Vortices and the Couette flow of helium II

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The critical Reynolds number at which the Couette flow of helium II becomes unstable is calculated at different temperatures from a linear stability analysis of the Hall-Vinen-Bekeravich-Khalatnikov equations. The results are in good agreement with a recent experiment of Swanson and Donnelly. The spatial structure of the flow at the onset is investigated and it is found to be very different from classical Taylor vortex flow.

Our concern is the stability of the Couette flow of helium II. Although this flow has been considered an important problem in fluid mechanics[1] since the pioneering work of Chandrasekhar and Donnelly in 1957,[2] progress has been slower than in classical Taylor-Couette flow, the reasons being the lack of direct flow visualization and the difficulties in understanding the governing equations of helium at temperatures below the lambda transition $T_\lambda = 2.172$ K. While Taylor vortices can be studied on the solid ground of the Navier-Stokes equation, much less is known about the Hall-Vinen-Bekeravich-Khalatnikov (HVBK) equations[3,4] which have been proposed to describe the motions of helium II. The HVBK equations, presented in modern form by Hills and Roberts,[5] generalize Landau's two-fluid model to situations in which quantized vortex lines are present in the flow and the macroscopic superfluid velocity $\mathbf{v}^s$ has $\nabla \times \mathbf{v}^s \neq 0$. The HVBK model adds two important physical ingredients to Landau's. The first is the mutual friction force[6] which couples normal-fluid and superfluid components via the vortex lines. It is a quantity which depends on temperature because it arises from the collisions between the cores of the superfluid vortices and the thermal excitations which make up the normal fluid. The second ingredient is the vortex tension[6] which causes vortex waves. Its relevance to the stability of the flow was recognized in 1963 by Mamaladze and Matinyan,[7] who studied the problem at temperature $T=0$. Since the relative normal-fluid and superfluid fractions are strong functions of temperature, it is clear that T is an important factor in determining the stability. The temperature also adds an extra variable to the parameter space of the classical Taylor-Couette problem and makes more difficult the comparison of the data available from different experiments.

A first attempt to use the HVBK equations to compute the stability at finite T and compare it with an experiment was carried out by Snyder[8] in 1974 with a negative result. In 1987 Barenghi and Jones[9] corrected Mamaladze and Matinyan's theory at $T=0$ and in a subsequent paper,[10] in 1988, they extended their work to finite T to analyze the results of the experiments of Donnelly[11] (1959) and of Wolf *et al.*[12] (1981). In these experiments two different methods were used. Donnelly detected the torque which is induced on the outer stationary cylinder by the rotating inner cylinder. A break in the relation between the Reynolds number of the inner cylinder and the torque at the single temperature $T=2.1$ K was found to be in promising agreement with the HVBK model. Wolf *et al.* used second sound, which seems the ideal technique because it probes the vortex lines directly. Unfortunately, analysis of their results showed a disagreement of more than one order of magnitude with Donnelly's. This discrepancy has been resolved very recently by Swanson and Donnelly,[13] who performed a series of careful second-sound measurements at temperatures which include the region just below T_λ. They found that the critical Reynolds number approaches the known value for the onset of classical Taylor vortex flow as $T \to T_\lambda$. This is a definite test because at $T=T_\lambda$ helium II becomes helium I, which is a classical Navier-Stokes fluid. Very probably Wolf *et al.*, who detected a number of successive transitions at increasing Reynolds number, missed the first bifurcation which destabilizes Couette flow.

The calculations presented in this article are motivated by Swanson and Donnelly's recent work. We want to settle the question of the quantitative agreement between the HVBK model and the experimental data at different values of temperature. Until now the evidence in favor of the HVBK model is based on a point at a single temperature in Donnelly's old 1959 experiment. Knowing the difficulty of flow visualization in helium II, we also want to explore the spatial configuration of the flow and compare it with its classical counterpart.

Let us consider two concentric cylinders of inner radius R_1, outer radius R_2, radius ratio $\eta = R_1/R_2$, and height h. The gap between the cylinders has width $\delta = R_2 - R_1$ and is filled with helium II at temperature T. We assume that the inner cylinder rotates at constant angular velocity Ω while the outer cylinder is stationary. At slow enough velocity helium is in a vortex-free state. Vortices appear[14] at the critical velocity

$$\Omega^* = [(1-\eta^2)\Gamma/\eta^2\pi\delta^2]\ln(2\delta/\pi a_0) ,$$

where Γ is the quantum of circulation and a_0 is the vortex core radius parameter. At $\Omega > \Omega^*$ helium is in the

Couette state. Using cylindrical coordinates (r,φ,z), the normal-fluid and superfluid velocities $\mathbf{v}^n$ and $\mathbf{v}^s$ are $\mathbf{v}^n=\mathbf{v}^s=(0,V,0)$ with $V=Ar+B/r$, $A=-R_1^2\Omega/(R_2^2-R_1^2)$, and $B=R_1^2R_2^2\Omega/(R_2^2-R_1^2)$. The vortex lines are aligned along the z axis with areal density $n=2|A|/\Gamma$ and distance $b=n^{-1/2}$ from each other. It is the stability of this Couette state which we investigate. We make use of the governing incompressible HVBK equations[10]

$$(\partial/\partial t)\mathbf{v}^n+(\mathbf{v}^n\cdot\nabla)\mathbf{v}^n=-\nabla\sigma_n+\nu_n\nabla^2\mathbf{v}^n+(\rho_2/\rho)\mathbf{F}\ ,$$

$$(\partial/\partial t)\mathbf{v}^s+(\mathbf{v}^s\cdot\nabla)\mathbf{v}^s=-\nabla\sigma_s+\nu_s\boldsymbol{\omega}^s\times\mathrm{curl}\hat{\boldsymbol{\omega}}^s-(\rho_n/\rho)\mathbf{F}\ ,$$

$$\nabla\cdot\mathbf{v}^n=0 \text{ and } \nabla\cdot\mathbf{v}^s=0\ ,$$

where

$$\mathbf{F}=\tfrac{1}{2}B\hat{\boldsymbol{\omega}}^s\times[\boldsymbol{\omega}^2\times(\mathbf{v}^n-\mathbf{v}^s-\nu_s\mathrm{curl}\hat{\boldsymbol{\omega}}^s)]+\tfrac{1}{2}B'\boldsymbol{\omega}^s\times(\mathbf{v}^n-\nu^s\mathrm{curl}\hat{\boldsymbol{\omega}}^s)$$

is the mutual friction force, $\nu_s=(\Gamma/4\pi)\ln(b/a_0)$ is the vortex tension, ρ_n and ρ_s are the normal-fluid and superfluid densities, $\rho=\rho_n+\rho_s$ is the helium density, η_n is the viscosity of the normal fluid and $\nu_n=\eta_n/\rho_n$ its kinematic viscosity, σ_n and σ_s are efficient pressures, $\boldsymbol{\omega}^s=\mathrm{curl}\mathbf{v}^s$ is the superfluid vorticity and $\hat{\boldsymbol{\omega}}^s=\boldsymbol{\omega}^s/|\boldsymbol{\omega}^s|$. The values of the temperature-dependent parameters B, B', a_0, ν_n, ρ, ρ_s, and ρ_n are discussed in the review by Barenghi, Donnelly, and Vinen.[6] The boundary conditions are that $v_r^n=v_r^s=v_z^n=0$ at $r=R_1$ and R_2, $v_\varphi^n=\Omega R_1$ at $r=R$, and $v_\varphi^n=0$ at $r=R_2$.

We make the usual simplifying assumption that the cylinders have infinite length and investigate the effects of small perturbations of the Couette flow solution having the form $\exp(ik'z+im\varphi+pt)$. We can make the simplifying assumption $m=0$ because we know from our previous investigation[10] that at least for T just below T_λ the most unstable perturbations are axisymmetric, as in the classical Taylor-Couette case. We linearize the HVBK equations around the Couette flow solution and derive four ordinary differential equations in r for v_r^n, v_φ^n, v_r^s, and v_φ^s, which we solve at given geometry, temperature, Reynolds number $N_{\mathrm{Re}}=\Omega R_1\delta/\nu_n$ and dimensionless axial wave number $k=k'\delta$ to determine the eigenvalue p. If $\mathrm{Re}(p)>0$ then Couette flow is unstable. The method of solution has already been described;[10] it suffices to say that it is based on a spectral collocation method, in which the perturbations are expanded over Chebyshev polynomials and the resulting linear system is solved by an NAG routine. Typically we truncate the expansions after 16 polynomials.

Figure 1 compares Swanson and Donnelly's results at different temperatures with our calculation of the critical Reynolds numbers $N_{\mathrm{Re},c}$ at the same radius ratio $\eta=0.976\,28$ and gap $\delta=0.0472$ cm used in the experiment. The general qualitative agreement is good: If the temperature is lowered below T_λ, $N_{\mathrm{Re},c}$ first rises, then drops again at lower T. The low-temperature behavior is as expected: The superfluid fraction increases rapidly at decreasing T (it changes from 0% at T_λ to 42% at $T=2$

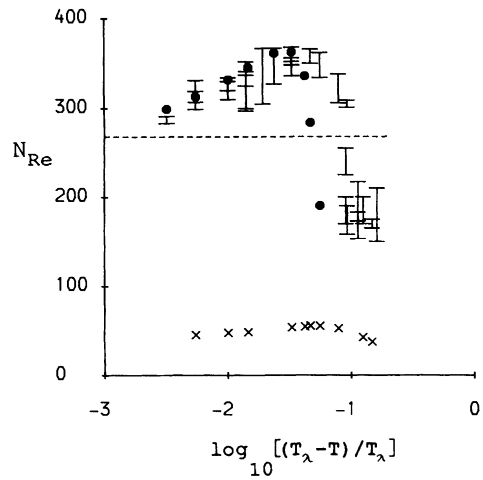

FIG. 1. Critical Reynolds numbers $N_{\mathrm{Re},c}$ as a function of reduced temperature $\ln_{10}[(T_\lambda-T)/T_\lambda]$. Error bars: data of Swanson and Donnelly; circles: present calculation; dotted line: classical value $N_{\mathrm{Re},c}=267.87$ in the limit $T\to T_\lambda$. The crosses denote the Reynolds numbers N_{Re}^* at which vortices appear in the gap.

K) and there is evidence[9] that for a pure superflow $N_{\mathrm{Re},c}=0$. The high-temperature result that helium II is more stable than helium I is remarkable; in this temperature region the theoretical values are inside the experimental error bars and we stress that there are no adjustable parameters in the theory. This is the strictest quantitative test that the HVBK model has ever been subjected to, and the result confirms past work[6] on the values of mutual friction and vortex core parameters.

To understand the discrepancy between theory and experiment in the low-temperature region, we study how the critical dimensionless wave number k_c changes with temperature. At $T=T_\lambda$, in the classical Taylor-Couette limit, the curve of the marginal states, defined by $N_{\mathrm{Re}}(p)=0$, is like a parabola in the N_{Re} vs k plane which has a minimum at $k=k_c=3.13$: The Taylor vortex flow which onsets at $N_{\mathrm{Re}}(k=k_c)=N_{\mathrm{Re}_c}$ has dimensionless wavelength $\lambda_c=2\pi/k_c\simeq 2$ and Taylor vortices consist of pairs of almost square cells. If the temperature is lowered, the curve of the marginal states moves to the left, its minimum k_c decreases, as shown in Fig. 2, and the cells become greatly elongated in the axial direction. Around $T\sim 2.07$ K the curve of the marginal states has moved so much to the left that its minimum is at $k=0$, the critical Reynolds number is $N_{\mathrm{Re},c}=N_{\mathrm{Re}}(k=0)$ and the flow is unstable to perturbations of infinitively long wavelength. Since no apparatus can have infinite cylinders, we conclude that at low temperatures our approximation which assumes the $\exp(ikz)$ dependence becomes invalid and end effects must be taken into account.

The correct way to proceed would be to solve the HVBK equations in $0<z<h$, $R_1<r<R_2$, instead of solving for the radial dependence only. This is, however, a difficult numerical task, and, above all, it is not clear which boundary conditions must be enforced at $z=0$ and $z=h$. We can somehow take the ends into account by as-

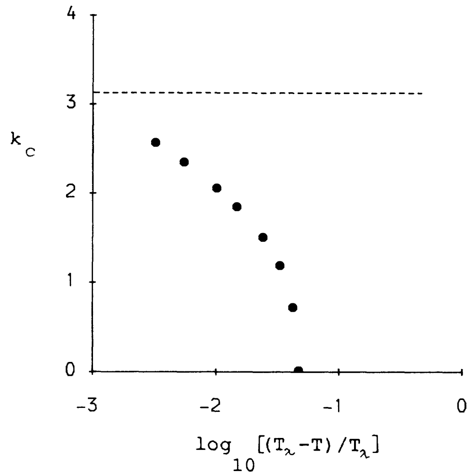

FIG. 2. Critical dimensionless axial wave number k_c as a function of reduced temperature.

suming that half a wavelength fits in the height $h=9.398$ cm of Swanson and Donnelly's apparatus. This corresponds to a minimum dimensionless wave number $k_{\min}=0.016$. The points for $T \le 2.07$ K in Fig. 1 refer to $k_{\min}$. The qualitative behavior is correct and $N_{\mathrm{Re},c}$ decreases with T, but it is not realistic to expect a quantitative agreement. The end conditions in Swanson and Donnelly's apparatus are ill defined and the gap at $z=0$ and $z=h$ is only partially closed. Moreover the vortex lines are probably more efficient than classical Taylor vortices in transmitting end effects to the middle section of the cylinders: The vortices extend from the top to the bottom of the apparatus and they can be pinned[16] or slide at the end caps depending on the roughness of the metal surfaces. The pinning effect is likely to make the flow more stable to overturning, and indeed the measured critical Reynolds numbers at low temperature are higher than the theoretical values. To settle the argument one should measure $N_{\mathrm{Re},c}$ in the low-temperature region at the same value of η and δ but for different h and experiment with end caps of different smoothness. We expect $N_{\mathrm{Re},c}$ at low T to depend on the aspect ratio h/δ.

To check the consistency of our calculations we compute the Reynolds number $N^*_{\mathrm{Re}}=\Omega^* R_1 \delta/\nu_n$ which corresponds to the first appearance of vortex lines in the system. Figure 1 shows that N^*_{Re} is much smaller than $N_{\mathrm{Re},c}$, as it should be. A second check consists of verifying the continuum approximation on which the HVBK model is based. At $T=2.16$ K our value of $N_{\mathrm{Re},c}$ corresponds to 14 300 vortex lines in the gap and more than seven rows of lines in the radial direction, which is probably enough. At $T=2.10$ K there are still seven rows and five rows at $T=2.05$ K. At some lower temperature one eventually reaches $N_{\mathrm{Re},c} \simeq N^*_{\mathrm{Re}}$; at that point the HVBK model breaks down and individual vortices must be taken into account.

We have already mentioned that a practical difference between the study of classical Taylor-Couette flow and helium II Couette flow is the lack of direct flow visualization: In the low-temperature environment one has to decide in advance what to look for, and a separate experiment must be set up for this purpose. The HVBK model can help us in gaining insight into the flow. In principle one should solve the full nonlinear HVBK equations, but experience of similar problems suggests that, unless another bifurcation occurs, the solution of the linearized equations is enough to describe the spatial configuration of the flow, at least for N_{Re} just above $N_{\mathrm{Re},c}$. Since second sound detects the vortex lines directly, the superfluid vorticity $\boldsymbol{\omega}^s$ is the quantity of major interest. We concentrate our attention on its azimuthal component ω^s_φ because it is zero in the Couette state. Figures 3(b)–3(e) show contour plots of ω^s_φ at $N_{\mathrm{Re}}=N_{\mathrm{Re},c}$ for different temperatures. It is instructive to compare ω^s_φ with the more familiar[15] azimuthal component of the vorticity in the classical Taylor-Couette problem, which is the same as ω^n_φ at T_λ, shown in Fig. 3(a). Clearly the flow of the superfluid component depends very much on T.

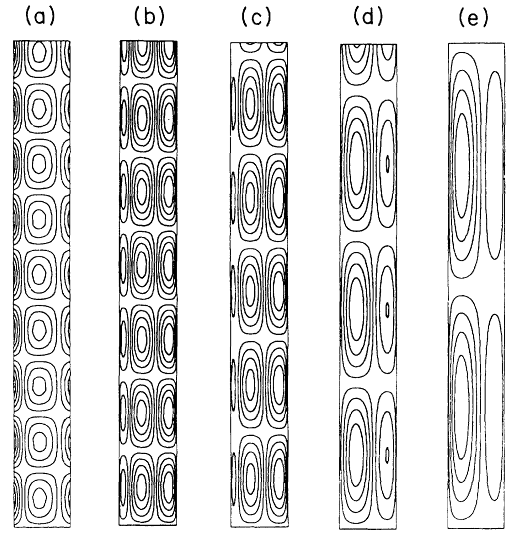

FIG. 3. Contour plots of ω^n_φ at $T=T_\lambda$ (a) and of ω^s_φ at $T=2.16$ (b), $T=2.14$ (c), $T=2.10$ (d), and $t=2.08$ (e).

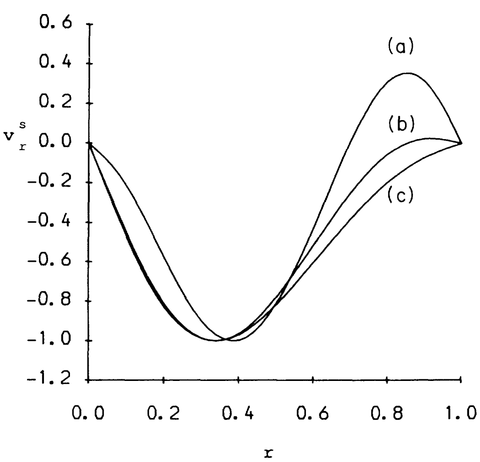

FIG. 4. Plots of v^s_r vs r at $T=2.16$ (a), $T=2.10$ (b), and $T=2.08$ (c). $r=0$ and $r=1$ correspond, respectively, to the inner and outer cylinder.

Just below T_λ, ω^s is still very similar to the classical distribution of vorticity. As the temperature is lowered the critical wave number decreases and the vortex cells become longer in the z direction. At the same time the radial structure becomes simpler. This is also evident from plots of v_r^s vs r at different values of T; see Fig. 4. The radial dependence of the normal fluid on the contrary does not depend much on T: The normal-fluid pattern simply becomes elongated in z as T decreases. The experimental challenge is to test these predictions by probing locally with second sound at different locations in between the cylinders.

We conclude that in the high-temperature region the HVBK model predicts successfully that helium II is more stable than helium I. The theoretical values of the critical Reynolds numbers are in good agreement with the measurements. In the low-temperature regime the comparison between theory and experiment is only qualitatively correct: The critical wavelength diverges and end effects, which the current theory can take only approximately into account, become significant. In this regime $N_{\mathrm{Re},c}$ should depend also on the aspect ratio h/δ. Finally our investigation of the flow pattern reveals that the superfluid cells are rather different from the classical Taylor vortices.

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