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Effects of quenched disorder on spin-$\frac{1}{2}$ quantum XXZ chains

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We analyze the zero-temperature phase diagram of the spin-$\frac{1}{2}$ quantum XXZ chain in the presence of weak disorder. The effects of various random perturbations are considered, including random fields and random exchange. For random perturbations that preserve the XY symmetry, we find a phase transition, as the anisotropy parameter is varied, from a ground state with quasi-long-range order (as in the pure system) to one in which typical correlation functions decay rapidly. The critical behavior at this transition is shown to be in the universality class of the Giamarchi-Schulz transition for one-dimensional bosons in a random potential. Random perturbations that break the XY symmetry are found always to destroy the quasi-long-range order of the ground state. Properties of the resulting random phases are also discussed.

I. INTRODUCTION

Classical disordered magnetic systems, such as the random-field Ising model and the Edwards-Anderson spin glass, have been studied extensively in recent years, but comparatively little is known about the effects of disorder in quantum spin systems, or in other many-body quantum systems. Weakly disordered quantum spin chains appear promising as model systems for a study of the interplay between randomness and quantum effects for a number of reasons. Firstly, because of the low dimensionality, quantum fluctuations are especially important in spin chains, which exhibit a rich variety of interesting behavior. In addition, a wide variety of techniques, including exact solutions and mappings to continuum field theories, are available for studying homogeneous spin chains. Finally, because of the marginally ordered nature of some of the phases in quantum spin chains, substantial progress can be made by studying weak disorder.

In this paper, we consider a class of spin-$\frac{1}{2}$ quantum systems with Hamiltonians

$$H=H_0+H_R$$

consisting of a nonrandom part, H_0, of the XXZ form

$$H_0=\sum_i (S_i^xS_{i+1}^x+S_i^yS_{i+1}^y+\Delta S_i^zS_{i+1}^z)\ , \tag{1.1}$$

which has XY symmetry, and various random parts, H_R. In the absence of disorder, the ground state of this system exhibits several different kinds of behavior as the anisotropy parameter Δ is varied. For $\Delta\leq -1$, the ground state has Ising-like ferromagnetic long-range order with all the spins completely aligned along the z axis. In the range $-1<\Delta\leq 1$, there is a gapless quasi-long-range-ordered phase in which the ground-state expectation values of the spin operators vanish, but spin correlations exhibit power-law decay with exponents that depend continuously on Δ. For $\Delta>1$, the ground state has long-range Ising-like antiferromagnetic order along the z axis with quantum fluctuations of the spins, so that $|\langle S_z\rangle|<\frac{1}{2}$.

We will primarily be concerned with the quasi-long-range-ordered phase with $|\Delta|<1$, for which we will introduce a perturbative renormalization group (RG) to study the effects of weak disorder. We use a simple generalization of Harris' criterion for the XXZ chain in this region, which relates the relevance of a random perturbation to correlation functions in the pure system of the operator, which couples to the randomness. Specifically, if we introduce a random perturbation of the form

$$H_R=\sum_i h_i\hat{O}_i\ , \tag{1.2}$$

where the h_i are independent, identically distributed random variables with zero mean and second moment D_h and the $\hat{O}_i$ are quantum operators depending on spin variables near site i whose ground-state expectation values in the pure system satisfy

$$\langle \hat{O}_i\hat{O}_j\rangle\sim|i-j|^{-2\zeta}\ , \tag{1.3}$$

then the RG eigenvalue λ_h, which determines the rescaling of D_h, satisfies

$$\lambda_h=1+2z-2\zeta\ , \tag{1.4}$$

where the dynamical exponent z equals one in the gapless phase of the XXZ spin chain. The factor of 2 multiplying the dynamical exponent in Eq. (1.4) arises from the time independence of the randomness so that in a space-time volume $L\times L^z$ the mean-square random perturbation scales as $L\times L^{2z}$. Thus if $\lambda_h<0$, the random couplings $\{h_i\}$ are irrelevant, and, provided that other relevant random (or uniform) couplings are not generated by renor-

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malization, no significant modifications of the pure system behavior will occur.

We study the effects of several kinds of weak disorder, specifically (i) a random transverse magnetic field in the z direction,

$$H_{\rm ZF}=\sum_i h_i^z S_i^z \ , \tag{1.5}$$

(ii) a random component in the planar exchange interaction,

$$H_{\rm PE}=\sum_i \delta J_i^{xy}(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y) \ , \tag{1.6}$$

(iii) a random component in the z-z exchange,

$$H_{\rm ZE}=\sum_i \delta J_i^z S_i^z S_{i+1}^z \ , \tag{1.7}$$

(iv) a random field in the X-Y plane,

$$H_{\rm PF}=\sum_i (h_i^x S_i^x + h_i^y S_i^y) \ , \tag{1.8}$$

and (v) a random XY symmetry-breaking exchange interaction,

$$H_{\rm PA}=\sum_i \gamma_i (S_i^x S_{i+1}^x - S_i^y S_{i+1}^y) \ . \tag{1.9}$$

We find that the first three perturbations, which preserve the X-Y symmetry, give rise to an identical phase transition in the disorder-exchange anisotropy plane from a ground state with quasi-long-range order (as in the pure system) to one with more rapid decay of spin correlations. We identify a critical anisotropy $\Delta=\Delta_c=-\frac{1}{2}$ as the exact value of the exchange anisotropy where the resulting phase boundary intersects the line of zero disorder, and we give a renormalization-group analysis of the critical behavior at the transition. In particular, we show that the transition is in the same universality class as the localization transition for one-dimensional (1D) bosons in a random potential first analyzed by Giamarchi and Schulz,[1] and that the disordered phase is characterized by a correlation length ξ, which diverges as the phase boundary is approached from the disordered side with the characteristic Kosterlitz-Thouless form

$$\xi \sim \exp[\text{const}/D^{1/4}(\Delta-\Delta_c)^{1/2}] \ , \tag{1.10}$$

where D is the mean-square strength of the disorder.

The random XY symmetry-breaking perturbations, Eqs. (1.8) and (1.9), always destroy the quasi-long-range order. This occurs for the random anisotropic planar exchange interaction $H_{\rm PA}$, Eq. (1.9), even though the generalized Harris' criterion, Eq. (1.4), suggest that there should be a regime where this type of disorder is irrelevant. We show how this discrepancy arises from higher-order terms in the RG.

The remainder of this paper is organized as follows. In the next section, we review the techniques used in treating the spin-$\frac{1}{2}$ XXZ chain without disorder and derive a functional integral expression for the partition function, which will be useful in the RG analysis. The RG treatment of the phase transition in the random case, as well as a discussion of the phase diagram, appear in Sec. III. We discuss the properties of the disordered phases of quantum spin chains in Sec. IV. Section V summarizes the results and raises questions for future study.

II. CONTINUUM FIELD THEORY FOR THE *XXZ* CHAIN

Since the long-distance properties of the system will determine the behavior for weak randomness, we may use a continuum description in phases with a continuous symmetry. In this section we review a continuum field theoretic description of the pure XXZ chain, which allows us to calculate correlation function exponents exactly in the region $|\Delta|<1$ in terms of a single parameter, which can be determined from the exact Bethe-ansatz solution.[2] Following Affleck[2] and earlier work by Luther and Peschel,[2] we obtain a representation of the quantum spin chain in terms of a *classical* 2D Gaussian model, which will prove convenient for the RG analysis of Sec. IV. We also derive continuum forms of the random perturbations, Eqs. (1.5)–(1.8).

We first perform a Jordan-Wigner transformation to obtain a representation of the spin chain in terms of spinless fermions on a lattice:

$$\begin{aligned} S_n^- &= \exp\left[i\pi \sum_{m<n} (S_m^z+\tfrac{1}{2}) \right] \Psi_n \ , \\ S_n^z &= \Psi_n^\dagger \Psi_n - \tfrac{1}{2} \ . \end{aligned} \tag{2.1}$$

Thus a fermion is present on a site when the spin is up and absent when the spin is down. The "tail" operator is needed to ensure that fermion operators at different sites anticommute.

When the anisotropy Δ equals zero, the Hamiltonian can be diagonalized exactly because the Jordan-Wigner fermions do not interact. The free fermion dispersion relation is $\varepsilon_k=-\cos(ak)$, where a is the lattice spacing, and the ground state is simply a half-filled band with Fermi points at $\pm\pi/2a$. Thus there are two types of low-energy excitations. An excitation that excites a fermion from just below to just above a given Fermi point carries momentum that vanishes linearly as the excitation energy goes to zero. Excitations across the Fermi sea carry a minimum momentum of $2k_F$. Thus there are important modes with momenta near 0 and $2k_F$, which must be retained in taking the continuum limit. We can rewrite the field Ψ as

$$\Psi_n=\sqrt{a}\,[e^{-ik_F x}\Psi_R(x)+e^{ik_F x}\Psi_L(x)] \ , \tag{2.2}$$

where $x=na$, $k_F=\pi/2a$, and Ψ_R and Ψ_L describe excitations near the right and left Fermi points, respectively. This decomposition is exact provided that the fields Ψ_R and Ψ_L only include the momenta $|k|<\pi/2a$. Thus excitations near each of the Fermi points are regarded as different species of particles, and in the ground state all the negative (positive) wave-vector states of Ψ_R (Ψ_L) are occupied. In the low-energy sector of the Hilbert space, the fields Ψ_L,Ψ_R are slowly varying, and the cutoff structure is therefore unimportant. We will impose a high-momentum cutoff Λ on the fields Ψ_R,Ψ_L with Λ satisfying $1/L \ll \Lambda \ll \pi/2a$, where L is the size of the system. If we assume that a description in terms of low-energy excitations near the Fermi points is valid even for nonzero Δ, then the fields Ψ_R, Ψ_L are again slowly varying, and the continuum limit may be taken by substituting the definition, Eq. (2.2), of the lattice fermions in terms of Ψ_R and Ψ_L into the Hamiltonian, Eq. (1.1), and keeping the lowest-order terms in a gradient expansion.

The result is

$$H = H_K + H_{\rm int} + H_{\rm umk} ,$$

where

$$\begin{aligned} H_K &= ia \int dx (\Psi_L^\dagger \partial_x \Psi_L - \Psi_R^\dagger \partial_x \Psi_R) , \\ H_{\rm int} &= a\Delta \int dx [(\Psi_L^\dagger \Psi_L)^2 + (\Psi_R^\dagger \Psi_R)^2 + 4\Psi_L^\dagger \Psi_L \Psi_R^\dagger \Psi_R] , \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} H_{\rm umk} = -a\Delta \int dx [&\Psi_L^\dagger(x)\Psi_R(x)\Psi_L^\dagger(x+a)\Psi_R(x+a) \\ &+ \Psi_R^\dagger(x)\Psi_L(x)\Psi_R^\dagger(x+a)\Psi_L(x+a)] . \end{aligned}$$

We have dropped a term in the Hamiltonian, which is quadratic in the fermions, because it is proportional to the conversed fermion number. The term $H_{\rm umk}$ results from umklapp scattering of two fermions across the Fermi sea in the same direction. Although the coefficients in Eq. (2.3) are correct only for small Δ, we shall see later in this section that by using the exact Bethe-ansatz solution we can obtain a representation of the XXZ chain valid for all anisotropies $-1 < \Delta \le 1$ in the phase with unbroken continuous symmetry.

Except for the random in-plane field, Eq. (1.8), the random perturbations, Eqs. (1.5), (1.6), (1.7), and (1.9), can all be expressed in terms of the fermions as local operators due to the cancellation of the tails entering in the Jordan-Wigner transformation. Thus their continuum forms can be similarly obtained, resulting in

$$\begin{aligned} H_{\rm ZF} &= \int dx [\eta(x)(\Psi_R^\dagger \Psi_R + \Psi_L^\dagger \Psi_L) + \rho(x)\Psi_L^\dagger \Psi_R + \rho(x)\Psi_R^\dagger \Psi_L] , \\ H_{\rm PE} &= \int dx (i\rho \Psi_L^\dagger \Psi_R - i\rho \Psi_R^\dagger \Psi_L) + \text{derivative terms} , \\ H_{\rm ZE} &= H_{\rm ZR} + \text{random quartic terms} , \\ H_{\rm PA} &= \int dx \{\eta(x) i(\Psi_L^\dagger \Psi_R^\dagger - \Psi_R \Psi_L) + i\rho(x)[\Psi_R^\dagger(x)\Psi_R^\dagger(x+a) - \Psi_L^\dagger(x)\Psi_L^\dagger(x+a)] \\ &\qquad - i\rho(x)[\Psi_L(x)\Psi_L(x+a) - \Psi_R(x)\Psi_R(x+a)]\} , \end{aligned} \tag{2.4}$$

where $\eta(x)$ and $\rho(x)$ are composed of the Fourier components of the coefficients $\{h_i\}$ of the appropriate random perturbation near zero and $2k_F$, respectively. For simplicity we have dropped the subscripts corresponding to the various random perturbations from η and ρ. The omitted terms in $H_{\rm ZE}$ correspond to a spatially random umklapp coupling, which can be shown to be irrelevant by arguments similar to those used in Sec. III. The remaining random perturbation, Eq. (1.8), cannot be expressed locally in terms of the fermions because of Jordan-Wigner tails:

$$H_{\rm PF} = \frac{1}{2} \sum_i \exp\left[i\pi \sum_{j<i} n_j\right] [(h_i^x - ih_i^y)\Psi_i^\dagger + (h_i^x + ih_i^y)\Psi_i] . \tag{2.5}$$

Later in this section we will show how the continuum limit of the tail operator may be obtained by transforming to a new set of variables which we now describe.

This final transformation of the spin Hamiltonian gives a description of the interacting fermion system in terms of bosonic density modulations. Such a formulation is advantageous because (i) the resulting boson Hamiltonian is harmonic and (ii) it enables one to write a functional integral over c-number fields, which will prove convenient for the RG treatment of Sec. III.

Mattis and Lieb[3] discovered that the operators

$$b_q^\dagger = \begin{cases} i\left[\dfrac{2\pi}{qL}\right]^{1/2} \Sigma_k c_{R,k+q}^\dagger c_{R,k} , & q > 0 \\ -i\left[\dfrac{2\pi}{|q|L}\right]^{1/2} \Sigma_k c_{L,k+q}^\dagger c_{L,k} , & q < 0 , \end{cases} \tag{2.6}$$

where $c_R^\dagger$ ($c_L^\dagger$) are fermion creation operators for the R (L) particles, satisfy Bose commutation relations in the idealized limit of an infinite, filled Fermi sea for the R and L particles. (In the actual model, these commutation relations hold only in the sector of Hilbert space, which contains neither particle nor hole excitations at distances greater than Λ from the Fermi points, where Λ is the cutoff on the fields Ψ_R, Ψ_L.) Furthermore, the boson excitations form a complete set of states for the fermions at fixed particle number.[4] Several field operators constructed from the Bose creation and annihilation operators will prove convenient for later use. Let us define

$$\begin{aligned} \Phi_R(x) &= \sum_{p>0} \left[\frac{2\pi}{pL}\right]^{1/2} (e^{-ipx} b_p + e^{ipx} b_p^\dagger) , \\ \Phi_L(x) &= \sum_{p<0} \left[\frac{2\pi}{|p|L}\right]^{1/2} (e^{-ipx} b_p + e^{ipx} b_p^\dagger) , \\ \Phi(x) &= \Phi_R(x) + \Phi_L(x) , \end{aligned} \tag{2.7}$$

and

$$\tilde{\Phi}(x)=\Phi_R(x)-\Phi_L(x) .$$

The momentum operator canonically conjugate to $\Phi(x)$ is

$$\Pi(x)=\sum_{p\,(\neq 0)}\left[\frac{|p|}{2\pi L}\right]^{1/2}(e^{-ipx}b_p-e^{ipx}b_p^\dagger) . \quad (2.8)$$

The fermion operators Ψ_R and Ψ_L are related to the fields Φ_R and Φ_L by

$$\Psi_R(x)=\frac{1}{\sqrt{L}}:\exp[-i\Phi_R(x)]: ,$$
$$\Psi_L(x)=\frac{1}{\sqrt{L}}:\exp[i\Phi_L(x)]: , \quad (2.9)$$

where the colons denote normal ordering. A concise derivation of this correspondence is contained in Ref. 4. The physical meaning of the Bose field Φ has been elucidated by Haldane[5] in work on one-dimensional harmonic fluids. In terms of the particles corresponding to the positions of the up spins, $(1/2\pi)\partial_x\Phi$ is the coarse-grained density fluctuation about the mean density. The actual expression for the particle density in the continuum is

$$n(x)=\left[n_0+\frac{1}{2\pi}\partial_x\Phi\right]\sum_m \exp im(\Phi+\pi x/a) , \quad (2.10)$$

where n_0 is the mean density, which is equal to $1/2a$ in the absence of a uniform z field or Ising ferromagnetic order. The sum over harmonics takes into account the discreteness of the particle number by allowing the density to be nonzero only when $\Phi(x)+2\pi n_0 x$ is an integer multiple of 2π. The field $-\Phi/2\pi n_0$ is thus a continuum limit of the Eulerian displacement coordinates of the bose particles away from their regularly spaced positions with separation $1/n_0$. Since the higher harmonics in Eq. (2.10) will turn out to be less relevant, it usually suffices to keep only the lowest-order terms in the sum; physically this corresponds to "smearing out" the particles.

The physical meaning of the field $\tilde{\Phi}$ can be understood by considering the bosonized form of the spin-raising operator S^+. From the above discussion of the field Φ, the tail operator may be expressed in the continuum limit as

$$\exp\left[i\pi\sum_{j<x/a}n_j\right]=\exp i(\pi x/2a+\Phi/2) . \quad (2.11)$$

[Note that the factors $e^{i\pi n_j}$ entering in the lattice form of the tail operator have the property that their squares are equal to the identity. Since the continuum form, Eq. (2.11), does not have this property, it will not necessarily yield the correct continuum forms of products of spin operators and must thus be used with some caution.] Using the relation, Eq. (2.1), between the spin operator and the fermions together with the continuum forms of Ψ and the tail operator, Eqs. (2.2) and (2.11), we obtain

$$S^+(x)=e^{-i\tilde{\Phi}/2}[1+(-1)^{x/a}e^{i\Phi}] , \quad (2.12)$$

which allows us to identify $-\tilde{\Phi}/2$ as the azimuthal angle of the spins. The effect of the factor in brackets (which for $\Phi=0$ corresponds to equally spaced up spins on alternate sites) is to make S^+ zero at the positions of the up spins, hence enforcing the hard-core constraint appropriate to spin $\frac{1}{2}$.

As shown by Mattis and Lieb, the kinetic part of the Hamiltonian can be expressed in terms of the boson operators:

$$H_K=a\sum_{p>0}|p|(b_p^\dagger b_p+b_{-p}^\dagger b_{-p}) . \quad (2.13)$$

Since H_{int} is also quadratic in the Bose operators, we can rewrite H_K+H_{int} in terms of Φ and Π:

$$H_K+H_{\text{int}}=\int dx[\tfrac{1}{2}\kappa_\pi\Pi^2(x)+\tfrac{1}{2}\kappa_\phi(\partial_x\Phi)^2] , \quad (2.14)$$

where, for small Δ,

$$\kappa_\pi=4\pi a(1-\Delta/\pi) ,$$
$$\kappa_\phi=(4\pi)^{-1}a(1+3\Delta/\pi) .$$

The first term on the right-hand side of Eq. (2.14), which is proportional to the square of the current, is the kinetic energy of the particles. The second term, proportional to the square of the density modulations, arises from the compressibility of the particle fluid.

Using Eq. (2.9), we rewrite the umklapp term in the Hamiltonian, Eq. (2.3), as

$$H_{\text{umk}}=g\int dx :\cos(2\Phi): , \quad (2.15)$$

where g is a function of Δ, which we will not calculate and the colons denote normal ordering.

A functional integral expression for the partition function, from which we can calculate ground-state correlations, can be obtained from the Hamiltonian, Eq. (2.13), in the usual manner:

$$Z=\int D\Pi\, D\Phi \exp\left[-\int dx\, d\tau[\tfrac{1}{2}\kappa_\pi\Pi^2+\tfrac{1}{2}\kappa_\phi(\partial_x\Phi)^2 - i\Pi\partial_\tau\Phi+g\cos(2\Phi)]\right] . \quad (2.16)$$

Performing the Gaussian functional integral over Π and rescaling τ to make the spin-wave velocity, $c=(\kappa_\phi\kappa_\pi)^{1/2}$, equal to one, we obtain

$$Z=\int D\Phi \exp\{-\int dx\, d\tau[\tfrac{1}{2}\kappa(\partial_\tau\Phi)^2+\tfrac{1}{2}\kappa(\partial_x\Phi)^2 +g\cos(2\Phi)]\} , \quad (2.17)$$

where the stiffness κ satisfies

$$\kappa^2=\frac{\kappa_\phi}{\kappa_\pi}=\frac{1+3\Delta/\pi}{(4\pi)^2(1-\Delta/\pi)} . \quad (2.18)$$

Thus, we have mapped the *XXZ* spin chain with anisotropy Δ onto a family of $(1+1)$-dimensional Gaussian models parametrized by the stiffness κ. The derivation from the microscopic Hamiltonian has yielded the relationship, Eq. (2.18), between κ and the anisotropy Δ in the lattice model. However, as typically happens in going from a lattice to a continuum model the short-wavelength behavior can renormalize the parameters of the model. Therefore, it is important to check the result, Eq. (2.18), against known properties of the lattice spin system. Fortunately, a Bethe-ansatz solution[6] exists for the spin-$\frac{1}{2}$ XXZ chain, which gives the scaling of the gap induced by a uniform XY symmetry-breaking exchange anisotropy with the strength of the anisotropy. From this it is possible to deduce the renormalized value of the stiffness κ in terms of the bare anisotropy Δ in the lattice model, yielding

$$\kappa=\frac{1}{2\pi}\left[1-\frac{1}{\pi}\cos^{-1}\Delta\right], \tag{2.19}$$

which agrees with our previous result, Eq. (2.18), up to second order in Δ.

The stiffness κ determines the exponents for the decay of various ground-state correlations functions. These correlation functions have the general form

$$\langle \exp\{i[a\Phi(x)+b\tilde{\Phi}(x)]\}\exp\{-i[a\Phi(0)+b\tilde{\Phi}(0)]\}\rangle \sim |x|^{-2(a^2/4\pi\kappa+b^2 4\pi\kappa)} \tag{2.20}$$

and may be calculated by using the definition, Eq. (2.7), of the fields $\Phi,\tilde{\Phi}$, in terms of creation and annihilation operators and performing a Bogolyubov transformation to diagonalize the Hamiltonian, Eq. (2.14). These correlation functions are related to ground-state spin correlations in the original XXZ model.

For the spin-$\frac{1}{2}$ XXZ chain without disorder, the power-law decay of ground-state correlations and the dependence of exponents on the anisotropy Δ imply the existence of a renormalization-group transformation, which maps the original lattice Hamiltonian, Eq. (1.1), onto a one parameter family of distinct fixed-point Hamiltonians for every distinct value of Δ in the interval $(-1,1)$, or equivalently, for κ in the interval 0 to $1/2\pi$. Such a transformation eliminates degrees of freedom with momenta in the range $\Lambda/b<k<\Lambda$ and rescales distances and times according to $x\to bx$ and $\tau\to b^z\tau$, where z is a dynamical exponent, equal to unity for the Lorentz-invariant fixed points of interest here. Operators are rescaled according to $O\to b^{-\zeta}O$, where ζ, the scaling dimension of the operator O, can be determined from the decay of ground-state correlations of O:

$$\langle O(x,\tau)O(y,\tau)\rangle\sim|x-y|^{-2\zeta}. \tag{2.21}$$

Thus we can use Eq. (2.20) to determine the scaling dimension of operators of the form $\exp(ia\Phi+ib\tilde{\Phi})$.

We now examine the effect of the umklapp term, Eq. (2.15). Power counting in the action in Eq. (2.17) reveals that the RG eigenvalue, λ_g, which describes the rescaling of the umklapp coupling, g, is related to ζ_g by

$$\lambda_g+\zeta_g=1+z, \tag{2.22}$$

where the dynamical exponent z equals one at the Gaussian fixed point. Using Eq. (2.19), which related the renormalized stiffness κ to the anisotropy Δ, we determine that H_{umk} is irrelevant for $\Delta<1$. The runaway from the Gaussian fixed line, which occurs for $\Delta>1$ is associated with the opening of a gap in the excitation spectrum and the onset of long-range Néel order.[2]

The final result we will need is the bosonized, continuum form of the random perturbations, Eqs. (2.4) and (2.5). Using the bosonization formula, Eq. (2.9), we find

$$\begin{aligned} H_{\text{ZF}}&=\int dx(\eta\partial_x\Phi+\rho e^{i\Phi}+\rho e^{-i\Phi}),\\ H_{\text{PF}}&=\int dx[(\eta_x+\rho_x e^{i\Phi}+\rho_x e^{-i\Phi})\cos(\tilde{\Phi}/2)\\ &\qquad+(\eta_y+\rho_y e^{i\Phi}+\rho_y e^{-i\Phi})\sin(\tilde{\Phi}/2)],\\ H_{\text{PA}}&=\int dx[\eta\sin\tilde{\Phi}+i\rho(e^{2i\Phi_R}-e^{-2i\Phi_L})\\ &\qquad-i\rho(e^{-2i\Phi_R}-e^{2i\Phi_L})], \end{aligned} \tag{2.23}$$

where η and ρ are proportional to the uniform and alternating parts of the coefficients of the appropriate random terms in the Hamiltonian. Since a random field in the z direction acts as a random potential coupled to the particle density, we have used the expression Eq. (2.10) for the density and kept only the leading harmonic in Φ. Because this harmonic carries the phase factor $e^{i\pi x}$, it couples to the alternating part of the random potential. Since $\tilde{\Phi}$ is twice the azimuthal angle of the spins, the terms $\cos\tilde{\Phi}/2$ and $\sin\tilde{\Phi}/2$ in H_{PF} represent random fields in the x and y directions, respectively. The term $\sin\tilde{\Phi}$ in H_{PA} has the symmetry under 180° rotations about the z axis appropriate for anisotropic planar exchange, Eq. (1.9). (We do not obtain $\cos\tilde{\Phi}$ as one would naively expect from the identification of $\tilde{\Phi}/2$ with the azimuthal angle because of the phase factors $e^{ik_F a}$ associated with fermions at adjacent sites.) The terms in H_{PA} coupling to the alternating part of the disorder are of the form $e^{i\Phi}e^{i\tilde{\Phi}}$; these involve a coupling between the alternating part of the density and the phase of the spins, which arises because of the hard-core constraint, which appeared, for example, in Eq. (2.12). The couplings to higher-order harmonics in the density will turn out to be less relevant, and we have, hence, dropped them.

III. EFFECTS OF WEAK RANDOMNESS

A. XY symmetric randomness

As we have shown in the preceding section, the dominant continuum forms of the random z field and random z exchange are identical and of the same form as H_{ZF} in Eq. (2.23). Thus the total imaginary-time action corresponding to these XY-symmetry-preserving random perturbations is

$$\begin{aligned} S\{\Phi\}=\int_{x,\tau}&\tfrac{1}{2}\kappa(\partial_\tau\Phi)^2+\tfrac{1}{2}\kappa(\partial_x\Phi)^2+g\cos(2\Phi)\\ &+\eta(x)\partial_x\Phi+\rho(x)e^{i\Phi}+\rho(x)e^{-i\Phi}. \end{aligned} \tag{3.1}$$

In order to perform calculations in the presence of quenched disorder, it is convenient to use the replica formalism and consider the disorder average of the nth power of the partition function. This has the functional integral representation

$$\overline{Z^n}=\int D\Phi_1\cdots D\Phi_n\,\overline{\exp\left[-\sum_{\alpha=1}^{n}S\{\Phi_\alpha\}\right]}\;, \tag{3.2}$$

where S is the imaginary-time action, Eq. (3.1), and the overbar denotes a disorder average. This formalism allows one to compute various disorder averages of ground-state expectation values. For example, for arbitrary operators $O^{(1)}$ and $O^{(2)}$,

$$\overline{\langle O^{(1)}\rangle\langle O^{(2)}\rangle}=\lim_{n\to 0}\langle\langle O_\alpha^{(1)}O_\beta^{(2)}\rangle\rangle\;, \tag{3.3}$$

where $\alpha\neq\beta$ and the double brackets refer to averaging with respect to the functional integral in Eq. (3.2).

We perform the average in Eq. (3.2) in two steps. Using the Gaussian nature of the randomness, we first average the exponential in Eq. (3.2) over ρ to obtain

$$S_{\text{eff}}=\sum_{\alpha=1}^{n}\int_{x,\tau}[\tfrac{1}{2}\kappa(\partial_\tau\Phi_\alpha)^2+\tfrac{1}{2}\kappa(\partial_x\Phi_\alpha)^2 +g\cos(2\Phi_\alpha)+\eta(x)\partial_x\Phi_\alpha] -\sum_{\alpha,\beta}D_\rho\int_{x,\tau,\tau'}\cos[\Phi_\alpha(x,\tau)-\Phi_\beta(x,\tau')] \tag{3.4}$$

as our (intermediate) replicated action. Before performing the remaining disorder average over η, we follow a procedure analogous to the treatment of 1D bosons in Ref. 7 and eliminate the term linear in the spatial derivative in Eq. (3.4) by shifting the field Φ:

$$\Phi(x)\to\Phi(x)=\frac{1}{\kappa}\int_{-\infty}^{x}dy\,\eta(y)\;. \tag{3.5}$$

This shift affects only the Umklapp term in Eq. (3.4), which becomes

$$g\cos\left[2\Phi_\alpha-\frac{2}{\kappa}\int_{-\infty}^{x}\eta\right]\;. \tag{3.6}$$

The remaining disorder average over η must be performed perturbatively in powers of g. Since the umklapp term is irrelevant in the range of anisotropy Δ in which we are interested, this perturbative approach is justified. The leading term in the average over η is of order g^2 and is given by

$$-\sum_{\alpha,\beta}\frac{g^2}{2}\int_{x,x',\tau,\tau'}\cos[2\Phi_\alpha(x,\tau)-2\Phi_\beta(x',\tau')]f(x-x') \tag{3.7}$$

where

$$f(x-x')=\exp\left[-\frac{2D_\eta}{\kappa^2}|x-x'|\right]\;.$$

Since the field Φ exhibits quasi-long-range order at the pure system fixed point, we expect Φ to vary slowly over the length scale κ^2/D_η, which characterizes the decay of the function $f(x-x')$. Therefore we Taylor expand $\Phi_\beta(x')$ around x and keep only the leading term. This gives our final result for the replicated action in the presence of a random z field or z exchange:

$$\overline{S_{\text{eff}}}=\sum_{\alpha=1}^{n}\int_{x,\tau}[\tfrac{1}{2}\kappa(\partial_\tau\Phi_\alpha)^2+\tfrac{1}{2}\kappa(\partial_x\Phi_\alpha)^2] -\sum_{\alpha,\beta}D_\rho\int_{x,\tau,\tau'}\cos[\Phi_\alpha(x,\tau)-\Phi_\beta(x,\tau')] +\frac{g^2\kappa^2}{2D_\eta}\cos[2\Phi_\alpha(x,\tau)-2\Phi_\beta(x,\tau')]\;. \tag{3.8}$$

Thus the randomness has induced a pairwise interaction between the replicas. For random planar exchange, the replicated action is identical except that, since averaging over $\eta(x)$ is unnecessary in this case, the second harmonic term, proportional to g^2, is not generated in this manner.

We now examine the effects of the interreplica couplings in Eq. (3.8). As we have discussed in Sec. II, the Hamiltonian of the pure spin-$\frac{1}{2}$ XXZ chain flows to a distinct fixed-point Hamiltonian for every value of Δ in the interval $(-1,1)$. In the absence of randomness, the replicated action, Eq. (3.8), with D_ρ and g set equal to zero, is also a fixed point of a RG transformation in which each replica scales independently. The scaling dimension of the operator

$$\cos[n\Phi_\alpha(x,\tau)-n\Phi_\beta(x,\tau')] \tag{3.9}$$

at this fixed point is simply twice that of the operator $\cos(n\Phi)$ at the pure system fixed point. Thus, by power counting, we find the RG eigenvalue, λ_n, which describes the rescaling of the coupling constant of the operator, (3.9):

$$\lambda_n=1+2z-2\zeta_n^p\;, \tag{3.10}$$

where the dynamical exponent z equals one and

$$\zeta_n^p=\frac{n^2}{4\pi\kappa} \tag{3.11}$$

is the scaling dimension of the pure system operator $\cos(n\Phi)$, determined from Eq. (2.20). Since $\lambda_2<0$ everywhere in the gapless phase, the second random term, proportional to g^2, in Eq. (3.8) is irrelevant and we may drop it.[8] However, λ_1, the eigenvalue of D_ρ in Eq. (3.8) is positive for $-\frac{1}{2}<\Delta\leq 1$, indicating that D_ρ is relevant in this region and even an infinitesimally small amount of randomness carries one away from the line of fixed points governing the power law phase of the pure system. For $-1<\Delta<-\frac{1}{2}$, the randomness is irrelevant, and thus for small nonzero values of D_ρ, we expect a zero-temperature quantum transition between a power-law phase similar to that of the pure system and a randomness-dominated phase as the exchange anisotropy Δ is varied through some critical value Δ_c, which approaches $-\frac{1}{2}$ at zero disorder. We will discuss properties of the random phase in Sec. IV. However, before we do so, we analyze the critical behavior at the transition.

The critical properties can be analyzed through a perturbative renormalization-group for weak disorder. Such a RG treatment has been carried out by Giamarchi and Schulz[1] for the closely related problem of boson localization in a random potential. One can use a momentum shell integration to rederive the RG recursion relations obtained by Giamarchi and Schulz, yielding

$$\frac{\partial D_\rho}{\partial l}=\left[3-\frac{1}{2\pi\kappa}\right]D_\rho ,$$
$$\frac{\partial \kappa}{\partial l}=\mu D_\rho , \tag{3.12}$$

where $l=\ln b$ and μ is a positive constant. For $\kappa<1/6\pi$, the recursion equations have a stable line of fixed points at $D_\rho=0$. Writing $\kappa=1/6\pi+\varepsilon$, then to lowest order in ε and D_ρ the recursion relations become

$$\frac{\partial D_\rho}{\partial l}=\varepsilon D_\rho ,$$
$$\frac{\partial \varepsilon}{\partial l}=D_\rho/2 , \tag{3.13}$$

where we have dropped all the unimportant constants. The flows conserve the quantity (ε^2-D_ρ), and thus the flow lines are parabolas in the ε-D_ρ plane. The left half of the critical parabola $\varepsilon^2-D_\rho=0$ marks the phase boundary between the power-law phase and the disordered phase. Flows with starting values to the left of this critical parabola terminate on the fixed line at $D_\rho=0$. Thus the only effect of the disorder in this region is to renormalize the stiffness κ and, hence, the exponents for spin-spin correlation functions. For a system with physical values of the anisotropy Δ and the disorder corresponding to the starting values D_0 and ε_0, the flows will terminate on the fixed line at $\varepsilon_R=-(\varepsilon_0^2-D_0^2)^{1/2}$ and correlation functions will decay with exponents characteristic of the anisotropy corresponding to ε_R rather than the physical anisotropy Δ.

Flows with starting values to the right of the phase boundary are carried outside the region of validity of the perturbative RG to a randomness-dominated regime. We expect that there is some length scale, ξ, associated with the disordered phase, which diverges as the phase boundary is approached from the disordered regime. As discussed in the next section, with a random z field, ξ can be identified as the correlation length of the decay of spin correlation functions. For random exchange disorder, it appears that such an identification is not possible, and it is necessary to interpret ξ just as a crossover length to disordered behavior. We will discuss further the physical interpretation of this length scale in terms of properties of the disordered phase in Sec. IV. In either case, we can use the recursion relations to determine the form of the divergence of ξ by renormalizing up to a scale l^* at which the flows, which started near the critical parabola, are well into the disordered regime where ξ is of order one. Assuming that ξ depends smoothly on ε and D_ρ in this region, we can use the transformation law for ξ

$$\xi(l^*)=e^{l^*}\xi(0) \tag{3.14}$$

to determine the dependence of ξ on the bare values of the parameters. For a path in the phase diagram at fixed disorder, D_0, one obtains

$$\xi\sim \exp[\text{const}/D_0^{1/4}(\Delta-\Delta_c)^{1/2}] , \tag{3.15}$$

the standard Kosterlitz-Thouless form.[9]

The correlation length also diverges as the $D=0$ axis is approached from within the disordered regime with $\Delta>-\frac{1}{2}$. The form of this divergence may be obtained by renormalizing to a scale at which the strength of the disorder is of order unity so that ξ will be a smooth function of the disorder. Outside of the critical region (i.e., for $D^*/\varepsilon^2\gg 1$, where D^* is the renormalized value of D at which ξ becomes only weakly dependent on the strength of the randomness), we may neglect the renormalization of the stiffness κ by the randomness, yielding

$$\xi=D_0^{-\phi_S} ,$$

where

$$\phi_S=(3-1/2\pi\kappa)^{-1} \tag{3.16}$$

is the crossover exponent for XY-symmetric disorder.

We now discuss the full phase diagram in the presence of XY-symmetric randomness. For sufficiently anisotropy Δ, the pure system becomes ferromagnetic, and the continuum description in terms of the field Φ breaks down. The behavior of the random system in this region depends on whether there is random field or random bond disorder. For a sufficiently small random component in the exchange interactions of the form Eq. (1.6) or Eq. (1.7), the Ising-like ferromagnetic order is expected to persist, although the magnetization may be diminished by quantum fluctuations. For a random z field, however, the Imry-Ma argument indicates that the ferromagnetic state is unstable. Therefore, we expect a transition to a disordered phase as Δ becomes sufficiently negative.

At the other extreme, for Δ large and positive, an arbitrarily weak random z field will also destroy the Ising antiferromagnetic phase. However, weak random exchange is not expected to destroy the antiferromagnetic order so that there will be a second phase boundary passing through $\Delta=1, D=0$ from a disordered phase to an Ising antiferromagnet. The shapes of the phase boundaries near the Heisenberg ferromagnetic point $\Delta=-1$ can be found for weak randomness by considering the balance between the random and elastic parts of the energy. Since the transition in the pure system at $\Delta=-1$ is first order, the randomness couples differently in the two phases, which have opposite signs of

$$\delta_F\equiv\Delta+1 . \tag{3.17}$$

For a random z field the energy density for weak randomness is of order $D|\delta_F|^{-1/2}$. In the XY phase this arises from canting of the spins, while in the Ising field it is due to the breaking up of the ferromagnetic phase into domains of length $|\delta_F|/D$ separated by walls of width $|\delta_F|^{-1/2}$, which have energy $|\delta_F|^{1/2}$. By balancing the random energy with the energy density difference $|\delta_F|$ between the two nonrandom phases, we see that the phase boundary between the disordered and power-law phases has the form

$$D_c\sim|\delta_F|^{3/2} . \tag{3.18}$$

This result can also be derived formally by considering

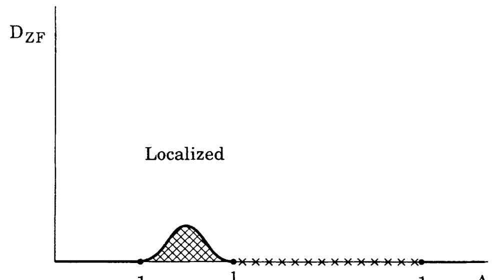

FIG. 1. Schematic phase diagram for a random z field with mean-square magnitude $D_{\rm ZF}$ as a function of the anisotropy Δ. The quasi-long-range-ordered phase exists in the cross-hatched region and along the line $D_{\rm ZF}=0, -1<\Delta\leq 1$. The Ising ferromagnetic and antiferromagnetic phases occur only in the absence of randomness at $D_{\rm ZF}=0$ for $\Delta<-1$ and $\Delta>+1$, respectively.

the appropriate action near the ferromagnetic point. Since the controlling ferromagnetic fixed point has no fluctuations, the spin fields must not rescale. We must, therefore, instead rescale Planck's constant (just as temperature is rescaled near ordered fixed points) as $\hbar'=b^{-\theta}\hbar$. The appropriate dynamical rescaling with $z=2$ (since $\omega\sim k^2$) to keep the spin-wave stiffness fixed then implies $\theta=1$. The anisotropy, which couples to $S_x^2+S_y^2$, then scales as $\delta_F'=b^2\delta_F$. When using replicas, the effective action with the random field contains a factor $1/\hbar^2$ so that the rescaling of

$$D'_{\rm RF}=b^{-2\theta+2z-1}D_{\rm RF}=b^3D_{\rm RF}\ .$$

Combined with the rescaling of δ_F, this again yields the result Eq. (3.18). From the calculations near the XY fixed line, it would appear that randomness drives the system towards larger effective Δ. Thus we guess that the RG flows will go toward larger Δ, leading to the left part of the phase diagram shown in Fig. 1.

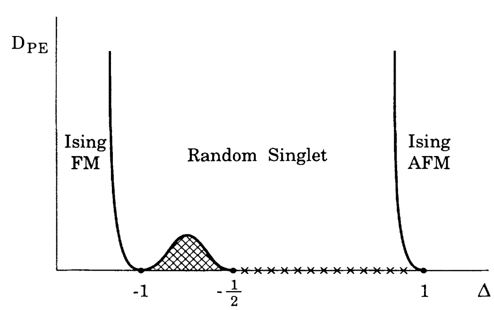

FIG. 2. Schematic phase diagram for XY-symmetric random exchange with mean square magnitude $D_{\rm PE}$ (or similarly $D_{\rm ZE}$) as a function of anisotropy Δ. The XY phase exists in the shaded region and along the $D_{\rm PE}=0$ line for $-1<\Delta\leq 1$. Both Ising ferromagnetic and antiferromagnetic phases also exist as shown. In between lies a disordered random singlet phase discussed in the text in which all susceptibilities are divergent.

The effects of random anisotropic exchange are quite similar: the difference of random exchange energies between the two phases is of order $|\delta J_z|L^{1/2}$ in a section of length L, while the energy to bend the local order from the XY plane to the z direction is of order $1/L$ at the Heisenberg point. The energy density from balancing these terms is of order $D^{2/3}$, which is comparable to δ_F for $D_c\sim|\delta_F|^{3/2}$, as in the random z-field case. Now, however, there is an extra phase present: the ordered Ising ferromagnetic phase. A natural guess is that the disordered phase will always intervene between the XY and Ising phases due to the destruction of correlations by randomness-induced intervening segments of the other phase. Both phase boundaries will have the same form near the bicritical point at $\Delta=-1$, as given by Eq. (3.18) and shown in Fig. 2. An interesting open problem is the nature of the phase transition from the Ising ferromagnet to the disordered phase.

Near the Heisenberg antiferromagnetic point, $\Delta=+1$, the situation is somewhat different. In the pure system, there is a Kosterlitz-Thouless-like transition from the power-law XY to an ordered Ising antiferromagnet as

$$\delta_A\equiv\Delta-1 \tag{3.19}$$

is varied through zero.[2] The correlation length diverges as $\xi_A\sim\exp(\delta_A^{-1/2})$ as the transition is approached. For a random z field both phases will be destroyed by randomness and there is no phase boundary. Anisotropic random exchange, on the other hand, is relevant in the XY phase for $\delta_A\leq 0$ but irrelevant in the Ising phase. Thus there will be a phase boundary between the Ising antiferromagnetic phase and the disordered phase coming out of the point $D=\delta_A=0$. The renormalization by the randomness of Δ towards higher values as in Eq. (3.12) implies that random exchange should favor the Ising phase. Since the eigenvalue of D in Eq. (3.12) is 2 at the Heisenberg fixed point, the phase boundary should have the form

$$D_c\equiv\xi_A^{-2}\equiv e^{-|\delta_A|^{-1/2}}\ . \tag{3.20}$$

The schematic phase diagram is shown in Fig. 2. The nature of the phase transition from the Ising antiferromagnet to the disordered phase is also an intriguing open problem.

B. XY-symmetry-breaking randomness

We now consider the effects of various random perturbations, which break the XY symmetry. A random field in the XY plane, given by $H_{\rm PF}$, Eq. (1.8), couples to operators of the form $e^{i\Phi/2}$, which have scaling dimension $\zeta=\pi\kappa$. The second moment of the random field thus has the eigenvalue $\lambda_{\rm PF}=3-2\zeta$, from Eq. (1.4), which is positive everywhere in the gapless phase. Therefore a random planar field is relevant and always destroys the ground-state quasi-long-range order, as could have been anticipated.

The effects of the random XY-symmetry-breaking component, Eq. (1.9), of the exchange interaction are rather more subtle. As we have shown in Sec. II, the continuum form of this perturbation has two components. The alternating part of the random exchange, $\rho(x)$, couples to operators of the form $\exp(2i\Phi_R)$ and $\exp(2i\Phi_L)$. Using Eq. (2.20) for the ground-state correlations of these operators in the pure system, we find their scaling dimension is $\zeta=(4\pi\kappa)^{-1}$, which is greater than two everywhere in the gapless phase. Together with Eq. (1.4) relating the dimension of pure system operators to the RG eigenvalue of the second moment of the random coupling, this suggests that the alternating part of this random exchange interaction is irrelevant everywhere in the gapless phase. A similar analysis of the uniform part of the random perturbation, which couples to operators of the form $\exp(i\tilde{\Phi})$, reveals that it is irrelevant only in the range $\Delta > \sqrt{1/2}$. Since both components of the random perturbation are irrelevant in this range, it would appear that the quasi-long-range order of the pure system persists for Δ in this region. However, examination of higher-order terms generated by renormalization reveals that this is not the case. At second order, the operators coupled to the uniform and alternating parts of the random perturbation combine to give cross terms such as $e^{i\tilde{\Phi}}e^{2i\Phi_L}=e^{i\Phi}$, which are identical to the operators appearing in the case of XY-symmetric randomness. Since these operators are relevant for all $\Delta > -\frac{1}{2}$ and $e^{i\tilde{\Phi}}$ is relevant for $\Delta < \sqrt{1/2}$, we conclude that the XY-symmetry-breaking random exchange, Eq. (1.9), is relevant *everywhere* in the power-law phase of the spin-$\frac{1}{2}$ XXZ chain.

Because the runaway from the zero-disorder fixed line is caused by different operators depending on whether or not $\Delta < \sqrt{1/2}$, the crossover behavior in the presence of random planar anisotropy is more complicated than for XY-symmetric randomness. The RG flow equations for the disorder in this case are

$$\frac{\partial D_{\mathrm{SB}}}{\partial l}=\lambda_{\mathrm{SB}}D_{\mathrm{SB}} \, , \qquad \frac{\partial D_{\mathrm{S}}}{\partial l}=\lambda_{\mathrm{S}}D_{\mathrm{S}}+\alpha D_{\mathrm{SB}}^2 \, , \tag{3.21}$$

where

$$\lambda_{\mathrm{SB}}=3-8\pi\kappa \, , \qquad \lambda_{\mathrm{S}}=3-1/(2\pi\kappa)$$

are the eigenvalues of the operators coupled to the XY-symmetric and XY-symmetry-breaking randomness, respectively, and $D_{\mathrm{S}},D_{\mathrm{SB}}$ are the second moments of the XY-symmetric, effective random exchange and the symmetry-breaking random planar anisotropy, respectively. The coefficient of the quadratic term, α, in Eq. (3.21) is an unimportant constant whose value we have not computed. The second moments satisfy the initial conditions

$$D_{\mathrm{S}}(0)=0 \, , \qquad D_{\mathrm{SB}}(0)=D_{\mathrm{PA}} \, . \tag{3.22}$$

Solution of the flow equations reveals that if $\lambda_{\mathrm{SB}} < \lambda_{\mathrm{S}}/2$, the crossover is controlled by the XY-symmetric operator, $e^{i\Phi}$, which is generated. In this case, the behavior of the correlation length for weak randomness is

$$\xi \sim D_{\mathrm{PA}}^{-\phi_{\mathrm{PA}}} \, ,$$

where

$$\phi_{\mathrm{PA}}=2/\lambda_{\mathrm{S}} \tag{3.23}$$

is twice the crossover exponent for XY-symmetric randomness, Eq. (3.16). When $\lambda_{\mathrm{SB}} > \lambda_{\mathrm{S}}/2$, the crossover is controlled by the symmetry-breaking randomness and

$$\phi_{\mathrm{PA}}=1/\lambda_{\mathrm{SB}} \, . \tag{3.24}$$

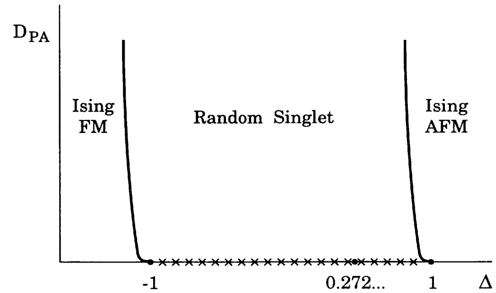

FIG. 3. Schematic phase diagram for XY-symmetry-breaking randomness with mean-square strength D_{PA}. The quasi-long-range ordered phase exists only on the line $D_{\mathrm{PA}}=0$ for $|\Delta|<1$. The crossover exponent for small randomness has a discontinuity in its derivative at $\Delta \simeq 0.272$.

Thus the crossover exponent for random in-plane anisotropy has a discontinuity in its derivative at $\Delta=0.2720\ldots$. As for the case of random exchange anisotropy, for $|\Delta|>1$ the ordered Ising phases are expected to persist with a small amount of random planar anisotropy or random planar field as shown in Fig. 3. The phase boundaries will have the same shape as in the random exchange case discussed earlier.

IV. PROPERTIES OF THE RANDOM PHASES

We have shown that a weak random field in the z direction and weak XY-symmetric random exchange interactions give rise to phase transitions from a quasi-long-range ordered ground-state resembling that of the pure system to disordered ground states. Although the critical behavior at these transitions is identical for both random field and random exchange disorder, we will see that the disordered phases in the two cases appear quite different. The fundamental reason for this difference is the symmetry of the full Hamiltonian in the random exchange case under $S_z \rightarrow -S_z$ and the antiferromagnetic tendencies, which combine to give some exact degeneracies. This symmetry is absent when a random field in the z direction is present. In this section we review previous work on disordered quantum spin chains relating to the properties of the disordered phases, and discuss these and some additional results.

A. Random z-field disorder

The spin-$\frac{1}{2}$ XXZ chain in a random field in the z direction is equivalent to a system of lattice fermions with nearest-neighbor interactions in a random potential. The point $\Delta=0$ is particularly simple, since interactions between the fermions vanish and the ground state can be described in terms of single-particle eigenfunctions, which are always exponentially localized for site-diagonal disorder in one dimension. In the ground state all the negative energy eigenstates are occupied, and hence the Fermi level lies at the band center, $\varepsilon=0$. Therefore, the density of states, $\rho(\varepsilon)$, near the Fermi level plays an important role in determining low-temperature properties.

For the Lloyd model, in which the random on-site energies have a Cauchy distribution, the density of states, $\rho(\varepsilon)$, can be calculated exactly and is smooth and positive at all energies.[10] This behavior is believed to hold for generic smooth distributions of the on-site energies in one dimension. Since there is a nonzero density of states at the Fermi level, the excitation spectrum in a random z field is gapless. The low-energy excitations are localized with a characteristic size given by the wave-function localization length at the Fermi level.

The low-temperature behavior of the uniform z susceptibility, χ_{zz}, defined by

$$\chi_{zz}=\frac{1}{N}\lim_{h_z\to 0}\left[\frac{\partial}{\partial h_z}\sum_{i=1}^{N}\langle S_i^z\rangle\right] \tag{4.1}$$

is also determined by $\rho(\varepsilon)$. Since the total magnetization in the z direction is simply the number of excess fermions above the Fermi level, the zero-temperature limit of χ_{zz} is proportional to the density of states at the Fermi energy. Thus χ_{zz} approaches a finite, nonzero constant as $T\to 0$, as it does in the quasi-long-range-ordered phase.

Klein and Perez[11] have used the correspondence between the $\Delta=0$ XXZ chain in a random transverse field and fermions in a random potential to examine the decay of the ground-state spin correlation function

$$C_{ij}=\langle S_i^+S_j^-\rangle \ . \tag{4.2}$$

They prove rigorously that C_{ij} decays exponentially with probability one in the limit of large separations $|i-j|$. Although this result may appear obvious, it is actually rather nontrivial due to the effects of the tail operators, which enter in the Jordan-Wigner transformation of Eq. (2.1).

Another consequence of the low-energy localized excitations in the disordered phase is that the in-plane susceptibility diverges at zero temperature even though the spatial correlations decay exponentially. This is because the susceptibility is given by an integral over *both* space and imaginary time:

$$\chi_{xx}=\frac{1}{N}\int d\tau\sum_{i,j}\tfrac{1}{2}\langle S_i^+(\tau)S_j^-(0)\rangle \ . \tag{4.3}$$

At fixed imaginary time difference, τ, the integrand decays exponentially in spatial separation $|i-j|$, but for small spatial separations the correlations decay slowly in time as $\exp(-\varepsilon\tau)$ in regions in which there is a low-energy localized excitation with energy ε. In particular the spatial average of

$$\overline{\langle S_i^+(\tau)S_i^-(0)\rangle}=\int d\varepsilon\,\rho(\varepsilon)e^{-\varepsilon|\tau|}\sim\frac{\rho(0)}{\tau} \tag{4.4}$$

is dominated by the low-energy excitations yielding a χ_{xx}, which diverges as

$$\chi_{xx}(T)\sim\ln\left[\frac{1}{T}\right] \tag{4.5}$$

at low temperatures.

If Δ is nonzero, then the equivalent Fermi system is interacting, and one cannot use the noninteracting results. This phase is, however, equivalent to a "Fermi glass" with short-range interactions, which have been studied by a number of authors.[12] For strong random fields the behavior can be understood by perturbing about the limit of no hopping; as for the noninteracting case, one expects low-lying localized excitations, in this case quasiparticles (which are well defined in the limit of low energies in spite of the interactions). These quasiparticles will have a constant density of states at zero energy, $\rho_Q(0)$. We can thus use similar arguments to the noninteracting case to yield a constant χ_{zz} and a logarithmically divergent χ_{xx} at low temperatures. We must now, however, distinguish between the quasiparticle density of states, $\rho_Q(\varepsilon)$, which determines χ_{zz} and the single-particle density of states, $\rho(\varepsilon)$, which determines χ_{xx}. The latter differs by "wave-function renormalization" from the former, but, with localized excitations, this will only give rise to a constant numerical factor in χ_{xx}. The localization length of the excitations near the Fermi level yields the characteristic length, ξ, for the decay of spatial correlations of the spins. It is this length that will diverge as the transition to the quasi-long-range-ordered phase is approached, with the form of the divergence given by Eq. (3.15).

B. Random exchange disorder

The behavior of the spin chain with random exchange is rather different. Again, most of the known results pertain to the $\Delta=0$ spin chain with a random component in J_{xy}. This corresponds to a tight-binding model of free fermions with random nearest-neighbor hopping but no random potential so that the system has an exact particle-hole symmetry. As in the random field case, there exist special distributions of the randomness for which the density of states can be calculated exactly. In particular, for several generalized Poisson distributions, $\rho(\varepsilon)$ can be calculated using a method introduced by Dyson[13] to study a harmonic chain with random masses and spring constants. For these distributions the density of states is singular at the band center, where it behaves as

$$\rho(\varepsilon)\sim\frac{1}{|\varepsilon\ln^3\varepsilon|} \ . \tag{4.6}$$

A general argument due to Eggarter and Riedinger[14] indicates that this form is generic for purely off-diagonal randomness in one dimension. The form of the singularity implies that the susceptibility, χ_{zz}, is divergent at low temperatures,

$$\chi_{zz} \sim \frac{1}{T \ln^2 |T|} , \tag{4.7}$$

in marked contrast to the random field case.

As in the random field case, the excitation spectrum in the random exchange model is also gapless because $\rho(\varepsilon)$ is nonzero at the band center. However, the single-particle wave functions at very small energies are qualitatively very different. Away from the band center the wave functions are exponentially localized, but as the band center is approached the localization length, $l(\varepsilon)$ diverges. Theodorou and Cohen[15] have examined the form of this divergence for the Dyson model and find

$$l(\varepsilon) \sim -\ln \varepsilon . \tag{4.8}$$

Since the Fermi level in the spin chain lies at $\varepsilon = 0$, the low-energy excitations are only weakly localized, and hence one would not expect strictly exponential decay of spatial correlations in this disordered phase. By assuming a single characteristic length, $l(\varepsilon)$, for the properties of the low-energy wave functions, one finds a schematic form for the average correlation function

$$\overline{C_{ij}(\tau)} \sim \int_0^\infty d\varepsilon\, \rho(\varepsilon) e^{-\varepsilon|\tau|} e^{-|i-j|/l(\varepsilon)} . \tag{4.9}$$

The average equal time correlation function then decays as

$$\overline{C_{ij}} \sim \frac{1}{|i-j|^2} , \tag{4.10}$$

i.e., a power-law decay even in the disordered phase. The in-plane susceptibility at low temperatures can also be readily found to be

$$\chi_{xx}(T) \sim \frac{1}{T \ln^2 T} , \tag{4.11}$$

which is of the same form as χ_{zz}. Note that one factor of $\ln T$ in Eq. (4.11) arises from the spatial integral over the correlation function. As we shall see below, it is not clear that the low-energy excitations are well characterized by a single length; nevertheless the form of the average correlation function in Eq. (4.10) is likely to be correct. Note, however, that the *typical* correlation function will decay much more rapidly, since the average is dominated by rare events.

For $\Delta \neq 0$, the properties of the disordered phase are less well understood. Several authors[16] have used approximate real-space renormalization-group methods to conclude that the qualitative behavior is very similar to the XY case with $\Delta = 0$. Indeed it is likely that the form, Eq. (4.6), of the density of states, which determines the z susceptibility remains the same for all $|\Delta| < 1$ for strong randomness. One of the authors[17] has recently shown this to be the case using an asymptotically exact renormalization group. This behavior would probably even persist for $\Delta \geq 1$ with a transition to Ising antiferromagnetic order at some critical $\Delta_c > 1$ (which approaches unity for weak randomness).

The basic picture of this phase is that of tightly coupled singlet pairs of spins. Strong exchange bonds, J_{ij}, will pair nearest-neighbor spins, while other spins can be coupled over long distances via virtual excitations of the intervening pairs. The arguments of Eggarter and Reidinger[14] and Dasgupta and Ma[16] suggest that the effective interaction between spins with separation R, which are not paired with the intervening spins, will be of order $\exp - \sqrt{R}$. This implies that at a temperature T there will be a density of spins $1/R(T) \sim 1/\ln^2 T$, which are still unpaired and hence contribute to the susceptibility, yielding a χ_{zz} of the form of Eq. (4.11). The existence of these almost free spins arise from the degeneracy of the ground state for chains with an odd number of spins. Since the total S_T^z is a good quantum number and the Hamiltonian is symmetric under $S^z \rightarrow -S^z$, any odd length chain will have a degenerate pair of ground states with $S_T^z = \pm\frac{1}{2}$. In the uniform phase the difference between these two states will be spread over the whole chain, but in the presence of sufficiently strong randomness the difference will primarily be localized on one or a small group of spins. If two such chains are joined together, the free spins will pair together to form a low-energy singlet-triplet excitation.

The spatial correlations in this picture do not decay uniformly: the "almost free" spins may be anomalously strongly correlated to spins far away. The probability of a pair of spins at separation R being strongly correlated is of order $1/R^2$, which is just proportional to the probability that *both* are free at temperature $T(R)$. Thus the contribution of these pairs to the average spatial correlation function yields

$$\overline{C_{ij}} \sim \frac{1}{|i-j|^2} ,$$

the same form as found previously from the assumption of a single length scale. Although the single length scale assumption appears to be incorrect, the contribution to $\overline{C_{ij}}$ from the strongly coupled pairs is of the same order as that from the exponential tail in Eq. (4.9). Note that this behavior cannot occur in the presence of a random (or uniform) z field, since in that case there will no longer be exact degeneracies. The half-integer spin is also important in giving rise to isolated spins: integer spins can, in some circumstances, pair half with each of two other spins, although there should also be circumstances in which they will form random singlet phases.

As the phase boundary between the random singlet phase and the quasi-long-range-ordered antiferromagnetic phase is approached, there will be a characteristic length ξ above which the decay of correlations become broadly distributed. This length, which is likely to characterize the typical long-distance decay in the disordered phase, should diverge as the transition is approached, as given by Eq. (3.15). A better understanding of the "random singlet" phase will have to await further work.

C. XY-symmetry-breaking randomness

We finally consider the disordered phases, which will result from in-plane random fields or random anisotropies.

In the case of random in-plane anisotropy, the Hamiltonian is still invariant under 180° rotations about the z axis as well as $S_z \to -S_z$. The total z component of the spin, S_T^z, is thus still a good quantum number mod 2. This implies that odd length chains will again have exact degeneracies. We thus again expect, at least for some range of parameters, a random singlet phase with divergent susceptibilities of the form discussed above for the random exchange case. The X and Y susceptibilities still have similar divergent contributions because they connect states with S_T^z differing by ± 1, which can therefore be almost degenerate.

In the case of random planar field, S_T^z is no longer a good quantum number although the system is still symmetric under $S^z \to -S^z$ (or more precisely a rotation of 180° about the z axis followed by time reversal). The absence of S_T^z conservation gives rise to local level repulsion, which suppresses the large low-energy density of states, which exists in the more XY-symmetric cases. This should make the susceptibilities finite at zero temperature at least for strong random fields. However, this will not always be the case. If the Ising anisotropy is sufficiently large, but not large enough to cause a transition to a long-range Ising ordered phase, then regions of the system with a stronger tendency towards Ising ferromagnetic order can support low-energy excitations between states, which are locally approximately given by even and odd combinations of mostly up and mostly down spins (or the two staggered combinations for the antiferromagnetic case). In such a "Griffiths phase," these regions can give rise to a strongly divergent χ_{zz} (or staggered susceptibility) if they occur with sufficient spatial density. They would not, however, appear to give rise to divergent χ_{xx} or χ_{yy}, since the two states differ by a large number of spin flips. This behavior has been found explicitly for the exactly solvable case with $\Delta \to -\infty, h_i^y = 0$, and a random h_i^x with magnitude of order $|\Delta|$.[18] A schematic phase diagram for random planar fields is shown in Fig. 4. Note however, that although χ_{zz} diverges at some value of the anisotropy before the Ising ordered phase (shown as a dashed line in Fig. 4), this does not represent a true phase boundary.

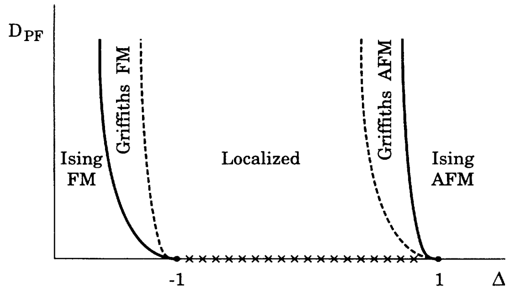

FIG. 4. Schematic phase diagram for random fields in the XY plane with mean-square strength D_{PF}. The "Griffiths phases" are not ordered but have divergent susceptibility or staggered susceptibility in the z direction. The dashed lines represent the boundaries of the regions with divergent χ_{zz}, but they are not true phase boundaries.

V. CONCLUSIONS

In this section we briefly summarize some of our main results and raise questions for further study on random quantum spin chains.

We have used a generalized Harris' criterion along with a perturbative RG to study the phase diagram and critical behavior near the phase boundary of the spin-$\frac{1}{2}$ XXZ chain with several kinds of weak randomness. For random fields and random exchange disorder, which preserve the XY symmetry, we have found a transition, as the anisotropy Δ is varied, from a quasi-long-range-ordered ground state resembling that of the pure system to one in which typical correlation functions decay rapidly. Although the critical behavior at this transition is identical for the random field and random exchange cases, the resulting random phases are quite different. In the case of a random z field, the disordered phase is expected to resemble a "Fermi glass" in which the elementary excitations are localized and spin correlations decay exponentially, even though the susceptibility χ_{xx} is infinite. For XY-symmetric random exchange, on the other hand, one finds a "random singlet" phase in which spins are tightly coupled in singlet pairs. In contrast to the random field case, where elementary excitations are well localized, these singlet pairings can occur over large distances, giving rise to strongly divergent susceptibilities, χ_{xx} and χ_{zz}, at $T \to 0$. Further work on this random singlet phase would be useful, especially since analogous phases also occur for spin-$\frac{1}{2}$ systems in higher dimensions such as the insulating and metallic phases of phosphorus doped silicon.[19]

In addition to the phase transition from the quasi-long-range-ordered XY phase induced by randomness, there also exist randomness-induced transitions from states with long-range Ising ferromagnetic or antiferromagnetic order. At this point, only one of these transitions is understood at all[18] and the nature of other such transitions—especially those in the presence of other conserved quantities—is a very interesting question for future research.

Finally, it should also prove interesting to investigate the behavior of higher spin chains in the presence of quenched disorder. Affleck and Haldane[20] have constructed critical theories for higher half-odd-integer spin chains via non-Abelian bosonization, and further progress along these lines and those discussed in this paper should be possible.

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