RAPID COMMUNICATIONS

PHYSICAL REVIEW B VOLUME 45, NUMBER 3 15 JANUARY 1992-I

Relationship between resistance, localization length, and inelastic-scattering length

Y. B. Band
Department of Chemistry, Ben-Gurion University, Beer Sheva, Israel

H. U. Baranger
AT&T Bell Laboratories, 600 Mountain Avenue, Murray Hill, New Jersey 07974

Y. Avishai
Service de Physique Theorique, Centre d'Etudes Nucléaires Saclay, F91191 Gif sur Yvette, France
and Department of Physics, Ben-Gurion University, Beer Sheva, Israel
(Received 19 June 1990; revised manuscript received 4 November 1991)

An integral equation for the average resistance of a wire sample of length L, $R(L)$, is obtained in terms of the probability density function for inelastic scattering and the average quantum resistance due to elastic scattering within the sample. This equation yields the average sample resistance as a function of the sample length L, the electron localization length ξ, and the inelastic-scattering length $\lambda(T)$, which depends on temperature. For the metallic regime, $L>\xi>\lambda(T)$, and the insulating regime, $L>\lambda(T)>\xi$, analytic expressions for the average resistivity $\rho(T)$ in terms of ξ and $\lambda(T)$ are obtained. Our approach allows a unified treatment of both regimes. The relationship of the present results to those of Abrahams *et al.* and Thouless on the conductance of thin wires is discussed.

The measurement of dc resistance of a sample as a function of sample length L and temperature T provides much information on the nature of the sample and is therefore of fundamental importance. In this paper we present a simple formulation to calculate the average resistance, $R(L,T)$, of a thin wire sample of length L in terms of the probability density function for inelastic scattering, $P(x,L,T)$ (defined below), and the average sample resistance $r(x)$ due to elastic scattering within the sample. The latter quantity is the resistance determined from the quantum mechanical transmission and reflection amplitudes for elastically scattered waves traversing a length x,[1] averaged over distributions of elastic-scattering centers.[2,3] The former quantity is the probability density function for inelastic scattering not to occur in the region $[0,x]$ and to occur in the region $[x,x+dx]$ in a sample of length L. The main result of this work applies directly to conduction in insulators, semiconductors,[4] metallic glasses,[5] and metals.[6] It also bridges the gap between the recent work on the ballistic electronic conductance of small samples[7] and studies at higher temperatures and/or long sample lengths in which inelastic collisions become important. Band and Avishai[2] derived the expression for $R(L,T)$ in terms of the probability density function $P(x,L,T)$ and an average elastic-scattering resistance $r(x)$ for the special case of a periodically varying one-dimensional potential which elastically scatters electrons, with and without inclusion of randomization of the periodic potential (i.e., averaging over randomizations). Here we generalize the result to an arbitrary average elastic-scattering potential and thereby obtain a more general expression.

The mutual effects of elastic and inelastic scattering on resistance have been considered by many authors.[8-16] The model we develop includes the effects of phase randomizing inelastic scattering on the electronic propagation in an average elastic potential. We initially assume that the major effect of an inelastic collision is phase randomization of the electronic wave function; following Refs. 15 and 16, backscattering due to inelastic scattering is not included (backscattering due to elastic scattering is). We provide a comparison of our results with these references. We then phenomenologically add a contribution to the resistance from proper inelastic-scattering effects using Matthiessen's rule,[17] which also accounts for inelastic backscattering.

The generalization of the expression[2] for $R(L,T)$ is given by

$$R(L,T)=\int_0^L dx_1 P(x_1,L,T)\left[r(x_1;0)+\int_{x_1}^L dx_2 P(x_2-x_1,L-x_1,T)\right.$$
$$\left.\times\left\{r(x_2;x_1)+\int_{x_2}^L dx_3 P(x_3-x_2,L-x_2,T)[r(x_3;x_2)+\cdots]\right\}\cdots\right]. \tag{1}$$

This formula is obtained employing the following phenomenological argument. Without inelastic collisions, the resistance $R(L,T)$ equals the average resistance (averaged over the distribution of elastic-scattering sites[2,3]), $r(L;0)$. If an inelastic collision occurs at x_1, the phase of the electronic wave function is disturbed and the resistance is computed by adding in series the resistance $r(x_1;0)$ and the resistance of the remainer of the path.[3] We define the probability of proceeding to x_1 without suffering a phase-breaking collision and then suffering a phase-breaking collision at x_1 as $P(x_1,L,T)$. $r(x_1;0)$ must be multiplied by the probability $P(x_1,L,T)$ and an integral over all possible x_1 from zero to L must be taken. Now, let us consider the whole random walk of the electron. The electron starts from $x=0$ and proceeds to x_1 without suffering an inelastic collision, and suffers its first inelastic collision at x_1 with probability $P(x_1,L,T)$. The resistance associated with this random walk is $r(x_1;0)$. The electron then proceeds from x_1 to x_2 without suffering an inelastic collision, and suffers its second inelastic collision at x_2 with probability $P(x_2,L,T)$, and so on. The resistance of each of these random walks must be added in series. Equation (1) expresses the series summation of resistances weighted by the appropriate probability distributions. In what follows we shall assume translational invariance of the average elastic resistance, i.e., $r(x_2;x_1)=r(x_2-x_1)$. It should be stressed that the assumption of translational invariance for $r(x_2;x_1)$ is valid only upon taking an impurity average of the sample, and the approach used here is appropriate only for systems where fluctuation phenomena in the transport are of no relevance.

It is a simple matter to convert the series in Eq. (1) into the integral equation

$$R(L)=\int_0^L dx\,P(x,L)[r(x)+R(L-x)]\,, \tag{2}$$

(we have dropped the explicit T dependence). This is a universal form for $R(L)$ in terms of an arbitrary density function $P(x,L)$ and an elastic collision resistance $r(x)$, in the sense that it holds regardless of the detailed form of $r(x)$ and $P(x,L)$. Given $P(x,L)$ and $r(x)$, the resistance $R(L)$ can be calculated by numerically solving the integral equation.

While Eq. (2) is valid for all L, it is of interest to use Eq. (2) to obtain an expression for $R(L)$ for the case of large L, i.e., $L\gg\xi,\ell(T)$, where ξ is the electron localization length and $\ell(T)$ is the temperature-dependent inelastic-scattering length, since this is the usual limit for macroscopic samples. In this limit, the resistance is linearly proportional to the length of the sample with the constant of proportionality called the resistivity, $R(L,T)=\rho(T)L$. We can easily obtain an expression for the resistivity ρ as a function of ξ and $\ell(T)$ for this macroscopic limit. Substituting the expression $R(L)=\rho L$ into Eq. (2), we obtain

$$\rho L=\int_0^L dx\,P(x,L)r(x)+\rho L-\rho\langle x\rangle\,, \tag{3}$$

where

$$\langle x\rangle=\int_0^L dx\,P(x,L)x\,. \tag{4}$$

Thus,

$$\rho(T)=\frac{\int_0^L dx\,P(x,L)r(x)}{\int_0^L dx\,P(x,L)x}=\frac{\langle r\rangle}{\langle x\rangle}\,. \tag{5}$$

We would now like to specify a probability density function $P(x,L)$ and a resistance $r(x)$. For a *conductor*, $\ell(T)\ll\xi$, the number of electrons that have not suffered a collision decays exponentially with length as $\exp[-x/\ell(T)]$ where $\ell(T)$ is the inelastic-scattering length. The exponential character originates from the dynamics of population decay, $dn/dx=-N\sigma n$, where n is the electron density unaffected by inelastic collisions, $N(T)$ is the density of the phonons at temperature T and σ is the inelastic cross section for electron-phonon collisions, and $\ell(T)^{-1}=N(T)\sigma$. Clearly, the probability density for large L also decays exponentially; for finite L, the normalized probability density is

$$P(x,L)=\ell(T)^{-1}\exp[-x/\ell(T)]+\exp[-L/\ell(T)]\delta(x-L+\epsilon)\,, \tag{6}$$

where ϵ is a positive infinitesimal number. The last term guarantees a normalized proability distribution and assures that if no inelastic collision occurs in the sample, $R(L)=r(L)$. It is consistent with the ideas introduced by Büttiker regarding resistance of perfect leads.[8] In the localization regime, the randomization of the elastic potential yields localization of the electronic wave function and results in an exponential form of the average resistance for large x given by[15]

$$r(x)=r_0[\exp(x/\xi)-1]\,. \tag{7}$$

This form of the average elastic resistance vanishes as $x\rightarrow 0$, as required.

An explicit expression for the resistivity can now be obtained using Eq. (6) for the probability density function and Eq. (7) for the elastic-scattering resistance in Eq. (5) for ρ. Assuming $R(L,T)=\rho(T)L$, we obtain the following expression for the resistivity as a function of ξ and $\ell(T)$,

$$\rho(T)=\frac{(r_0/\xi)(1-\exp\{-L/\ell(T)[1-\ell(T)/\xi]\})}{[1-\ell(T)/\xi]\{1-\exp[-L/\ell(T)]\}}\,. \tag{8}$$

We conclude from Eq. (8) that the resistivity varies inversely with ξ for $\ell(T)/\xi\ll 1$, the resistivity depends on both ξ and $\ell(T)$ for $\ell(T)/\xi<1$, and it diverges exponentially with the power of the exponential dependent on $L/\ell(T)$ for $\ell(T)/\xi>1$ [hence the assumption $R(L,T)=\rho(T)L$ is not warranted for $\ell(T)/\xi>1$, but see below for modifications in this regime]. Figure 1 plots $\rho(T)/(r_0/\xi)$ vs $\ell(T)/\xi$ for a number of values of $L/\ell(T)$. For $\ell(T)/\xi\ll 1$, the dependence on $\ell(T)/\xi$ is indeed negligible. For $\ell(T)/\xi<1$, $\rho(T)$ depends on both $\ell(T)$ and ξ independently, but does not depend on $L/\ell(T)$ (i.e., the resistivity is independent of sample length). Note, however, that inelastic collisions have been used only to inhibit localization effects in our analysis so far. We have yet to include the effects of inelastic collisions contributing directly to the resistance.

For an *insulator*, $\ell(T)>\xi$, the hopping mechanism is weak and the resistance increases dramatically. In this localized regime, the "probability of proceeding to x_1 without suffering a phase-breaking collision and then suffering a phase-breaking collision at x_1," $P(x_1,L,T)$, must be modified. The probability $P(x_1,L,T)$ is no longer independent of the localization length. The form of $P(x_1,L,T)$ necessary to take into account Mott variable range hopping in a thin wire is

$$P(x,L)=C\exp[-x(\ell^{-1}+2\xi^{-1})-\Delta(x)/k_BT]+\exp[-x(\ell^{-1}+2\xi^{-1})-\Delta(x)/k_BT]\delta(x-L+\epsilon)\,. \tag{9}$$

Here, $\Delta(x)$ is the mean hopping energy for hopping a distance x, $\Delta(x)=(xWdn/dE)^{-1}$, where W is the width of the thin wire and dn/dE is the density of states per unit energy.[18] The $\Delta(x)$ term in the exponential accounts for the fact that the probability $P(x,L)$ must go to zero at small x (the probability of finding a site with appropriate hopping energy becomes small), the ℓ^{-1} term is unimportant in the $\ell(T)>\xi$ regime, and the $2\xi^{-1}$ term originates from the site overlap integral.[18] The resulting resistance is linear in L for $L>\ell(T)$, and using Eq. (5) the resistivity can be analytically determined in terms of modified Bessel functions. It can be shown to behave asymptotically as Mott variable range hopping in one dimension, $\rho(T)\sim\exp(a[\xi W(dn/dE)k_BT]^{-1/2})$, where a is a numerical constant. However, we should note that the variable-range-hopping resistance in 1D may be dominated by the lowest probability hop and the average arguments used by Mott do not account for this.

We emphasize that the probability in Eq. (9) allows for a unified treatment of metal and insulator regimes and a smooth transition between them.

Previously, Abrahams *et al.*[15] and Thouless[16] considered the conductance of thin wires in the regime $\ell(T)\ll\xi$. They obtained the result that for a fully phase-coherent sample of size L

$$R_{\mathrm{coh}}(L,T)=\frac{\sigma_0^{-1}L}{1-L/\xi}\,, \tag{10}$$

where σ_0 is the conductance times an appropriate short sample length[16] (see below). For a system of size L larger than $\ell(T)$, the resistance can be obtained by adding up the resistance of an appropriate number of phase-coherent lengths, i.e., lengths of the order $\ell(T)$. This gives

$$R(L,T)=R_{\mathrm{coh}}[L=\ell(T),T]\frac{L}{\ell(T)}=\frac{\sigma_0^{-1}L}{1-\ell(T)/\xi}\,, \tag{11}$$

and, hence,

$$\rho(T)=\frac{\sigma_0^{-1}}{1-\ell(T)/\xi}\,. \tag{12}$$

Taking $\sigma_0=r_0^{-1}\xi$, i.e., using ξ as the appropriate short sample length, we see that this equation, which results from the analysis in Refs. 15 and 16, is equivalent to that obtained from Eq. (8) for $\ell(T)<\xi\ll L$. Our method provides a more rigorous means of adding up the resistance of the phase coherent lengths than in Eq. (11), and produces the same result when $\ell(T)<\xi\ll L$.

We now consider the direct contribution of inelastic-scattering collisions to the resistance. From Matthiessen's rule,[17] when two scattering mechanisms (elastic and inelastic) contribute to the resistance, $\rho(T)=\rho^{(1)}(T)+\rho^{(2)}(T)$. In the metallic regime, Eq. (8) furnishes the resistivity $\rho^{(1)}(T)$ due to elastic scattering, as affected by inelastic scattering limiting localization. The simple formulation of Ohm's law gives $\rho^{(2)}(T)=m/[ne^2\tau(T)]=k_F/ne^2\ell(T)$ due to proper inelastic scattering, where m is the electron effective mass, n is the electron density, k_F is the Fermi momentum, and τ is the inelastic collision time. Thus, we finally obtain

$$\rho(T)=\frac{r_0/\xi}{1-\ell(T)/\xi}+\frac{k_F/ne^2}{\ell(T)}\,,\quad \text{for } \ell(T)<\xi<L\,. \tag{13}$$

In this argument we assumed for simplicity that the transport mean free path from inelastic processes is the same as the total phase-breaking mean free path; this assumption can easily be relaxed.

Knowledge of the temperature dependence of $\ell(T)$ allows determination of the temperature dependence of the resistivity. For example, in Fig. 2 we plot $\rho(T)/(r_0/\xi)$ as determined by Eq. (13) versus temperature using the

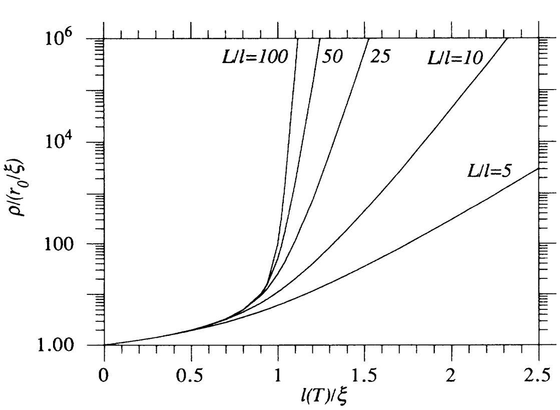

FIG. 1. $\rho(T)/(r_0/\xi)$ vs the ratio of the inelastic-scattering length to the localization length as obtained from Eq. (10). The results in the region $\ell(T)/\xi<1$ satisfy the assumption of resistivity independent of sample length.

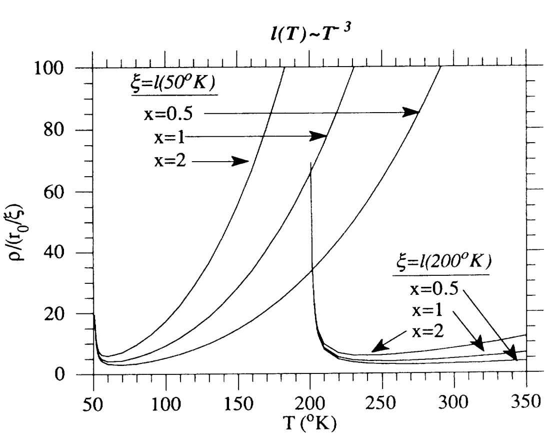

FIG. 2. Calculated $\rho(T)/(r_0/\xi)$ vs T, assuming $\ell(T)\sim T^{-3}$, for two values of the localization length, $\xi=\ell(200\text{ K})$ and $\xi=\ell(50\text{ K})$. For $T<200$ K and $T<50$ K, respectively, the resistivity is exponentially increasing and is length dependent.

dependence $\ell(T) \sim T^{-3}$, for two different values of the localization length corresponding to $\xi = \ell(200\text{ K})$ and $\xi = \ell(50\text{ K})$, and three different values of the dimensionless ratio $x = k_F/ne^2 r_0$. For temperatures near those for which $\ell(T) = \xi$, the resistivity decreases rapidly with increasing temperature due to the first term on the right-hand side of Eq. (13), and the material is insulating for lower temperatures. For high temperatures, the second term in Eq. (13) dominates and the resistivity increases with increasing temperature. Note that the resistivity goes through a minimum as a function of temperature. Determination of the parameters appearing in the present theory may be possible from the deconvolution of experimentally determined temperature dependence of the resistivity of samples.

We would like to thank M. Kaveh, Z. Ovadiahu, and M. Büttiker for valuable suggestions. This work is partially supported by a grant from the U.S.-Israel Binational Science Foundation.

---

[1] R. Landauer, IBM J. Res. Dev. **1**, 223 (1957); Philos. Mag. **21**, 863 (1970); M. Ya Azbel, J. Phys. C **14**, L225 (1981); P. W. Anderson, Phys. Rev. **109**, 1492 (1958); D. S. Fisher and P. A. Lee, Phys. Rev. B **23**, 6851 (1981); C. M. Soukoulis and E. N. Economou, Phys. Rev. Lett. **46**, 618 (1981).

[2] Y. B. Band and Y. Avishai, Phys. Rev. B **34**, 3429 (1986).

[3] P. Erdos and R. C. Herndon, Adv. Phys. **31**, 65 (1982).

[4] N. F. Mott, Philos. Mag. B **58**, 369 (1988).

[5] M. A. Howson and B. L. Gallagher, Phys. Rep. **170**, 265 (1988).

[6] F. Seitz, *The Modern Theory of Solids* (Dover, New York, 1987).

[7] See C. W. J. Beenakker and H. van Houten, in *Solid State Physics*, edited by H. Ehrenreich and D. Turnbull (Academic, New York, 1991), Vol. 44, and references therein.

[8] See, for example, M. Büttiker, Phys. Rev. B **33**, 3020 (1986).

[9] J. L. D'Amato and J. M. Pastawski, Phys. Rev. B **41**, 7411 (1990).

[10] S. Datta, Phys. Rev. B **40**, 5830 (1989).

[11] A. Stern, Y. Aharonov, and Y. Imry, Phys. Rev. A **41**, 3436 (1990).

[12] N. Wingreen, K. W. Jacobson, and J. W. Wilkins, Phys. Rev. Lett. **61**, 1396 (1988).

[13] S. Feng, Phys. Lett. A **143**, 400 (1990).

[14] S. Hershfield, Phys. Rev. B **43**, 11 586 (1991).

[15] E. Abrahams, P. W. Anderson, D. C. Licciardello, and T. V. Ramankrishnan, Phys. Rev. Lett. **42**, 673 (1979); P. A. Lee and T. V. Ramankrishnan, Rev. Mod. Phys. **57**, 287 (1985).

[16] D. J. Thouless, Solid State Commun. **34**, 683 (1980); Phys. Rev. Lett. **39**, 1167 (1977); J. Non-Cryst. Solids **35-36**, 3 (1980).

[17] N. W. Ashcroft and N. D. Mermin, *Solid State Physics* (Holt, Rinehart and Winston, New York, 1976); R. Kubo and T. Nagamiya, *Solid State Physics* (McGraw-Hill, New York, 1969), p. 120.

[18] N. F. Mott, J. Non-Cryst. Solids **1**, 1 (1968); A. Miller and E. Abrahams, Phys. Rev. **120**, 745 (1960); T. G. Castner, in *Hopping Transport in Solids*, edited by M. Pollak and B. Shklovskii (North-Holland, Amsterdam, 1991), pp. 7-9.