull scaling with large moduli is somehow "anomalous." Although Weibull himself states that[1] "it is utterly hopeless to expect a theoretical basis for distribution functions of random variables such as strength properties of materials. . . ," the present model is one physically plausible theory for such distribution functions which, in fact, can describe the very behavior Weibull sought to capture with his empirical distributions.

Noted added in proof. The use of Eqs. (6) and (7) to explore "breakdown" behavior, as mentioned in the text below Eq. (20), appeared in Ref. 7.

APPENDIX

For linear cracks, i.e., one element at each tip of any crack ($z=2$), the linking of two cracks of sizes c' and c'' sharing a single crack tip element to form a crack of size $c'+c''+1$ at some enhanced rate can be accounted for exactly. We denote the stress enhancement at the element between a c' and c'' crack as $\sigma_{c',c}$ and the enhanced growth rate as $\alpha_{c',c''}=zr(\sigma_{c',c''})$. There are then three processes involving an arbitrary size c crack which must be followed as additional failed elements are introduced into the system.

(1) First, there is the simple growth from size $c-1$ to c, which occurs only if the element adjacent to the element at crack tip is also unfailed. The change in N_c with break fraction p due to this process is simply

$$\left.\frac{dN_c}{dp}\right|^{(1)} = \frac{1}{1-p} N_{c-1}\alpha_{c-1}\left[1-\frac{1}{1-p}\sum_{c'=1} N_{c'}\right] . \tag{A1}$$

The second term on the right-hand side accounts for the probability that the adjacent element is part of another crack, with the factor $1/(1-p)$ to avoid double counting of the crack tip site. The overall factor of $1/(1-p)$ arises simply from the fact that there are only $(1-p)$ unfailed elements available to break at any p.

(2) The second process is the growth of a size c crack to any larger size, whether linking is involved or not. The change in $N_c(p)$ due to this process is

$$\left.\frac{dN_c}{dp}\right|^{(2)} = -\frac{1}{1-p} N_c\left[\alpha_c(1-p)+\frac{1}{1-p}\sum_{c'=1}\alpha_{c,c'}N_{c'}\right] . \tag{A2}$$

The first term on the rhs is simply the growth from $c\to c+1$, while the second term is the growth by linking to all other sizes.

(3) The third process is the formation of a size c crack by the linking of two smaller cracks, and is only operable for cracks of size $c\geq 3$. The change in N_c with p for this linking process is

$$\left.\frac{dN_c}{dp}\right|^{(3)} = \frac{1}{1-p}\left[\frac{1}{2}\right]\left[\frac{1}{1-p}\sum_{c'=1}^{c-2}\alpha_{c-c'-1,c'}N_{c-c'-1}N_c\right] . \tag{A3}$$

The factor of $\frac{1}{2}$ accounts for double counting in the sum over crack size.

The above three processes are the only possible contributions to the size c crack population. Linking of more than two cracks is a higher-order function of dp and can be neglected. Thus, the nucleation and growth equations are the sum

$$\frac{dN_c}{dp} = \left.\frac{dN_c}{dp}\right|^{(1)} + \left.\frac{dN_c}{dp}\right|^{(2)} + \left.\frac{dN_c}{dp}\right|^{(3)} . \tag{A4}$$

The evolution of the unfailed, nontip elements $N_0(p)$ is simply given by

$$N_0(p)=N_T-\sum_c (z+c)N_c+\tfrac{1}{2}z\left[\sum_c cN_c\right]^2 , \tag{A5}$$

where the last term accounts for the fact that some cracks share the same tip elements.

Notice that the original equations (6) and (7) without linking are obtained by dropping all terms proportional to products of $N_{c'}N_{c''}$, including terms coming from $p=\sum_{c=1}cN_c$, and hence linking is explicitly second order in the crack densities. The evolution in time $N_c(t)$ is obtained from the evolution $N_c(p)$ in p, given by Eqs. (A1)–(A5), by considering the total rate of crack growth. The time increment Δt required to add one more failed element ($p\to p+1/N_T$) is the inverse of the rate of crack growth

$$\frac{\Delta p}{\Delta t}=N_0\alpha_0+(1-p^2)\left[\sum_c \alpha_c N_c\right]+\tfrac{1}{2}\sum_{c,c'}\alpha_{c,c'}N_cN_{c'} , \tag{A6}$$

where the first term on the rhs of (A6) is the rate of formation of single isolated cracks, the second term is the rate of growth of existing cracks without linking, and the final term is the rate of growth due to crack linking. The time evolution is then obtained by integrating dt/dp up to the defect density p of interest to obtain the time $t(p)$ at that point.

For the random-depletion problem, i.e., no enhancement of the crack growth or linking rates, the solutions of (A1)–(A5) for $N_c(p)$ are simply

$$N_0(p)=(1-p)^3 ,$$

$$N_c(p)=(1-p)^2p^c , \quad c\geq 1 ,$$

and with $dp/dt=1-p$, the time to break p elements is just

$$t=-\ln(1-p) ,$$

which is $t\simeq p$ for small p, as expected.

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