Weak-localization correction to the number density of superconducting electrons

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(Received 28 June 1991)

Anticipating that the backscattering processes that lead to weak localization in normal metals will occur also in dirty superconductors, we have evaluated the effect of these processes on the number density of superconducting electrons. We find a reduction in this density similar in form to the reduction in conductivity for the normal metal, except that the superconducting gap now provides the lower energy cutoff instead of the inelastic scattering rate. This result can be derived heuristically from the conductivity sum rule, and also from the use of an exact-eigenstates method. The localization effect leads to a weakening of the superconducting state to a point at which order-parameter amplitude fluctuations become important.

I. INTRODUCTION

The phenomenon of weak localization in normal metals has been extensively investigated, both theoretically and experimentally, over the past decade and is now well understood.[1-3] It arises from the coherent scattering of electrons from impurities in the metal: A constructive interference between electrons scattered backward via different paths reduces the current at fixed voltage. This process has provided a quantitative explanation for decreases in the conductivity of thin films proportional to the logarithm of the absolute temperature T and for the positive magnetoconductivities found in such systems.

The same quantum interference process must occur in superconductors. Here the backscattering involves superconducting quasiparticles and their attendant coherence factors, but it is reasonable, nonetheless, to expect observable consequences. Even though the zero-frequency conductivity is infinite in a superconductor, the effects might occur in the number density of superconducting electrons, $n_s(T)$. This is made plausible by the following scaling argument.[4,5] In the absence of weak localization, in the dirty limit $\Delta \ll \Gamma = \tau^{-1}$, where Δ is the superconducting energy gap and Γ is the elastic-scattering rate, one knows that $n_s(T)$ scales with the dc conductivity in the normal state, $\sigma(\omega=0)$:

$$\frac{n_s(T)}{\sigma(\omega=0)} = \frac{m\pi}{e^2}\Delta \tanh\left[\frac{1}{2}\beta\Delta\right], \tag{1}$$

where m and e are the electron's mass and charge and β is the reciprocal of the absolute temperature. If this scaling were to persist into the weak-localization regime, one would indeed see weak-localization effects in superconductors.

The motivation for considering such processes lies in recent experimental and theoretical work on the superconductor-insulator transition[5-11] (for a recent review, see Ref. 12.) Strong enough disorder appears to destroy superconducting order eventually, leaving an insulator, just as disorder is known to destroy metallic behavior. A theory[13] of the noninteracting metal-insulator transition was obtained by combining scaling ideas with weak-localization calculations. Proceeding by analogy, we might expect such calculations also to be important in the superconducting case.

In this paper we perform a rigorous calculation—the exact superconducting analog of the normal-metal weak-localization calculation—and show that there is indeed a weak-localization correction to n_s. Ma and Lee[5] calculated forms for n_s in the strong-disorder regime from the normal conductivity $\sigma(\omega)$ using a relation valid for one-body potentials derived, for example, by de Gennes.[14] However, they did not use this method to calculate the weak-localization correction, although it is possible to do this (as we show later). The weak-localization result turns out to be flat in the low-temperature limit, which makes experimental observation unlikely. It is of theoretical importance because it shows how backscattering, although not capable of extinguishing superconductivity, weakens it enough for other fluctuation effects to become important. These may then lead to the destruction of the superconductivity.

In Sec. II we sketch the superconducting weak-localization calculation. (Some details are relegated to the Appendix.) The material here is essentially of a technical nature. These sections may be skipped by those interested only in the meaning and observable consequences of the calculation, which are discussed in Sec. III. There we show that the number density of superconducting electrons, $n_s(T)$, does scale with zero-frequency conductivity $\sigma(\omega=0)$, in agreement with an argument, which we summarize, based on the conductivity sum rule. Possible difficulties of interpretation in the superconducting state are discussed.

II. OUTLINE OF CALCULATION

As is by now well known, backscattering effects are contained in the so-called maximally crossed diagrams, identified by Langer and Neal.[15] In what follows we evaluate the sum of these diagrams, shown in Fig. 1(a),

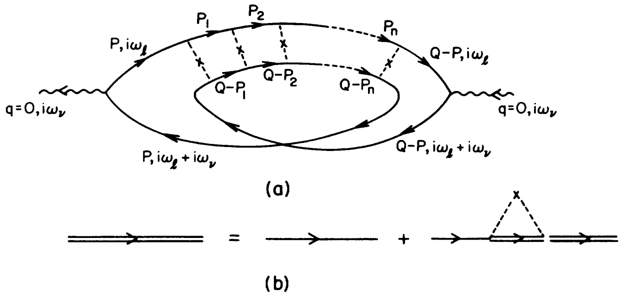

FIG. 1. (a) Maximally crossed conductivity diagram leading to backscattering corrections calculated in the text; (b) impurity dressing of electron propagators used in (a).

with superconducting electron propagators, including impurity scattering, as indicated in Fig. 1(b). This calculation is the exact superconducting analog of the normal weak-localization calculation, which can be found in, for example, Bergmann.[1] As in the normal case, this diagram is the lowest-order term in a $1/k_F l$ expansion, as discussed in the Appendix. The dressed-electron Green function of Fig. 1(b) is given by[16]

$$G(p,i\omega_l)=\frac{\overline{i\omega_l}+\varepsilon_p\tau_3+\bar{\Delta}\tau_1}{\overline{i\omega_l}^{\,2}-\varepsilon_p^2-\bar{\Delta}^2}\ , \tag{2a}$$

where

$$\overline{i\omega_l}=i\omega_l\left[1+\frac{i}{2\tau}\frac{1}{[(i\omega_l)^2-\Delta^2]^{1/2}}\right],\qquad \bar{\Delta}=\Delta\left[1+\frac{i}{2\tau}\frac{1}{[(i\omega_l)^2-\Delta^2]^{1/2}}\right]. \tag{2b}$$

Since we are interested in calculating $n_s(T)$, we perform the calculation for zero external Matsubara frequency $i\omega_\nu$ and zero momentum Q, although, for convenience, we first keep the frequency nonzero. The contribution of the nth diagram to the electromagnetic response kernel is

$$\begin{aligned}K_{\alpha\beta}^{(n)}(q=0,i\omega_\nu)=&\frac{e^2}{m^2}\frac{1}{\beta}\sum_l\int\frac{d^3p}{(2\pi)^3}\frac{d^3Q}{(2\pi)^3}p_\alpha(Q-p)_\beta(\Gamma/2\pi)^{n+1}\\ &\times\prod_{i=1}^{n}\int\frac{d^3p_i}{(2\pi)^3}\mathrm{Tr}[G(p,i\omega_l+i\omega_\nu)G(p,i\omega_l)\tau_3G(p_1,i\omega_l)\tau_3\cdots G(p_n,i\omega_l)\tau_3G(Q-p,i\omega_l)\\ &\times G(Q-p,i\omega_l+i\omega_\nu)\tau_3G(Q-p_1,i\omega_l+i\omega_\nu)\tau_3\cdots\tau_3\\ &\times G(Q-p_n,i\omega_l+i\omega_\nu)\tau_3]\ ,\end{aligned} \tag{3}$$

where $K_{\alpha\beta}$ relates the current response J_α to the applied electromagnetic potential A_β via

$$J_\alpha(q,\omega)=-K_{\alpha\beta}(q,\omega)A_\beta(q,\omega)\ . \tag{4}$$

From the normal-metal result, we expect the greatest contribution to come from zero total momentum Q, and so outside the trace we set $Q=0$. We also expect most contributions to $\int d^3p_i$ to come from near the Fermi surface, and so we substitute

$$\int d^3p_i/(2\pi)^3\rightarrow N(0)\int d\varepsilon\tfrac{1}{2}\int_{-1}^{1}dx_i,$$

where $N(0)$ is the density of states at the Fermi surface and $x_i=\cos\theta_i$, with θ_i the angle between p and Q. Finally, noting that, by isotropy, $p_\alpha(Q-p)_\beta\rightarrow-\frac{1}{3}k_F^2\delta_{\alpha\beta}$ and defining $K_{\alpha\beta}=K\delta_{\alpha\beta}$, we find

$$\begin{aligned}K^{(n)}(0,i\omega_\nu)=&-\frac{ne^2}{2m\beta}\sum_l\int\frac{d^3p}{(2\pi)^3}\frac{d^3Q}{(2\pi)^3}\frac{[N(0)\Gamma/2\pi]^{n+1}}{N(0)^2}\\ &\times\prod_{i=1}^{n}\int d\varepsilon_i\frac{dx_i}{2}\mathrm{Tr}[G'(p)G(p)\tau_3G_1\tau_3G_2\cdots\tau_3G_n\tau_3\\ &\times G(Q-p)G'(Q-p)\tau_3G_1'\tau_3G_2'\cdots\tau_3G_n'\tau_3]\ .\end{aligned} \tag{5}$$

Here all the G''s have the frequency $z'=i\omega_l+i\omega_\nu$ and all the G's have the frequency $z=i\omega_l$. Further, in G_n' the momentum is $Q-p_n$, and in G_n the momentum is p_n.

In order to do the p_i integration, one must move G_i next to G_i'. This would appear to be a problem because matrices do not necessarily commute. However, from the explicit form (2a), we easily see that the τ_3G's commute among themselves:

$$\begin{aligned}\tau_3G_i\tau_3G_j=&\frac{(\bar{z}+\varepsilon_i\tau_3-\bar{\Delta}\tau_1)(\bar{z}+\varepsilon_j\tau_3+\bar{\Delta}\tau_1)}{(\bar{z}^2-\varepsilon_i^2-\bar{\Delta}^2)(\bar{z}^2-\varepsilon_j^2-\bar{\Delta}^2)}\\ =&D^{-1}[(\bar{z}^2+\varepsilon_i\varepsilon_j-\bar{\Delta}^2)+\bar{z}(\varepsilon_i+\varepsilon_j)\tau_3+(\bar{z}\bar{\Delta}-\bar{z}\bar{\Delta})\tau_1+i(\varepsilon_i+\varepsilon_j)\bar{\Delta}\tau_2]\\ =&\tau_3G_j\tau_3G_i\ .\end{aligned} \tag{6}$$

The same is obviously true for the τ_3G''s. Thus we can rearrange the factors as follows:

$$\mathrm{Tr}[G_p'G_p(\tau_3G_1)(\tau_3G_2)\cdots(\tau_3G_n)(\tau_3G_{Q-p})\tau_3(\tau_3G_{Q-p}')(\tau_3G_n')\cdots(\tau_3G_2')(\tau_3G_1')\tau_3]$$
$$=\mathrm{Tr}[(\tau_3G_{Q-p}')\tau_3G_p'G_p(\tau_3G_{Q-p})(\tau_3G_1)(\tau_3G_2)\cdots(\tau_3G_n)\tau_3(\tau_3G_n')\cdots(\tau_3G_2')(\tau_3G_1')\tau_3]$$
$$=\mathrm{Tr}[(\tau_3G_{Q-p}'\tau_3G_p'G_p\tau_3G_{Q-p})G_1\tau_3G_2\cdots\tau_3G_nG_n'\tau_3\cdots\tau_3G_2'\tau_3G_1']\ . \tag{7}$$

Now, at the center, we have G_nG_n', and we can perform the p_n integral. We define the p_n integral by

$$C=(\Gamma/2\pi)\int_{-1}^{1}\frac{dx}{2}\int d\varepsilon\, G(p,z)G(Q-p,z)\ , \tag{8}$$

and evaluate this in the Appendix, arriving at

$$C=\alpha\left[1+\frac{z}{\Delta}\tau_1\right]\ , \tag{9}$$

where

$$\alpha=-\frac{z^2+z\Delta\tau_1}{[1-2i\tau(z^2-\Delta^2)^{1/2}](z^2-\Delta^2)}\times\left[1-\frac{DQ^2\tau}{[1-2i\tau(z^2-\Delta^2)^{1/2}]^2}\right]\ . \tag{10}$$

Above, D is the diffusion constant; $D=v_F^2\tau/d$ in d-dimensions. Unfortunately, our result for C, being of the form $\alpha[1+(z/\Delta)\tau_1]$, does not commute with τ_3. This prevents us from writing down the sum of all diagrams as a geometric series, such as occurs in the normal-state calculation, in C. However, carrying on with the calculation, we see that, in the trace, we now have

$$\cdots\tau_3G_{n-1}\tau_3\alpha\left[1+\frac{z}{\Delta}\tau_1\right]\tau_3G_{n-1}'\tau_3\cdots$$
$$=\cdots\tau_3G_{n-1}\alpha\left[1-\frac{z}{\Delta}\tau_1\right]G_{n-1}'\tau_3\cdots\ . \tag{11}$$

Using the properties of Pauli matrices to move G_{n-1} through the $\alpha[1-(z/\Delta)\tau_1]$ term yields

$$G_{n-1}\alpha\left[1-\frac{z}{\Delta}\tau_1\right]G_{n-1}'$$
$$=\alpha G_{n-1}G_{n-1}'-\alpha\frac{z}{\Delta}\tau_1\bar{G}_{n-1}G_{n-1}'\ , \tag{12}$$

where we define $\bar{G}$ by

$$\bar{G}=\frac{\bar{z}-\varepsilon\tau_3+\bar{\Delta}\tau_1}{\bar{z}^2-\varepsilon^2-\bar{\Delta}^2}\ . \tag{13}$$

Then performing the p_{n-1} integral yields $\alpha C-\alpha(z/\Delta)C'$, where C' is defined by

$$C'=(\Gamma/2\pi)\times\int_{-1}^{1}\frac{dx}{2}\int d\varepsilon\frac{(\bar{z}-\varepsilon_p\tau_3-\bar{\Delta}\tau_1)(\bar{z}+\varepsilon_{Q-p}\tau_3+\bar{\Delta}\tau_1)}{(\bar{z}^2-\varepsilon_p^2-\bar{\Delta}^2)(\bar{z}^2-\varepsilon_{Q-p}^2-\bar{\Delta}^2)}\ . \tag{14}$$

C' is also evaluated in the Appendix, giving the result

$$C'=\alpha\left[\frac{z^2}{\Delta^2}+\frac{z}{\Delta}\tau_1\right]\ . \tag{15}$$

In this way we can successively perform all the p_i integrals. For the $n=1$ diagram, we just found the contribution $C=\alpha[1+(z/\Delta)\tau_1]$, and for the $n=2$ diagram we now obtain

$$\alpha^2\left[1+\frac{z}{\Delta}\tau_1\right]-\alpha\frac{z}{\Delta}\tau_1\alpha\left[\frac{z^2}{\Delta^2}+\frac{z}{\Delta}\tau_1\right]$$
$$=\alpha^2\left[1-\frac{z^2}{\Delta^2}\right]\left[1+\frac{z}{\Delta}\tau_1\right]\ . \tag{16}$$

One can easily prove by induction that for the nth-order diagram, one gets

$$\alpha^n\left[1-\frac{z^2}{\Delta^2}\right]^{n-1}\left[1+\frac{z}{\Delta}\tau_1\right]\ . \tag{17}$$

So we get a geometric series after all and, adding it up, obtain

$$S=-\frac{\Delta^2\{[1-2i\tau(z^2-\Delta^2)^{1/2}]^2-DQ^2\tau\}[1+(z/\Delta)\tau_1]}{(z^2-\Delta^2)([1-2i\tau(z^2-\Delta^2)^{1/2}]^3-\{[1-2i\tau(z^2-\Delta^2)^{1/2}]^2-DQ^2\tau\})}\ . \tag{18}$$

Substituting this into Eq. (4), we get an expression for $K(0,0)$, which can also be written as n_se^2/m, thereby defining the superconducting number density:

$$\frac{n_se^2}{m}=-\frac{ne^2}{4\pi\tau mN(0)\beta}\sum_l\int\frac{d^3p}{(2\pi)^3}\frac{d^3Q}{(2\pi)^3}\mathrm{Tr}(\tau_3G_{Q-p}'\tau_3G_p'G_p\tau_3G_{Q-p}\tau_3S)\ . \tag{19}$$

This is calculated in the Appendix, giving

$$\frac{n_se^2}{m}=\frac{2\pi De^2\tau}{m\beta}\sum_Q\sum_l\frac{\Delta^2}{(z^2-\Delta^2)[1-2i\tau(z^2-\Delta^2)^{1/2}]^3}\frac{[1-2i\tau(z^2-\Delta^2)^{1/2}]^2-DQ^2\tau}{[1-2i\tau(z^2-\Delta^2)^{1/2}]^3-\{[1-2i\tau(z^2-\Delta^2)^{1/2}]^2-DQ^2\tau\}}\ . \tag{20}$$

In the dirty limit $\Gamma \gg \Delta$, we finally obtain, from the Appendix, the extremely simple result

$$\frac{\delta n_s(T)e^2}{m} = -\frac{4De^2}{\beta}\sum_l \frac{\Delta^2}{\omega_l^2+\Delta^2}\sum_Q \frac{1}{DQ^2+2(\omega_l^2+\Delta^2)^{1/2}} \ . \tag{21}$$

This is the correction to the number density of superconducting electrons due to weak-localization effects, the quantity we wished to evaluate. We should compare this to the normal-state result for the correction to the conductivity, given by

$$\delta\sigma(0) = -\frac{2De^2}{\pi}\sum_Q \frac{1}{DQ^2+\Gamma_i} \ . \tag{22}$$

In Eq. (22) the lower cutoff in Q is given by $DQ^2=\Gamma_i$, where Γ_i is the inelastic-scattering rate. In the superconducting case [Eq. (21)], the lower cutoff is given by $DQ^2=2\Delta$. We can include the inelastic scattering in the superconducting case by evaluating the frequency-dependent response,

$$K(0,i\omega_\nu) = -\frac{2De^2}{\beta}\sum_l \sum_Q \left[1+\frac{\Delta^2-\omega_l(\omega_l+\omega_\nu)}{(\omega_l^2+\Delta^2)^{1/2}[(\omega_l+\omega_\nu)^2+\Delta^2]}\right]\frac{1}{DQ^2+(\omega_l^2+\Delta^2)^{1/2}+[(\omega_l+\omega_\nu)^2+\Delta^2]^{1/2}} \ , \tag{23}$$

and replacing $i\omega_\nu \to \Gamma_i$ to obtain, for the case $\Gamma_i \gg \Delta$,

$$\frac{\delta n_s e^2}{m} = -\frac{4De^2}{\beta}\sum_l \frac{\Delta^2}{\omega_l^2+\Delta^2}\sum_Q \frac{1}{DQ^2+2(\omega_l^2+\Delta^2)^{1/2}+\Gamma_i} \ . \tag{24}$$

Thus, for the superconducting case, it appears that we have $\sum_Q 1/DQ^2$ with lower cutoff Q_1,

$$Q_1 = \begin{cases} 1/l_\phi \ , & \Gamma_i \gg \Delta \\ \sqrt{D/2\Delta} \ , & \Gamma_i \ll \Delta \ , \end{cases} \tag{25}$$

where $l_\phi=\sqrt{D\tau_i}$ is the inelastic diffusion length and $\tau_i=\Gamma_i^{-1}$ is the inelastic lifetime. Since our electron wave function incorporates elastic scattering off impurities, the lower length scale in the problem, which has disappeared because of the expansion in Q, is the elastic mean free path $l=v_F\tau$. This leads to an upper momentum cutoff $Q_2=1/l$. Performing this sum and using

$$\frac{1}{\beta}\sum_l \frac{1}{\omega_l^2+\Delta^2} = \frac{\tanh(\frac{1}{2}\beta\Delta)}{2\Delta} \ , \tag{26}$$

we obtain approximate $T=0$ expressions for $\delta n_s e^2/m$, which are listed in Table I.

At $T=0$ we can obtain the exact results by expanding the integral in terms of the small parameter Δ/Γ. The exact results are very similar to the ones in Table I in that the leading two terms are the same, but the numerical coefficients are different in some cases. This is to be expected, and we should also be wary about coefficients arising from the upper Q cutoff because of its arbitrary nature.

III. PHYSICAL MEANING

Equation (21) is the central result of our calculation. We shall now try to elucidate the physical meaning of this result and the formulas derived from it in Table I.

A. Conductivity sum rule

From our result for $\delta n_s(T)$ and the known results for the BCS superconducting number density,[17] the weak-localization correction to conductivity[1] $\delta\sigma(0)$, and the Drude conductivity $\sigma(0)$, we find that, at $T=0$ in the limit $\Gamma_i \gg \Delta$,

$$\frac{\delta n_s(T)}{\delta\sigma(0)} = \frac{n_s(T)}{\sigma(0)} = \frac{m\pi}{e^2}\Delta \ . \tag{27}$$

Thus we see that n_s still scales with $\sigma(0)$ when we include the weak-localization corrections in the limit $\Gamma_i \gg \Delta$, but not in the limit $\Gamma_i \ll \Delta$.

One can understand the scaling of n_s with $\sigma(0)$ by using a "missing-area" argument[18] based on the conductivity sum rule. In the superconducting case at $T=0$, the conductivity is zero up to $\omega=2\Delta$, while the normal conductivity is finite. In the superconducting case, therefore, one loses some area under the $\sigma(\omega)$ curve, which shows up as area in the $\delta(\omega)$ pole at the origin. In fact, the two

TABLE I. Formulas for correction to n_s, the number density of superconducting electrons.

Number of dimensions	$\delta n_s e^2/m$: $\Gamma_i \ll \Delta$	$\delta n_s e^2/m$: $\Gamma_i \gg \Delta$
0	$-\frac{2De^2}{\hbar}$	$-\frac{2De^2\Delta\tau_i}{\hbar}$
1	$-\frac{2e^2\Delta}{\pi\hbar}\left[\left[\frac{D}{2\Delta}\right]^{1/2}-l\right]$	$-\frac{2e^2\Delta}{\pi\hbar}(l_\phi-l)$
2	$\frac{e^2\Delta}{2\pi\hbar}\ln(2\Delta\tau)$	$-\frac{e^2\Delta}{2\pi\hbar}\ln\left[\frac{\tau_i}{\tau}\right]$
3	$-\frac{e^2\Delta}{\pi^2\hbar}\left[\frac{1}{l}-\left[\frac{2\Delta}{D}\right]^{1/2}\right]$	$-\frac{e^2\Delta}{\pi^2\hbar}\left[\frac{1}{l}-\frac{1}{l_\phi}\right]$

areas are proportional, not equal:

$$\frac{\pi n_s e^2}{2m} = \frac{\pi^2}{4} \int_0^{2\Delta} d\omega\, \sigma(\omega) \; . \tag{28}$$

For $\Gamma_i \gg \Delta$ this leads to direct proportionality between n_s and $\sigma(0)$, which persists even when weak localization is included. In the limit $\Gamma_i \ll \Delta$, one finds that $\delta\sigma(\omega)$ is not constant over the frequency region up to 2Δ—the conductivity varies on a frequency scale of order Γ_i, being given by

$$\delta\sigma(\omega) = -\frac{2De^2}{\pi} \int_0^{1/l} \frac{d^dQ}{(2\pi)^d} \frac{1}{DQ^2 - i\omega + \Gamma_i} \; . \tag{29}$$

This can be exactly evaluated for $d=0,1,2,3$, and the low- and high-frequency behavior is summarized in Table II.

One finds that the sum rule is still obeyed to leading order in Δ/Γ in the limit $\Gamma_i \ll \Delta$ for dimension $d \geq 2$. For $d<2$ the sum rule reproduces the correct functional form, but gives the wrong coefficient. This is because the frequency dependence in $\delta\sigma(\omega)$ goes as $\omega^{(d-2)/2}$, and integrating this up to 2Δ gives a correction term that is $O((\Delta/\Gamma)^{(d-2)/2})$, which is subdominant only for $d>2$. This argument does not apply to the $d=2$ case, which has logarithmic dependence, and so the success of the sum rule in this case is the only "nontrivial" case. Note that, in all cases, the sum rule always reproduces the correct functional forms—it is only numerical coefficients it get wrong. In short, our results can essentially be reproduced by the missing-area argument.

The correct physical limit is $\Gamma_i \ll \Delta$ as superconductivity cannot persist in the opposite limit. This is because the electrons which would form a Cooper pair are then only coherent over a distance $l_\phi = \sqrt{D\tau_i} \ll \xi = \sqrt{\xi_0 l} \approx \sqrt{D/\Delta}$, where ξ is the dirty coherence length and ξ_0 the BCS coherence length. Thus the electron wave functions are dephased over a region smaller than the size of a Cooper pair and the superconducting state cannot form. That this is the physical limit is confirmed in two-dimensional (2D) thin films where one finds that the experimental parameters satisfy $\Gamma \gg \Delta \gg \Gamma_i$. All of the results in the physical limit $\Gamma_i \ll \Delta$ are not temperature dependent at low temperature since Δ is flat here, and so this correction cannot easily be seen experimentally, unlike the situation for weak localization in the normal state.

We must also check that the elastic mean free path and inelastic diffusion lengths are the same in both the normal and superconducting states if we are going to compare formulas and limits directly. In the normal case, $l = v_F\tau$ is the elastic mean free path and $l_\phi = \sqrt{D\tau_i}$ is the inelastic diffusion length. Thus we need to see whether l and τ_i are different in normal and superconducting states. The elastic mean free path can easily be calculated from a Boltzmann-equation argument[19] and is found to be the same as in the normal case. The lifetime τ for low-lying quasiparticles increases in the superconducting state, but their velocity is lower than in the normal case by an exactly compensating factor:

$$\tau_s(\varepsilon_k) = \frac{E_k}{\varepsilon_k}\tau \; , \quad v_s(\varepsilon_k) = \frac{\varepsilon_k}{E_k} v_F \; ,$$
$$l_s(\varepsilon_k) = v_s(\varepsilon_k)\tau_s(\varepsilon_k) = v_F\tau = l \; . \tag{30}$$

The inelastic lifetime is more of a problem, because it depends on the particular inelastic scattering important in the system. In general, it will be expected to depend on the energy of the state being scattered. This introduces a calculational difficulty—the $\tau_i(\omega)$ has to be incorporated into the self-energy of the original Green functions. This problem has been encountered before in the study of thermal conductivity[20] in superconductors. There inelastic processes such as phonon scattering were introduced into the superconducting Green functions in an Eliashberg equation. One finds that, if we write

$$G(k,\omega) = \frac{1}{Z(\omega)\omega - \varepsilon_k\tau_3 - Z(\omega)\Delta(\omega)\tau_1} \; , \tag{31}$$

then the decay rate $\Gamma(\omega)$ is given by

$$Z_1\Gamma(\omega) = 2Z_2(\omega^2 - \Delta_1^2) - 2\Delta_1\Delta_2 Z_1 \; , \tag{32}$$

where $Z = Z_1 + iZ_2$ and $\Delta = \Delta_1 + i\Delta_2$. For the impurity Green function we used, this gives $\Gamma(\omega) = 1/\tau$, as one might expect. In general, the Green function will be very complicated and will probably only be known numerically. However, Ambegaokar and Woo[20] found that the decay rate due to inelastic phonon scattering in lead did not differ much between normal and superconducting cases, and so we may feel justified in using the same τ_i in both cases. Thus we feel confident in using the same parameters in both cases. Note that as long as superconductivity persists, we must still have the limit $\Gamma_i \ll \Delta$.

TABLE II. Formulas for correction to $\sigma(\omega)$, the frequency-dependent conductivity.

Number of Dimensions	$\delta\sigma(\omega)$ $\omega \ll \Gamma_i$	$\omega \gg \Gamma_i$
0	$-\frac{2De^2}{\pi\hbar}\frac{1}{\Gamma_i}$	$-\frac{2De^2}{\pi\hbar}\frac{\Gamma_i}{\omega^2}$
1	$-\frac{2e^2}{\pi^2\hbar}(l_\phi - l)$	$-\frac{2e^2}{\pi^2\hbar}\left[\left[\frac{D}{\omega}\right]^{1/2} - l\right]$
2	$-\frac{e^2}{2\pi^2\hbar}\ln(\frac{\tau_i}{\tau})$	$\frac{e^2}{2\pi^2\hbar}\ln(\omega\tau)$
3	$-\frac{e^2}{\pi^3\hbar}(\frac{1}{l} - \frac{1}{l_\phi})$	$-\frac{e^2}{\pi^3\hbar}\left[\frac{1}{l} - \left[\frac{\omega}{D}\right]^{1/2}\right]$

B. Physical picture

The weak-localization phenomena in superconductors can be understood in a more physical manner by using the picture developed by Bergmann[1] for weak localization in normal metals. This picture interprets the Langer-Neal diagrams as the interference terms between two processes involving the coherent backscattering of electrons—the scattering of electrons in a momentum state k to the momentum state $-k$. In the superconducting case, we can think of the localization as being due to the backscattering of the "bare" Cooper pairs. The backscattering process reduces the ability of these bare Cooper pairs to carry supercurrent and so reduces n_s. The conductivity diagram is one of a family of diagrams required for charge conservation in which current vertices are inserted into the maximally crossed diagrams in all ways. These other diagrams are of higher order and so need not be evaluated in our lowest-order calculation, but they clearly show the Cooper-pair propagator being dressed by backscattering processes to yield the final "dressed" Cooper pairs. It is the latter objects that propagate without scattering.

Calculating the change in n_s for such a process should lead to a term proportional to $\sum_Q 1/DQ^2$ as occurs in the normal case. Then, for example, in 2D we end up with a term of the form $\ln(Q_2/Q_1)$, where Q_1 and Q_2 are the upper and lower momentum cutoffs. The appropriate cutoffs then have to be provided by the theory.

In both the normal and superconducting states, the lower length scale is the elastic mean free path as this is the distance an electron moves before making a collision—or, put another way, the spatial size of the single-electron wave function. Thus the upper momentum cutoff is $Q_2=1/l$. In the normal state the upper length scale is the distance an electron diffuses before making a dephasing inelastic collision, $l_\phi=\sqrt{D\tau_i}$, which leads to a lower momentum cutoff of $Q_1=1/l_\phi$. This leads to a weak-localization correction to the conductivity in the 2D case proportional to $-\ln(\tau_i/\tau)$, as one finds from the diagrams. In the superconducting case the upper length scale is the size of the superconductor two-electron wave function, the coherence length $\xi=\sqrt{\xi_0 l}\approx\sqrt{D/\Delta}$, leading to a lower momentum cutoff $Q_1=1/\xi$. This then leads to a δn_s term proportional to $\ln(\Delta\tau)$ as was found earlier. Similar results obtain in all dimensions, since this picture just works by setting the appropriate length scales.

C. Exact eigenstate formula

For a system of particles in the presence of a one-body potential, such as that due to a set of impurities, the Hamiltonian is separable and we end up with a set of independent electrons with a energy spectrum ε_m. These levels are filled up in order of increasing energy up to some Fermi energy. Both the normal-state conductivity and the number density of superconducting electrons can then be related to matrix elements of the current operator and, thus, to each other. We then have a relation from which we can calculate superconducting properties given the normal-state ones.[21,22]

We will now derive the appropriate relations. The Green function for our system is

$$G(\mathbf{r},\mathbf{r}';i\omega_l)=\frac{\phi(\mathbf{r})\phi(\mathbf{r}')}{i\omega_l-\varepsilon_m}\ , \tag{33}$$

and we can substitute this into the usual formulas for conductivity[23] to get

$$K_{\alpha\beta}(\mathbf{r},\mathbf{r}',i\omega_\nu)=\left[\frac{e}{2m}\right]^2\frac{1}{\beta}\sum_l\sum_{m,n}\frac{1}{i\omega_l-\varepsilon_m}\,\frac{1}{i\omega_l+i\omega_\nu-\varepsilon_n}p_{nm\alpha}(\mathbf{r})p_{nm\beta}(\mathbf{r}')\ , \tag{34}$$

where we define

$$p_{nm\alpha}(\mathbf{r})=\phi_n(\mathbf{r})\frac{\partial}{\partial r_\alpha}\phi_m(\mathbf{r})-\phi_m(\mathbf{r})\frac{\partial}{\partial r_\alpha}\phi_n(\mathbf{r})\ . \tag{35}$$

We now approximate the energy levels by a continuous spectrum and change variables to $\varepsilon=\varepsilon_n$, $\varepsilon'=\varepsilon_m-\varepsilon_n$. Finally, setting $\sum_n\to 2N(0)\int d\varepsilon$ gives

$$K_{\alpha\beta}(\mathbf{r};\mathbf{r}';i\omega_\nu)=\left[\frac{e}{2m}\right]^2 2N(0)\frac{1}{\beta}\sum_l\sum_m\int d\varepsilon\frac{1}{i\omega_l-\varepsilon-\varepsilon'}\,\frac{1}{i\omega_l+i\omega_\nu-\varepsilon}p_{nm\alpha}(\mathbf{r})p_{nm\beta}(\mathbf{r}')\ . \tag{36}$$

Now, for the ε integration to be nonzero, we need ω_l and $\omega_l+\omega_\nu$ to have opposite sign, so that

$$K_{\alpha\beta}(\mathbf{r},\mathbf{r}';i\omega_\nu)=\left[\frac{e}{2m}\right]^2 2N(0)\frac{1}{\beta}\sum_l\sum_m 2\pi i\frac{1}{-i\omega_\nu-\varepsilon'}\Theta(-\omega_l)\Theta(\omega_l+\omega_\nu)p_{nm\alpha}(\mathbf{r})p_{nm\beta}(\mathbf{r}')\ , \tag{37}$$

and since

$$\frac{1}{\beta}\sum_l\Theta(-\omega_l)\Theta(\omega_l+\omega_\nu)=\frac{\omega_\nu}{2\pi} \tag{38}$$

and $\sigma(i\omega_\nu)=K(i\omega_\nu)/i\omega_\nu$, we finally obtain

$$\sigma_{\alpha\beta}(\mathbf{r},\mathbf{r}',i\omega_\nu)=\frac{N(0)e^2}{2m^2}\sum_m\frac{1}{i\omega_\nu+\varepsilon_m-\varepsilon_n}p_{nm\alpha}(\mathbf{r})p_{nm\beta}(\mathbf{r}')\ . \tag{39}$$

To get the zero-momentum conductivity, we simply average over all impurity configurations and over all spatial positions $\mathbf{r}$.

In the superconducting case we have a similar formula for the response function K,

$$K_{\alpha\beta}(\mathbf{r},\mathbf{r}';0)=\left(\frac{e}{2m}\right)^2\frac{1}{\beta}\sum_l\sum_{m,n}\mathrm{Tr}\left[\frac{1}{i\omega_l-\varepsilon_m\tau_3-\Delta\tau_1}\,\frac{1}{i\omega_l-\varepsilon_n\tau_3-\Delta\tau_1}\right]p_{nm\alpha}(\mathbf{r})p_{nm\beta}(\mathbf{r}')\ , \tag{40}$$

where the trace is over the Nambu-G'orkov matrix elements. Making the same substitutions as before leads to

$$K_{\alpha\beta}(\mathbf{r},\mathbf{r}';0)=2N(0)\left(\frac{e}{2m}\right)^2\frac{1}{\beta}\sum_l\sum_m\int d\varepsilon\frac{-\omega_l^2+(\varepsilon+\varepsilon')\varepsilon+\Delta^2}{(\varepsilon^2+\Delta^2+\omega_l^2)[(\varepsilon+\varepsilon')^2+\Delta^2+\omega_l^2]}p_{nm\alpha}(\mathbf{r})p_{nm\beta}(\mathbf{r}')\ . \tag{41}$$

Performing the ε integral gives the result

$$\frac{\pi\Delta^2}{i(\omega_l^2+\Delta^2)}\left[\frac{1}{2i(\omega_l^2+\Delta^2)^{1/2}+\varepsilon_m-\varepsilon_n}+\frac{1}{2i(\omega_l^2+\Delta^2)^{1/2}+\varepsilon_n-\varepsilon_m}\right], \tag{42}$$

and since these two terms give equal results after summing over ε_m, we end up with

$$K_{\alpha\beta}(\mathbf{r},\mathbf{r}';0)=\frac{N(0)e^2}{2m^2}\left(\frac{2\pi i}{\beta}\right)\sum_l\sum_m\frac{\Delta^2}{\omega_l^2+\Delta^2}\,\frac{1}{2i(\omega_l^2+\Delta^2)^{1/2}+\varepsilon_m-\varepsilon_n}p_{nm\alpha}(\mathbf{r})p_{nm\beta}(\mathbf{r}')\ . \tag{43}$$

When this is averaged over all impurity configurations and all spatial positions $\mathbf{r}$, this just yields n_se^2/m. We see that

$$\frac{n_se^2}{m}=\frac{2\pi}{\beta}\sum_l\frac{\Delta^2}{\omega_l^2+\Delta^2}\sigma[2i(\omega_l^2+\Delta^2)^{1/2}]\ . \tag{44}$$

By identical means we find a formula for $K(q=0,i\omega_\nu)$:

$$K(0,i\omega_\nu)=\frac{\pi}{\beta}\sum_l\left[1+\frac{\Delta^2-\omega_l(\omega_l+\omega_\nu)}{(\omega_l^2+\Delta^2)^{1/2}[(\omega_l+\omega_\nu)^2+\Delta^2]^{1/2}}\right]\sigma\{i(\omega_l^2+\Delta^2)^{1/2}+i[(\omega_l+\omega_\nu)^2+\Delta^2]^{1/2}\}\ . \tag{45}$$

If we now insert the known form for the frequency-dependent normal-state conductivity [Eq. (29)] into Eqs. (44) and (45), we reproduce our earlier results [Eqs. (21) and (23)]. The exact eigenstate method can be used to obtain any superconducting result from the corresponding normal result in the case of a system with a one-body potential—however, it cannot be used to discuss phonon and Coulomb interactions, whereas the diagrammatic approach can. Thus the diagrammatic approach is not made obsolete by the exact-eigenstate approach. Equivalent exact-eigenstate formulas have been used in previous papers by Ma and Lee,[5] Kotliar and Kapitulnik,[11] and Ramakrishnan.[22]

IV. CONCLUSIONS

We have proved that there is indeed a weak-localization correction to the number density of superconducting electrons. It does not have a strong temperature dependence at low temperature as a result of the upper length scale being the coherence length rather than the inelastic dephasing length. Let us see at which point we would naively expect the extinction of superconductivity. In the case of a 2D film, n_s is given by

$$n_s=\pi n\Delta\tau+\frac{m\Delta}{2\pi}\ln(4\Delta\tau)\ . \tag{46}$$

If we assume this holds in the strong-localization region, as we increase the elastic-scattering rate Γ, n_s decreases and equals zero at a value of Γ equivalent to normal-state resistance,

$$R_\square=\frac{\pi^2}{2}\left(\frac{\hbar}{e^2}\right)\Big/\ln\left(\frac{E_F}{2\Delta}\right), \tag{47}$$

similar to Eq. (15) in Ref. 22. Similarly, in the 3D case we have, for n_s,

$$n_s=\pi n\Delta\tau-\frac{m\Delta}{\pi^2 l}\ , \tag{48}$$

giving a normal-state conductivity of

$$\sigma=\frac{1}{(3\pi^5)^{1/2}}\left(\frac{e^2}{\hbar}\right)k_F\ , \tag{49}$$

for destruction of superconductivity. The results [Eqs. (47) and (49)] should be compared to the results corresponding to the Ioffe-Regel criterion[24] in 2D and 3D:

$$\sigma_{\min}^{2D}=\left(\frac{1}{2\pi}\right)\frac{e^2}{\hbar}\ ,\qquad \sigma_{\min}^{3D}=\left(\frac{1}{3\pi^2}\right)\frac{e^2}{\hbar}k_F\ . \tag{50}$$

We see that the results have the same scale in both cases—this is not surprising since these are the only dimensionally correct quantities in the strong-localization limit. If we could believe the above formulas, this would indicate that localization phenomena in metals and superconductors should occur on the same resistance scale.

The simple arguments given above are not valid, since we are using a weak-perturbation result far beyond its realm of validity. A quick glance at the exact-eigenstate formula shows that given any nonzero normal-state resistance, we will obtain a nonzero n_s. This is because we have ignored fluctuations in the amplitude of the order parameter Δ in deriving the exact-eigenstate formula. Such fluctuations have also been ignored in the diagrammatics since we have not considered how impurities affect the magnitude of Δ, which would involve solving an Eliashberg equation in the presence of impurities. Takagi and Kuroda[25] have performed such a calculation, but a fully self-consistent procedure is lacking. However, as Ramakrishnan[22] has pointed out, the above arguments do yield the resistance scale at which crossover of superconducting coherence and localization lengths occurs. At this point local fluctuations in the order parameter become important, and it is these which finally extinguish the superconductivity.

It seems to be borne out that localization in the superconducting case occurs on the resistance scales predicted above by our simple model. There seem to be essentially two types of experimental systems—granular and homogeneous.[26] In the homogeneous case T_c decreases as the normal-state resistance increases until superconductivity is extinguished; in the granular case T_c does not vary much, but the phase coherence between grains is destroyed, leading to loss of superconductivity. In other words, in the homogeneous case the amplitude of the order parameter is depressed; in the granular case its phase coherence is destroyed. Both granular[6–8] and homogeneous[9,10] thin films show a transition to insulating behavior at resistance $R_\square$ of order 10 kΩ. It has been suggested[9,27] that both types of films may make the transition at the universal resistance $R_\square = h/4e^2$. Similarly, both granular[28] and homogeneous[29] bulk superconductors show suppression of superconductivity when the normal-state resistivity ρ is of order ρ_M, the Mott resistivity.

It has been suggested[9,30,27] that both types of thin-film superconductors may make the superconducting-to-insulating transition at a universal resistance of order $R_\square = h/4e^2$. Our theory does not agree with this as the expression for the transition resistance is sample dependent. The data for thin homogenous films of Pb and Al in Ref. 9 suggest nonuniversality, but the whole question of universality remains an open one.

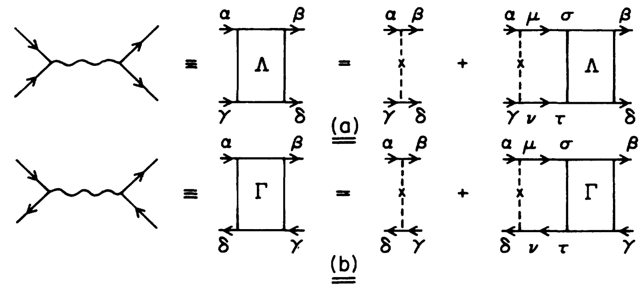

FIG. 2. (a) Cooperon and (b) the diffusion propagators in the superconducting case.

The weak-localization results derived above can be regarded as the onset of the transition from superconductor to insulator, in that they show how superconducting transport is depressed by backscattering processes. These processes are not sufficient to extinguish fully superconductivity—this can only be accomplished by including fluctuations in the order-parameter amplitude. This requires a self-consistent analysis of localization effects on the superconducting interaction itself, which will be the subject of future work.

ACKNOWLEDGMENTS

One of us (V.A.) thanks the Aspen Center for Physics and the AT&T Bell Laboratories for hospitality. We particularly thank A. J. Millis, L. S. Levitov, and T. V. Ramakrishnan for their comments on earlier versions of this work. We also thank R. A. Buhrman, A. J. Sievers, and R. C. Dynes for useful discussions. V.A. receives partial support from the NSF under Grant No. DMR-88 15828. RAS is grateful for financial support from the Olin Foundation.

APPENDIX

Calculating C and C'

We first evaluate C, which is given by

$$C = (\Gamma/2\pi)\int d\varepsilon \int_{-1}^{1} \frac{dx}{2} G(k,z)G(Q-k,z) . \quad \text{(A1)}$$

Now, using the expansion of ε_{Q-k} for small Q near the Fermi surface

$$\varepsilon_{Q-k} = \frac{(Q-k)^2}{2m} \approx \frac{k^2}{2m} - \frac{Q\cdot k}{m} = \varepsilon_k - Qv_F x , \quad \text{(A2)}$$

we get

$$\begin{aligned} G(k,z)G(Q-k,z) &= \frac{\bar{z}+\varepsilon_k\tau_3+\bar{\Delta}\tau_1}{\bar{z}^2-\varepsilon_k^2-\bar{\Delta}^2}\frac{\bar{z}+\varepsilon_{Q-k}\tau_3+\bar{\Delta}\tau_1}{\bar{z}^2-\varepsilon_{Q-k}^2-\bar{\Delta}^2} \\ &= \frac{(\bar{z}+\varepsilon_k\tau_3+\bar{\Delta}\tau_1)[\bar{z}+(\varepsilon_k-Qv_Fx)\tau_3+\bar{\Delta}\tau_1]}{(\varepsilon_k^2-\bar{z}^2+\bar{\Delta}^2)[(\varepsilon_k-Qv_Fx)^2-\bar{z}^2+\bar{\Delta}^2]} \\ &= \frac{[\bar{z}^2+\varepsilon(\varepsilon-Qv_Fx)+\bar{\Delta}^2]+\bar{z}(2\varepsilon-Qv_Fx)\tau_3+2\bar{z}\bar{\Delta}\tau_1+i(Qv_Fx)\bar{\Delta}\tau_2}{(\varepsilon^2-\bar{z}^2+\bar{\Delta}^2)[(\varepsilon-Qv_Fx)^2-\bar{z}^2+\bar{\Delta}^2]} \end{aligned} \quad \text{(A3)}$$

so that

$$C=\frac{1}{2\pi\tau}\int_{-1}^{1}\frac{dx}{2}\int d\varepsilon\frac{[\bar{z}^2+\varepsilon(\varepsilon-Qv_Fx)+\bar{\Delta}^2]+\bar{z}(2\varepsilon-Qv_Fx)\tau_3+2\bar{z}\bar{\Delta}\tau_1+i(Qv_Fx)\bar{\Delta}\tau_2}{(\varepsilon^2-\bar{z}^2+\bar{\Delta}^2)[(\varepsilon-Qv_Fx)^2-\bar{z}^2+\bar{\Delta}^2]} . \tag{A4}$$

The τ_2 and τ_3 components vanish as a result of oddness in the integrands, and upon performing the ε integral in the normal manner we obtain

$$C=-\frac{i}{2\varepsilon_1\tau}\int_{-1}^{1}\frac{dx}{2}\,\frac{\bar{\Delta}^2+2\bar{\Delta}\bar{z}\tau_1}{2\varepsilon_1[\varepsilon_1^2-\frac{1}{4}(Qv_Fx)^2]} , \tag{A5}$$

where we define $\varepsilon_1=(\bar{z}^2-\bar{\Delta}^2)^{1/2}$ with a positive imaginary part. We then expand this for small Q, since most contribution will come from near $Q=0$, to get

$$C=-\frac{i}{2\varepsilon_1^3\tau}\int_{-1}^{1}\frac{dx}{2}(\bar{\Delta}^2+\bar{\Delta}\bar{z}\tau_1)\left[1+\frac{(Qv_Fx)^2}{4\varepsilon_1^2}\right]$$

$$=-\frac{\Delta^2+z\Delta\tau_1}{[1-2i\tau(z^2-\Delta^2)^{1/2}](z^2-\Delta^2)}\left[1-\frac{DQ^2\tau}{[1-2i\tau(z^2-\Delta^2)^{1/2}]^2}\right] , \tag{A6}$$

where we have used

$$(\bar{z}^2-\bar{\Delta}^2)^{1/2}=(z^2-\Delta^2)^{1/2}+\frac{i}{2\tau} , \quad \bar{z}/\bar{\Delta}=z/\Delta . \tag{A7}$$

We next evaluate C' in the same manner as we evaluated C. By definition,

$$C'=\frac{1}{2\pi\tau}\int_{-1}^{1}\frac{dx}{2}\int d\varepsilon\frac{(\bar{z}-\varepsilon\tau_3+\bar{\Delta}\tau_1)[\bar{z}+(\varepsilon-Qv_Fx)\tau_3+\bar{\Delta}\tau_1]}{(\varepsilon^2-\bar{z}^2+\bar{\Delta}^2)[(\varepsilon-Qv_Fx)^2-\bar{z}^2+\bar{\Delta}^2]} , \tag{A8}$$

which can be expanded to give

$$C'=\frac{1}{2\pi\tau}\int_{-1}^{1}\frac{dx}{2}\int d\varepsilon\frac{[\bar{z}^2-\varepsilon(\varepsilon-Qv_Fx)+\bar{\Delta}^2]-(Qv_Fx)\bar{z}\tau_3+2\bar{z}\bar{\Delta}\tau_1-i(2\varepsilon-Qv_Fx)\bar{\Delta}\tau_2}{(\varepsilon^2-\bar{z}^2+\bar{\Delta}^2)[(\varepsilon-Qv_Fx)^2-\bar{z}^2+\bar{\Delta}^2]} . \tag{A9}$$

Again, the τ_2 and τ_3 components both vanish because of oddness in the integrands. Upon evaluating the ε integrals, we obtain

$$C'=-\frac{i}{2\varepsilon_1\tau}\int_{-1}^{1}\frac{dx}{2}\,\frac{\bar{z}^2+\bar{z}\bar{\Delta}\tau_1}{\varepsilon_1^2-\frac{1}{4}(Qv_Fx)^2} , \tag{A10}$$

which is exactly the same as C except $\bar{\Delta}^2+\bar{z}\bar{\Delta}\tau_1\rightarrow\bar{z}^2+\bar{z}\bar{\Delta}\tau_1$. Thus we finally obtain, after making the small Q expansion,

$$C'=-\frac{z^2+2z\Delta\tau_1}{[1-2i\tau(z^2-\Delta^2)^{1/2}][(z^2-\Delta^2)^{1/2}]}\left[1-\frac{DQ^2\tau}{[1-2i\tau(z^2-\Delta^2)^{1/2}]^2}\right] . \tag{A11}$$

Details of n_s calculation

From Sec. II we have

$$\frac{n_se^2}{m}=-\frac{ne^2}{4\pi\tau mN(0)}\sum_l\int\frac{d^3p}{(2\pi)^3}\frac{d^3Q}{(2\pi)^3}\,\mathrm{Tr}(\tau_3G'_{Q-p}\tau_3G'_pG_p\tau_3G_{Q-p}\tau_3S) , \tag{A12}$$

and performing the trace gives us

$$\mathrm{Tr}(\cdots)=-\frac{2\alpha}{(\varepsilon^2-\varepsilon_1^2)^2} . \tag{A13}$$

Performing the ε integral gives

$$\frac{n_se^2}{m}=\frac{2\pi De^2}{m\beta}\sum_Q\sum_l\frac{\Delta^2}{(z^2-\Delta^2)[1-2i\tau(z^2-\Delta^2)^{1/2}]^3}\,\frac{[1-2i\tau(z^2-\Delta^2)^{1/2}]^2-DQ^2\tau}{[1-2i\tau(z^2-\Delta^2)^{1/2}]^3-\{[1-2i\tau(z^2-\Delta^2)^{1/2}]^2-DQ^2\tau\}} . \tag{A14}$$

In the dirty limit, we set $\tau \to 0$, and so the sum becomes

$$\frac{n_s e^2}{m} = -\frac{4De^2}{\beta} \sum_l \frac{\Delta^2}{\omega_l^2+\Delta^2} \sum_Q \frac{1}{DQ^2+2(\omega_l^2+\Delta^2)^{1/2}} \ . \tag{A15}$$

$1/k_F l$ expansion

It is well known in the theory of weak localization that, in two dimensions, diagrammatic series form an expansion in the small parameter $1/k_F l$. [In $d \neq 2$ dimensions, the expansion parameter is $(1/k_F l)^{d-1}$.] To prove this we group diagrams so that we have an expansion in terms of diffusion and cooperon ladders and find that the order of a given term in the expansion parameter is the number of loops in the diagram. One then has only to calculate the interaction vertices for the new propagators. This is been done by Hikami[31] in order to map the localization problem onto a nonlinear σ model.

In the superconducting case diffusion and cooperon ladders add up in the same manner to yield effective propagators $\Gamma_{\alpha\beta\gamma\delta}$ and $\Lambda_{\alpha\beta\gamma\delta}$, which now have four matrix indices, as shown in Fig. 2. To calculate Λ, for example, we see that it satisfies

$$\Lambda_{\alpha\beta\gamma\delta} = \Gamma^0_{\alpha\beta\gamma\delta} + \Gamma^0_{\alpha\mu\gamma\nu}\Pi_{\mu\sigma\nu\tau}\Lambda_{\sigma\beta\tau\delta} \ , \tag{A16}$$

where

$$\Gamma^0_{\alpha\beta\gamma\delta} = \frac{1}{2\pi N(0)\tau}(\tau_3)_{\alpha\beta}(\tau_3)_{\gamma\delta} \tag{A17}$$

and

$$\Pi_{\mu\sigma\nu\tau} = \int \frac{d^3p}{(2\pi)^3} G_{\mu\sigma}(p,i\omega_l)G_{\nu\tau}(Q-p,i\omega_l+i\omega_\nu) \ . \tag{A18}$$

Solving the above yields

$$\Lambda = \frac{1}{4\pi N(0)\tau^2} \frac{1}{DQ^2+(\omega_l^2+\Delta^2)^{1/2}+[(\omega_l+\omega_\nu)^2+\Delta^2]^{1/2}} \left[\tau_3 \otimes \tau_3 - \frac{(z\tau_0-\Delta\tau_1)\otimes(z'\tau_0-\Delta\tau_1)}{(\omega_l^2+\Delta^2)^{1/2}[(\omega_l+\omega_\nu)^2+\Delta^2]^{1/2}} \right] , \tag{A19}$$

with a similar result for the diffusion case. Apart from the matrix structure and cutoff at energy 2Δ, this is of the same form as in the normal case and leads to a $1/k_F l$ expansion. The interaction vertices are affected only in that they pick up a matrix structure and smaller terms of order $\Delta\tau$—their order in $1/k_F l$ is unaffected.

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