Nearly antiferromagnetic Fermi liquids: An analytic Eliashberg approach

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I study a model in which quasiparticles on a two-dimensional square lattice interact via exchange of antiferromagnetic spin fluctuations, determining when one may write the Eliashberg equations for the normal-state self-energy and for T_c. For parameters which have been argued to be appropriate for high-T_c superconductors, I find the mass enhancement and scattering rate to vary over the Fermi surface by a factor of the ratio of the antiferromagnetic correlations length ξ to the lattice constant. The pairing kernel is dominated by high-frequency spin fluctuations, and so T_c in the BCS approximation is essentially independent of ξ, explaining previous numerical work. The analytic solution suggests normal-state self-energy effects may not lower T_c as severely as in other models of d-wave superconductivity.

The notion that superconductivity may be produced by exchange of spin fluctuations has a long history in condensed-matter physics. The p-wave spin-triplet superfluidity of ^{3}He is believed to be due to exchange of ferromagnetic spin fluctuations (paramagnons).[1] More recently, the exchange of antiferromagnetic spin fluctuations (antiparamagnons) has been argued to lead to spin-singlet higher-angular-momentum pairing in quasi-one-dimensional metals[2] and heavy-fermion compounds.[3] Rather detailed attempts have been made to describe the superconductivity of, e.g., the heavy-fermion compound UPt_3 using an Eliashberg-equation approach[4] and taking the experimentally determined dynamic spin susceptibility as input. One difficulty with the Eliashberg approach is that it neglects terms of order (ω_{SF}/E_F), where ω_{SF} is the frequency scale characteristic of the spin fluctuations and E_F is the Fermi energy of the quasiparticles. It is not clear whether the parameter ω_{SF}/E_F is in fact small in the heavy electron metals. Another difficulty is that there is as yet no derivation from a more microscopic theory of the ansatz that quasiparticles in a strongly correlated electronic system interact by exchanging Bose fluctuations with propagators given by measured spin or charge response functions. A third difficulty is that the properties (including the T_c) of a hypothetical magnetically-mediated superconductor described by the Eliashberg equations are much more sensitive to normal-state self-energy effects than are the properties of a conventional phonon-mediated superconductor.[5] In particular, the T_c is much lower for fixed coupling strength in the magnetically-mediated than in the phonon-mediated case and the sensitivity to pair breaking by inelastic scattering is much greater.

The discovery of high-temperature superconductivity in doped antiferromagnetic insulators has led to intense theoretical work on various models of superconductivity, among which are models very similar to the spin-fluctuation exchange models previously considered in the context of superfluidity in ^{3}He and superconductivity in heavy-fermion materials. Interest in these models has been bolstered by the discovery that nuclear magnetic resonance (NMR) experiments on the high-T_c superconductors may be interpreted as implying the presence of strong temperature-dependent antiferromagnetic correlations.[6] Several groups have explored the idea[7] that the superconductivity and the normal-state properties such as the linear resistivity in these materials may be understood in a model involving conventional bandlike electrons interacting by exchange of magnetic excitations. One approach has been to use the random phase approximation (RPA) to the Hubbard model to calculate various physical properties. Bulut, Scalapino, and Morawitz have calculated the average quasiparticle damping rate to second order in the Hubbard interaction, U.[8] Wermbter and Tewordt have used an approximation to the RPA susceptibility as a kernel in Eliashberg equations.[9] They approximated the susceptibility $\chi(q,\omega)$ by the separable form $\chi(q,\omega)=Q(q)\Psi(\omega)$ and further approximated $Q(q)$ by the second-order harmonic

$$Q(q)=Q_0+Q_1[\cos(q_x a)+\cos(q_y a)] ,$$

and then solved numerically for the average mass enhancement, quasiparticle scattering rate and T_c, concluding that T_c would be too low to be relevant to cuprate superconductivity. Other workers[10,11] have performed similar calculations using non-RPA models for the magnetic excitations. Closely related work has been stimulated by the "spin-bag" model of high-temperature superconductivity.[12] The physical picture underlying this model is of a small number of carriers doped into an ordered antiferromagnetic; these are argued to form polaronlike entities which pair, leading to superconductivity. Subsequent work[13] has focused on justifying this picture starting from the high doping limit, by studying models very similar to those of Refs. 8–11. However, in this work[13] the main focus is on structure in the one-electron spectral function which occurs at energies somewhat removed from the Fermi energy. I will argue in this paper that, when the Eliashberg equation I derive are valid, this structure is not important for low-energy quantities such as the one-electron mass enhancement.

In a very interesting recent paper[14] Monthoux, Balatsky, and Pines (MBP) have pursued this idea by formulating and solving numerically a gap equation for superconductivity due to exchange of antiferromagnetic spin fluctuations. They assumed that the propagator describing these spin fluctuations was given by the dynamical susceptibility inferred from NMR experiments. They wrote down a three-dimensional integral equation for the superconducting gap function. They did not include normal-state self-energy effects. Their calculation is technically interesting and different from what has been done previously because their spin-fluctuation propagator was taken to be very sharply peaked about a particular wave vector **Q** which does not span the Fermi surface. Thus this calculation includes an interplay between commensurate magnetic fluctuations and an incommensurate Fermi surface. Anther important feature of their calculation is that the spin-fluctuation propagator was *not* assumed to have the separable form $\chi(q,\omega)=Q(q)\Psi(\omega)$. By use of symmetry arguments and of several approximations they reduced the gap equation to a one-dimensional integral equation, which they solved numerically. They found several interesting results: (a) the superconducting gap transforms according to the $d_{x^2-y^2}$ representation of the symmetry group D_4 of the CuO_2 plane; (b) the numerically calculated T_c is relatively high and is well described by a BCS-like formula $T_c \sim \Gamma \exp(-1/\lambda)$, with λ an average interaction strength. T_c did not, in their numerical results, apparently depend on the long correlation length ξ or slow spin-fluctuation frequency ω_{SF} characterizing the antiferromagnetic fluctuations, but instead involved only the electron-spin-fluctuation coupling constant λ and the energy scale Γ characterizing spin fluctuations with momentum far from the antiferromagnetic wave vector.

Whether the $d_{x^2-y^2}$ state predicted by Monthoux, Balatsky, and Pines is actually observed in the high-T_c materials is unclear. The weak temperature dependence[15] of the penetration depth at temperatures much less than T_c is widely believed to be inconsistent with states (such as the $d_{x^2-y^2}$) having nodes in the superconducting gap, although this conclusion is not universally accepted.[16] However, other properties including the temperature dependences in the superconducting state of Knight shifts[17] and of NMR relaxation rate anisotropies have been argued[18] to be inconsistent with conventional *s*-wave pairing. Angle-resolved photoemission experiments[19] have not yet detected the gap nodes in the (1,1) direction occurring in a $d_{x^2-y^2}$ state.

In this paper I investigate analytically some theoretical questions raised by models in which exchange of antiferromagnetic spin fluctuations is the dominant quasiparticle interaction mechanism in the high-T_c materials, and in particular by the calculation of Ref. 14. I do not discuss the derivation of such models from the microscopic Hamiltonian. I do point out some previously unnoticed experimental consequences of the model. I believe an analytic treatment of the problem is useful because it elucidates the role played by the strong momentum dependence (and the interplay of this with the strong energy dependence) of the interaction, which was in previous work either averaged over or treated by a numerical method which involved approximations and was difficult to interpret. I study the electron self-energy of the model, showing that in the parameter regimes believed to be appropriate for the high-T_c material, the self-energy is given by equations very similar to the Eliashberg equations arising in the electron-phonon problem. I estimate the limits of validity of the Eliashberg-like equations, arguing that too near half filling the equations no longer apply and indicating what additional terms should be added. Where Eliashberg equations apply, I obtain and solve approximately the gap equation with and without self-energy effects and argue that self-energy effects in this model may be weaker than expected from other models of spin-fluctuation-mediated *d*-wave superconductivity, although a numerical treatment is required for a definite answer. I also obtain expressions for the mass enhancement and scattering rate, showing that these may have a dramatic variation over the Fermi surface.

The model I study is of electrons moving on a two-dimensional square lattice of lattice constant a, with some dispersion relation ε_k. These electrons are subject to an interaction

$$H_{\rm int} = -\frac{1}{2}\sum_{k,\omega_n} \mathbf{S}(\mathbf{k},\omega_n)\cdot\mathbf{S}(-\mathbf{k},\omega_n) I^2 \chi(\mathbf{k},\omega_n) \ . \quad (1)$$

Here ω_n is a Matsubara frequency, **S** is an electron-spin operator, I is a coupling constant, and χ, a spin susceptibility, requires further discussion. Monthoux, Balatsky, and Pines[14] write for the susceptibility as a function of real frequency for k in the first quadrant ($k_x, k_y > 0$)

$$\chi(k,\omega+i\delta)=\chi_0\Gamma\beta^{1/2}[1/(\Gamma_k - i\omega)] \ . \quad (2)$$

Here χ_0 is the measured $q=0$ static susceptibility, $\beta^{1/2}\cong\pi$ is a parameter, and Γ is a frequency scale characterizing spin fluctuations. To save writing I have redefined Γ so that, in the present work,

$$\Gamma=\Gamma_{\rm MBP}/\beta^{1/2}\pi \ . \quad (3)$$

In high-T_c materials it is claimed[6] that $\chi_0\Gamma\cong 0.1$. The wave-vector-dependent energy scale Γ_k is

$$\Gamma_k=\Gamma[1+(k-Q)^2\xi^2]/(\xi/a)^2 \ . \quad (4)$$

ξ is the correlation length characterizing the antiferromagnetic fluctuations. These are maximal at the wave vector $Q=(\pi/a,\pi/a)$. For k in other quadrants, one must shift Q appropriately.

The susceptibility in Eq. (2) is inferred from NMR experiments.[6] The NMR relaxation rates in high-T_c materials show anomalous temperature dependences. In Ref. 6 these were argued to arise from low-lying commensurate antiferromagnetic spin fluctuations. The total dynamic susceptibility was partitioned into a conventional Fermi-liquid-like contribution and an anomalous antiferromagnetic part which was assured to dominate at low frequencies for wave vectors near Q. The susceptibility in Eqs. (1) and (2) is taken to be the extension[20] to finite ω of the anomalous antiferromagnetic contribution of Ref. 6.

In particular, it is consistent with dynamic scaling with mean-field exponents and obviously obeys the Kramers-Kronig relation between real and imaginary parts of χ. However, it does not obey the total moment sum rule at $T=0$,

$$\sum_{k,\omega}\chi''(k,\omega)=\langle (S_1^{\alpha})^2\rangle<\tfrac{3}{4}\ , \tag{5}$$

because χ'' only vanishes as $1/\omega$ at high ω, so the sum on the left-hand side of Eq. (5) diverges logarithmically. The simplest way[20] to cure this divergence is to introduce a momentum-independent cutoff D and assume that $\chi''(k,\omega)$ is given by Eq. (2) for $\omega<D$ but that $\chi''(k,\omega)=0$ for $\omega>D$. Equation (5) then shows that one must have $D\sim\Gamma$. This cutoff means that the real part of χ is no longer given by Eq. (2), but is corrected by a term which is negligible for k near Q, but important for k far from Q. With the ansatz $\chi''(\omega)=0$ for $|\omega|>D$ and the standard spectral presentation for χ, I obtain for the Matsubara χ

$$\chi(k,i\Omega_n)=-\chi_0\Gamma\beta^{1/2}\frac{|\Omega_n|-(2\Omega_n/\pi)\tan^{-1}(\Omega_n/D)-\Gamma_k+(2\Gamma_k/\pi)\tan^{-1}(\Gamma_k/D)}{\Gamma_k^2-|\Omega_n|^2}\ . \tag{6}$$

Equation (6) is finite at $|\Omega_n|=\Gamma_k$ and, if Γ_k is much less than D, is quite well approximated by

$$\chi(k,i\Omega_n)\cong\chi_0\Gamma\beta^{1/2}\times\begin{cases}\dfrac{1}{\Gamma_k}\ , & |\Omega_n|<\Gamma_k\\ \dfrac{1}{|\Omega_n|}\ , & D>|\Omega_n|>\Gamma_k\\ \dfrac{D}{|\Omega_n|^2}\ , & |\Omega_n|>D\ .\end{cases} \tag{7a}$$

If Γ_k is comparable to D, then an adequate approximation is

$$\chi(k,i\Omega_n)\approx\chi_0\Gamma\beta^{1/2}\times\begin{cases}\dfrac{1-(2/\pi)\tan^{-1}\Gamma_k/D}{\Gamma_k/D}\ , & |\Omega|<D\\ D|\Omega_n|^2\ , & |\Omega|>D\ .\end{cases} \tag{7b}$$

One sees from Eqs. (7) that the characteristic frequency scale for χ is $D\sim\Gamma$. For the high-T_c materials values $\Gamma\cong 0.05$ eV have been quoted.[6,14] These values of Γ are much smaller than the conduction electron bandwidth.

Consider now the electron self-energy due to the electron-boson interaction. When the boson propagator is only weakly momentum dependent, the problem becomes essentially identical to the electron-phonon problem. The electron-phonon theory involves two parameters: a dimensionless coupling constant λ which may be of order unity, and a truly small parameter $\lambda(\omega_D/E_F)$. The smallness of the second parameter leads to two simplifications. One is that "vertex correction" diagrams are of order ω_D/E_F relative to noncrossing diagrams, and so may be neglected if $\lambda\omega_D/E_F\ll 1$. The terms of order $(\omega_D/E_F)^0$ may be shown to lead to a closed form self-consistent integral equation for the self-energy. The second, related simplification produced by the small value of ω_D/E_F is that only electronic states near the Fermi surface are important, so that one may integrate analytically over the magnitude of the electron momentum, leaving an equation involving Fermi surface quantities only. In the problem at hand, the characteristic energy of the boson at momentum q, Γ_q, is also much less than the bandwidth, leading one to suspect a similar approximation is feasible.[21]

Consider the two low-order perturbation theory diagrams for Σ shown in Figs. 1(a) and 1(b). Using the unperturbed Green function $G_0(k,i\omega_n)=(i\omega_n-\varepsilon_k)^{-1}$ and Eq. (6) for χ we find that the contribution to Σ from Fig. 1(a), Σ_{1a}, may be written (f and b are the Fermi and Bose distribution functions, respectively, and $g^2=I^2\chi_0\Gamma\beta^{1/2}$),

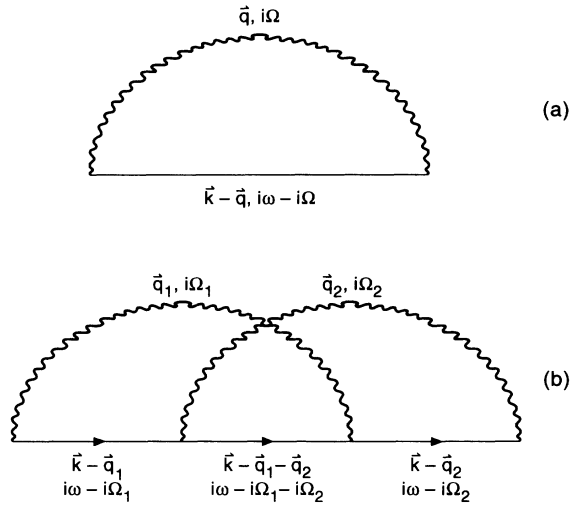

FIG. 1. Low-order perturbation-theory diagrams for the electron self-energy Σ. The solid line represents the electron propagator, the wiggly line the spin fluctuation. In the crossed diagram 1(b), if the external momentum **k** is on the Fermi surface, w may choose q_1 and q_2 so that two of the internal momenta, say, the first $(k-q_1)$ and third $(k-q_2)$, are also on the Fermi surface. The third line will then in general be displaced from the Fermi surface by a momentum of order the antiferromagnetic wave vector Q, leading to an energy denominator of order 2μ, where μ is the chemical potential measured from half filling.

$$\Sigma_{1a}(\mathbf{k},i\omega)=g^2\sum_q\int_{-D}^{D}\frac{dx}{\pi}\frac{x[b(x)+f(\varepsilon_q)]}{(\Gamma^2_{\mathbf{k}-\mathbf{q}}+x^2)(i\omega-x-\varepsilon_q)}. \tag{8}$$

The self-energy determines three quantities of possible physical interest: $\Sigma(\mathbf{k}, i\omega=0)$ gives the shift in the Fermi surface at $T=0$, $\partial\Sigma/\partial\omega$, and $\partial\Sigma/\partial k$ determine the mass enhancement, and $\Sigma''(\mathbf{k},\omega+i\delta)$ gives the scattering rate. As may be seen easily from (8), for $\partial\Sigma/\partial\omega$ and Σ'' the relevant values of ε_q are those within an energy of order Γ or D of the Fermi surface, while for $\Sigma(\mathbf{k}, i\omega=0)$ and $\partial\Sigma/\partial k$ the region near the Fermi surface does not contribute at all.

Consider first $\partial\Sigma/\partial\omega$ and Σ''. The constraint that ε_q be near the Fermi surface means that once the direction of q, θ_q, is fixed the magnitude of q may vary by an amount $\Delta q\sim x/v_F(\theta_q)$. But when q varies over this range then the other factor in the integral, $\chi''=x/x^2+\Gamma^2_{k-q}$, varies by an amount which we may estimate from Eq. (4) to be

$$\frac{\delta\chi''}{\chi''}\sim\frac{\Gamma^2_{\mathbf{k}-\mathbf{q}}}{\Gamma^2_{\mathbf{k}-\mathbf{q}}+x^2}\frac{(\mathbf{k}-\mathbf{q}-\mathbf{Q})\xi^2/a}{1+(\mathbf{k}-\mathbf{q}-\mathbf{Q})^2\xi^2}\frac{x}{v_Fa}. \tag{9}$$

Thus the relative change in χ'' is small, of order $N(\theta_q)D$, because if $(k-q-Q)\xi\gg1$ then all of the factors in (9) except x/v_Fa are of order unity and the integral is dominated by x of order D, while if $(k-q-Q)\xi\sim1$ then either $x\sim D/\xi^2$ or $\Gamma_{k-q}\ll x$. Very similar arguments apply to the scattering rate; thus for these quantities it is permissible, up to terms of order $N(\theta_q)D$, to integrate over the magnitude and angle independently, setting $|q|=q_F$ in Γ_{k-q}.

The renormalization of the Fermi surface, which is determined by $\Sigma(k, i\omega_n=0)$, is a different matter. The ε_q integral is not convergent so that whole zone is important. Further, the integrand is odd in ε_q, so the region near the Fermi surface cancels. In the usual Eliashberg theory $\Sigma(k, i\omega_n=0)=0$. In the present situation we expect it to be small (more precisely, since a uniform shift of the chemical potential is not important, we expect the variation around the Fermi surface of Σ to be small). I have calculated the leading order diagram numerically and found that it is indeed small relative to $\partial\Sigma/\partial\omega$, provided the model is sufficiently far from the nesting instability, as discussed below. Further, the renormalization of the Fermi surface should not affect the estimates of the mass enhancement or scattering rate appreciably.

The estimate for $\partial\Sigma/\partial k$ is very similar to that for $\Sigma(k, i\omega=0)$. By explicitly differentiating Eq. (8) with respect to k one may show that the integrand for $\partial\Sigma/\partial k$ is related to that for $\Sigma(k, i\omega=0)$ by a factor

$$\frac{\Gamma^2_{\mathbf{k}-\mathbf{q}}}{\Gamma^2_{\mathbf{k}-\mathbf{q}}+x^2}\frac{(\mathbf{k}-\mathbf{q}-\mathbf{Q})\xi^2/a}{1+(\mathbf{k}-\mathbf{q}-\mathbf{Q})^2\xi^2},$$

which is always less than unity. Thus if $\Sigma(k,i\omega=0)$ is negligible, so also is $\partial\Sigma(k,i\omega=0)/\partial k$. Henceforth I neglect $\partial\Sigma/\partial k$, and so determine the mass enhancement from $\partial\Sigma/\partial\omega$.

Now turn to the contribution of crossed diagrams such as Fig. 1(b). In the electron-phonon problem these are negligible because if we put two of the electron lines (say the first and third) on the Fermi surface, the second will in general be far from the Fermi surface. In the present case, because the interaction is sharply peaked in momentum space, this is not so. The momentum carried by a spin fluctuation is $q\sim Q+O(\xi^{-1})$. Thus we may estimate the energy of the intermediate electron line as $\varepsilon_{p_F}-q\sim\varepsilon_{p_F}-Q+v_F/\xi$. Now in a model with perfect nesting, $\varepsilon_{p_F}-Q=2\mu$, so the energy denominator appearing is not the Fermi energy E_F but $(2\mu+v_F/\xi)$, which is much less than E_F near half filling. Thus after integrating over the boson frequency we find that the crossed diagram is smaller than the leading diagram by a factor of order $\lambda D/(2|\mu|+v_F/\xi)$ with $\lambda=g^2N_0/D$. By using the expression for g^2 given above Eq. (8) and assuming $\chi_0\sim N_0$, this estimate becomes $(N_0I)^2\beta^{1/2}D/(2|\mu|+v_F/\xi)$. Thus the theory we discuss is only valid to leading (i.e., zeroth) order in $D/(2\mu+v_F/\xi)$. The density of states is either nondivergent or, in models with nesting, only logarithmically divergent as μ goes to zero. Thus, errors due to the approximation of integrating first over the energy [terms of order $N(\theta_p)D$] are less important than errors due to the neglect of crossed diagrams for a wide range of μ. The physics underlying these estimates is that the electronic energy v_F/ξ corresponding to the momentum scale of the boson ξ^{-1} is much greater than the typical boson frequency D, so the frequency dependence of the boson is more important than its momentum dependence.

To summarize, if the parameter $D/(2\mu+v_F/\xi)$ is sufficiently small, one may follow the usual Eliashberg procedure, obtaining the following equations of the self-energy:

$$[1-Z(\theta_k,i\omega_n)]i\omega_n=-\int\frac{d\theta_p}{2\pi}N(\theta_p)\pi T\sum_{\Omega_n}i\,\mathrm{sgn}(\Omega_n)I^2\chi(\mathbf{k}-\mathbf{p}, i\omega_n-i\Omega_n) \tag{10a}$$

and

$$W(\theta_k,i\omega_n)=-\int\frac{d\theta_p}{2\pi}N(\theta_p)\pi T\sum_{\Omega_n}\frac{W(\theta_p,\Omega_n)}{|\Omega_nZ(\theta_p,\Omega_n)|}I^2\chi(\mathbf{k}-\mathbf{p}, i\omega_n-i\Omega_n). \tag{10b}$$

Here the wave vectors $\mathbf{k}$ and $\mathbf{p}$ are restricted to lie on the Fermi surface, which is assumed to be topologically simple, θ_k is an angle parametrizing the Fermi surface, and $N(\theta_k)$ is the density of states at angle θ_k. These equations should hold for the model of Ref. 14, where the values $\Gamma=0.1$ and $\mu=-0.5$ (in units where $t=1$) were assumed. The linearized gap equation of Ref. 14 is equivalent to our Eq. (10b) except that they have set $Z=1$, have chosen to perform the frequency sum instead of the momentum sum, and have taken $D\to\infty$. Equations (10) may be solved straightforwardly on a computer. Equation (10a) is an integration and Eq. (10b) may be reduced to the inversion of a matrix defined on the Matsubara frequencies. Here I present an approximate analytic solution, based on Eqs. (7) for χ, which reveals the important energy scales.

Begin with the equation for the normal self-energy Z. One quantity of physical interest is the Fermi surface mass enhancement, $m^*/m-1=\lambda_{\mathbf{k}}$. Within the Eliashberg approach this is given by $\lim_{\omega\to 0}Z(k_F,\omega)\equiv\lambda_k$, because, as shown above, $\partial\Sigma/\partial k$ is negligible. It may be reasonably approximated by

$$\lambda_{\mathbf{k}}\simeq Z(k_F, i\omega_n=\pi T)=g^2\int\frac{d\theta_p}{2\pi}N_0(\theta_p)/\Gamma_{\mathbf{k}-\mathbf{p}} . \tag{11}$$

It is clear from Eq .(11) that for each k, two possible cases can arise: either for some $\mathbf{p}$ on the Fermi surface, $|\mathbf{k}-\mathbf{p}-\mathbf{Q}|\sim\xi^{-1}$, in which case $\lambda_{\mathbf{k}}$ may be large, or for all $\mathbf{p}$ on the Fermi surface, $|\mathbf{k}-\mathbf{p}-\mathbf{Q}|\xi\gg 1$, in which case $\lambda_{\mathbf{k}}$ will be small. Which case arises in any given circumstance depends upon the band filling and Fermi-surface shape, upon ξ, and possibly upon the position of $\mathbf{k}$ on the Fermi surface. A particularly interesting case occurs for the parameters used in Ref. 14, which are very similar to those used in other work and have been argued to be relevant to $YBa_2Cu_3O_7$. In this work the electronic dispersion was given by the tight-binding form $\varepsilon_k=-2(\cos k_x+\cos k_y)-\mu$, (we take the hopping $t=1$ and the lattice constant $a=1$) with $\mu=-0.5$. In addition, $\Gamma=0.1$ and $\xi=3$ were assumed. In this case for $\mathbf{k}=k_F(1,1)$ on the zone diagonal I find that the minimum of $(\mathbf{k}-\mathbf{p}-\mathbf{Q})$ occurs for $\mathbf{p}$ oppositely directed to $\mathbf{k}$, with $|\mathbf{k}-\mathbf{p}-\mathbf{Q}|\xi\sim 0.6$, implying a large $\lambda_{\mathbf{k}}$, while for $\mathbf{k}$ perpendicular to the zone face [$\mathbf{k}$ parallel to (1,0)] the minimum in $\mathbf{k}-\mathbf{p}-\mathbf{Q}$ occurs for $\mathbf{p}$ nearly parallel to (0,1) and the value of $|\mathbf{k}-\mathbf{p}-\mathbf{Q}|\xi\sim 4$ implying a small $\lambda_{\mathbf{k}}$. We may make a more accurate estimate by defining $\Gamma_{\min}(\theta_k)$ as the minimum as p ranges over the Fermi surface of $\Gamma(\xi^{-2}+(\mathbf{k}-\mathbf{p}-\mathbf{Q})^2)$. The dependence on angle away from the minimum value is quadratic. Indeed, if for fixed θ_k the minimum value of Γ occurs at an angle $\theta_{p\,\min}$ then I find (not writing the θ_p dependence of p_F),

$$\Gamma(\theta_k,\theta_p)=\Gamma_{\min}(\theta_k,\theta_{p\,\min})+\tfrac{1}{2}\Gamma p_F^2(\theta_p-\theta_{p\,\min})^2 . \tag{12}$$

Inserting (12) into (11) and integrating yields [not explicitly writing the dependence of $\Gamma_{\min}(\theta_k,\theta_{p\,\min})$],

$$\lambda_{\mathbf{k}}=\frac{g^2N_0}{2\pi(\frac{1}{2}\Gamma_{\min}\Gamma p_F^2)^{1/2}}\tan^{-1}\left[\frac{\Gamma p_F^2}{2\Gamma_{\min}}\right]^{1/2} . \tag{13}$$

Equation (13) implies a mass anisotropy around the Fermi surface of order ξ for the parameters given, because if $\mathbf{k}$ is along (11), $\Gamma_{\min}\sim\Gamma a^2/\xi^2$, but for $\mathbf{k}$ along (1.0), $\Gamma_{\min}\sim\Gamma$. Apparently similar results have been obtained by Monthoux in a numerical calculation.[22] We now consider the imaginary part of the self-energy. Analytically continuing Eq. (10a) we find

$$\Sigma''(\mathbf{k},\omega+i\delta)=\int\frac{d\theta_p}{2\pi}N_0(\theta_p)\int\frac{dx}{\pi}[b(x+\omega)+f(x)]\chi''(\mathbf{k}-\mathbf{p},x) . \tag{14}$$

Consider first $T=0$. The integral over x may be performed leaving

$$\Sigma''(\mathbf{k},\omega+i\delta)=\frac{g^2}{\pi}\int\frac{d\theta_p}{2\pi}N_0(\theta_p)\ln[\omega^2+\Gamma_{\mathbf{k}-\mathbf{p}}^2)/\Gamma_{\mathbf{k}-\mathbf{p}}^2 . \tag{15}$$

As with $\lambda_{\mathbf{k}}$, different behaviors occur in different circumstances. The various regimes are shown in Fig. 2. If $\omega\ll\Gamma_{\min}$ and $\Gamma_{\min}\ll 1$ then one may expand the log and perform the integral obtaining

$$\Sigma''(\mathbf{k},\omega+i\delta)=\frac{g^2N_0}{4\pi}\left[\frac{\Gamma_{\min}}{\frac{1}{2}\Gamma p_F^2}\right]^{1/2}\frac{\omega^2}{\Gamma_{\min}^2} . \tag{16}$$

This is the expected Fermi liquid behavior. Note, however, that the scattering rate is smaller by a factor of $(\Gamma_m/\Gamma)^{1/2}$ than the naive Fermi-liquid estimate $\omega^2/\Gamma_{\min}^2$.

If $\Gamma_{\min}(\theta_k)\ll\omega\ll\Gamma$ then for those θ_p's for which $\omega<\Gamma_{\mathbf{k}-\mathbf{p}}$, the scattering rate is $\ln(\omega/\Gamma_{\mathbf{k}-\mathbf{p}})$, so that integrating over this range gives

$$\Sigma''(\mathbf{k},\omega+i\delta)\sim g^2N_0(\omega/\Gamma)^{1/2} . \tag{17}$$

The regime of p's for which $\omega>\Gamma_{\mathbf{k}-\mathbf{p}}$ gives a similar contribution. Finally, for $\omega\gg\Gamma$,

$$\Sigma''(\mathbf{k},\omega+i\delta)\sim g^2N_0\ln(\omega/\Gamma) . \tag{18}$$

Very similar behavior occurs as a function of temperature; one just replaces ω by $2\pi T$ in Eqs. (16)–(18). Note, however, in the low-T regime that $\Sigma''\sim T^2/\Gamma\Gamma_{\min}$, and that it is possible that $\Gamma_{\min}\sim\xi^{-2}\sim T^{-1}$ (cf. Ref. 6). Thus if ξ is T dependent, and if k and ξ are such that $\Gamma_{\min}\sim\xi^{-2}$, the low-$T$ dependence ($T<\Gamma_{\min}$) is $\Sigma\sim T$.

For the parameters considered in Ref. 14, $\Gamma_{\min}$ for $\mathbf{k}$ parallel to (1,1) is less than the temperature, while $\Gamma_{\min}$

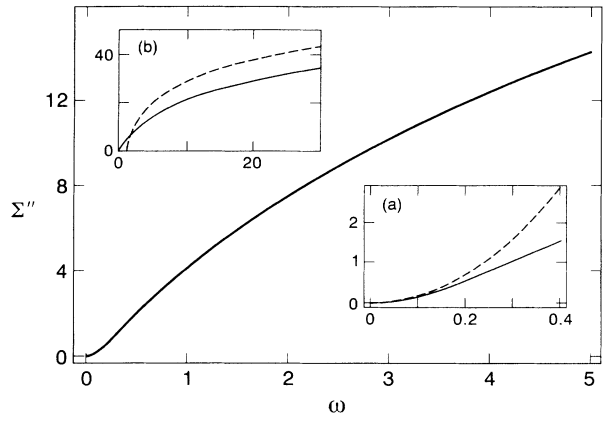

FIG. 2. Frequency dependence of imaginary part of self-energy for **k** along (1,1) calculated from Eqs. (12) and (15), with frequency units such that $\frac{1}{2}\Gamma p_F^2=1$, and parameters chosen so that $\Gamma_{\min}=0.2$ and $g^2N_0/2\pi^2=1$. For sufficiently small ω, $\Sigma''\sim\omega^2$; for intermediate ω, $\Sigma''\sim\omega^{1/2}$ crossing over to $\Sigma''\sim\ln\omega$ for larger values of ω. The insets display the small [inset (a)] and large [inset (b)] ω regimes more clearly; in each case the solid line is calculated from Eq. (15) and the dashed lines from the approximations in the text.

for **k** parallel to (1,0) is greater, so that for k parallel to (1,1) $\Sigma''\sim T^{1/2}$ while for **k** parallel to (1,0) $\Sigma''\sim T^2$. Because the range of **k**'s about (1,1) for which $\Sigma''\sim T^{1/2}$ itself scales as $T^{1/2}$, these equations lead to Fermi-surface-averaged self-energy varying as T, as previously found.[8–11,14,23,24]

These considerations are not directly applicable to the reduced T_c materials $La_{1.85}Sr_{0.15}CuO_4$ and $YBa_2Cu_3O_{6.6}$ because the parameter χ_0 [cf. Eq. (2)] deduced from NMR experiments is strongly temperature dependent,[25,26] implying[20] the development of a gap in the spin excitation spectrum. For these materials a naive application of Eq. (18) would lead to a Σ'' which decreased with temperature more rapidly than T. However, in Ref. 16 it is argued that for these materials onset of a spin gap implies that the form for $\chi(q)$ assumed in Eq. (2) is unlikely to be valid for ω of order room temperature or greater. The analysis presented here thus must be modified in order to treat these materials. Monthoux, Balatsky, and Pines[14] argue that one should simply introduce a temperature dependence of the coupling parameter I^2 to compensate for the temperature dependence of χ_0. In our view a more systematic treatment of the effects of the onset of the spin gap is required.

The variation of mass and scattering rate over the Fermi surface should be observable in angle-resolved photoemission. Their absence would rule out this model. The energy resolution required is

$$\Delta E \lesssim \Gamma_{\min}\sim 10\ \text{meV}\ , \tag{19a}$$

and the momentum resolution required is (a is the lattice constant)

$$\Delta k \lesssim \sqrt{(2k_F-Q)^2+\xi^{-2}}\sim 0.5/a\ . \tag{19b}$$

The large variation over the Fermi surface of the real and imaginary parts of the self-energy implies that vertex corrections to the conductivity will be large. Thus the naive inference[23,24] of a linear-in-T resistivity from the linear-in-T average self-energy is not necessarily valid.

We now turn to the superconducting properties described by Eq. (10b). The superconducting gap function W must transform according to an irreducible representation (which we label by α) of the crystal symmetry group in this case D_4. We choose to write

$$W(\theta_k,i\omega_n)=\Delta^\alpha(\theta_k,i\omega_n)P_\alpha(\theta_k)\ . \tag{20}$$

Here $P_\alpha(\theta_k)$ is the partner function corresponding to the α irreducible representation and $\Delta^\alpha(\theta_k,i\omega_n)$ is a function with the full symmetry of the crystal. The P_α are normalized so that

$$\int\frac{d\theta_k}{2\pi}P_\alpha^*(\theta_k)P_\beta(\theta_k)=\delta_{\alpha\beta}\ . \tag{21}$$

We consider first the weak-coupling limit, where we set $Z(k,i\omega_n)=1$. As may be seen from Eq. (13), the condition for the applicability of this limit is, roughly, $g^2N_0/\sqrt{\Gamma_{\min}\Gamma}\ll 1$; this is a very stringent condition. We also make the BCS approximations, which amount to neglecting the frequency dependence of the gap function and setting $i\omega_n=0$ on the right-hand side of Eq. (10b). The sum on $i\Omega_n$ may be performed; the resulting gap equation is

$$W(\mathbf{k})=-g^2\int\frac{d\theta_p}{2\pi}\frac{N(\theta_p)}{\Gamma_{\mathbf{k}-\mathbf{p}}}\phi_{\mathbf{k}-\mathbf{p}}(T)W(\mathbf{p})\ . \tag{22}$$

Here $\phi_{\mathbf{k}-\mathbf{p}}(T)\approx\ln(D/T)$ is a slowly varying function of $\mathbf{k}-\mathbf{p}$. We first consider a gap function transforming as the trivial representation $P_\alpha=1$. In this case the angular integrals are dominated by the region where $\Gamma_{\min}(\mathbf{k})$ is smallest, i.e., by **k** parallel to (1,1). We may analyze the problem as we did for $\lambda_{\mathbf{k}}$. We find [assuming that $\Delta(\mathbf{k})$ does not vary too strongly about the Fermi surface] a pairing interaction which is *repulsive* (unfavorable for superconductivity) and which scales as $g^2N_0\ln[\Gamma/\max(\Gamma_{\min},\Omega)]$. Because the integrals are dominated by the regions for which $\mathbf{k}-\mathbf{p}$ is near Q, the same conclusion holds for the d_{xy} partner function, because $P_{d_{xy}}=1$ if $\theta_k=\pm(\pi/a,\pi/a)$ and $P_{d_{xy}}=-1$ if $\theta_k=\pm(\pi/a,-\pi/a)$. Thus, as pointed out by Monthoux, Balatsky, and Pines,[14] within weak coupling the only possible symmetry for the gap function is $d_{x^2-y^2}$, for which P_α vanishes for **k** parallel to Q. To understand the structure of the gap equation in this case we again consider two cases; **k** parallel to (1,1) and **k** parallel to (1,0). If **k** is parallel to (1,1) the gap function $W(\mathbf{k})$ vanishes, as does the **p** integral in Eq. (22). If **k** is parallel to (1,0) the gap function is nonzero and we may write

$$W(\mathbf{k})=-g^2N_0D_n\phi(T)\int\frac{d\theta_p}{2\pi}\frac{1}{\Gamma_{\min}(\mathbf{k})+\Gamma p_F^2\theta_p^2}W(\mathbf{p})\ . \tag{23}$$

The θ_p integral in Eq. (23) is dominated by the $\mathbf{p}$'s near where $\Gamma_{\min}(\mathbf{k}-\mathbf{p})$ is minimal. As we have seen, this involves $\mathbf{p}$'s near (0,1). For these $\mathbf{p}$'s, $W(\mathbf{p})$ and $W(\mathbf{k})$ are of opposite signs; the interaction is therefore of the correct sign to produce superconductivity. The $\mathbf{p}$ integral may be performed, yielding essentially the same result as we previously obtained for the mass enhancement for $\mathbf{k}$ parallel to (1,0). Of course, for the parameters believed to be relevant for high-T_c superconductivity, only spin fluctuations of momentum farther than ξ^{-1} from the antiferromagnetic point enter. This explains the numerical result of Ref. 14 that T_c was independent of ξ in this model. It is not correct to think of the strong low-lying antiferromagnetic correlations as producing superconductivity in this model.

I now consider the effect of relaxing the condition $g^2 N_0/\sqrt{\Gamma_{\min}\Gamma} \ll 1$. First, I discuss including the normal-state self-energy. Note that the gap function vanishes where the normal self-energy is largest; this will probably lead to a finite region of gapless superconductivity near (1,1) (in contrast to the gapless point suggested by the weak-coupling approach) but may not reduce T_c too much. On the other hand, in the region where the gap function is large, the paring interaction is basically of the same strength as the interaction entering normal-state self-energy, thus the parameter g defined in Ref. 5 may be close to unity. Normal-state self-energy effects were found in Ref. 5 to be particularly large only for $g < \frac{1}{2}$. Thus I believe that normal-state self-energy effects will not affect qualitatively the results presented here.

The superconductivity found in the present calculation is likely to be different from that found in other d-wave superconductors in one interesting way. Because the nodes in the gap occur at the points on the Fermi surface where the interactions are strongest, it seems likely that the low-lying quasiparticle excitations would have a large mass, so that the change in superfluid density due to these thermal excitations could be unusually small at low T, possibly reconciling the penetration depth results[11] with a d-wave order parameter.

One important caveat should be mentioned. Because the pairing kernel for the $d_{x^2-y^2}$ state was found to be dominated by the large $\omega, q \neq Q$ region, one may question the validity of including only the interacting due to the low-lying antiferromagnetic spin fluctuations of Eqs. (1) and (2) in the pairing kernel. The higher-lying Fermi-liquid-like terms will have a small additional effect on the normal-state self-energy calculation, and will affect the pairing calculation in two ways: Because the energy scale is higher, the cutoff Γ will be increased, but because the momentum dependence is weaker, the relative strength of the $d_{x^2-y^2}$ interaction will be decreased. Numerical work is required to determine which effect dominates.

It is interesting to consider the relationship of the results presented here to those obtained in the spin-bag calculations.[12,13] The spin-bag results have two aspects. One is a pairing mechanism involving local suppression of antiferromagnetic order.[12] The expression of this mechanism in the highly doped, metallic case is argued to involve crossed diagrams related to that shown in Fig. 1(b).[13] The second aspect involves a "pseudogap" structure in the single-particle spectral function.[13] The results of this paper imply that in the parameter regime $(g^2 N_0 D)(D/(2|\mu| + v_F/\xi) \ll 1$ in which the Eliashberg description applies, the crossed diagrams cannot contribute to pairing. The pseudogap structure of the single-particle spectral function[13] is a slightly different matter. The calculations exhibiting this pseudogap involved evaluating the diagram shown in Fig. 1(a); this is the leading-order approximation to the Eliashberg equations. Thus the pseudogap may occur in the Eliashberg equations. From the results of Refs. 13 one may see that the pseudogap occurs at an energy $\sim 2|\mu|$ from the Fermi surface. From the arguments given in this paper or in Ref. 13, one may see that this is the energy of a final state produced by scattering an initial state at the Fermi surface by the antiferromagnetic wave vector Q. The results presented here show that if the Eliashberg condition $(N_0 I)^2 D/(2|\mu|) \ll 1$ is satisfied, this peak is at too high an energy to be relevant for the mass enhancement, scattering rate, or superconductivity.

In conclusion, I have shown that the model in which quasiparticles interact by exchanging spin fluctuations which are described the phenomenological susceptibility obtained from analysis of NMR experiments may be studied analytically via the Eliashberg equations, if the chemical potential measured from half filling is not too small. I wrote down the relevant Eliashberg equation in a form I hope is convenient for future numerical work and obtained an effective mass and scattering rate which vary dramatically around the Fermi surface, being largest in the (1,1) direction. I also showed that the d-wave superconductivity found in this model[14] is due essentially to high-energy spin fluctuations and not to the strong low-lying AF fluctuations producing the mass enhancement and scattering.

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