

Viscosity of the entangled-vortex-liquid phase

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The entangled-flux-line liquid phase of high-temperature superconductors is considered. This is a phase of directed, mutually avoiding polymerlike flux lines which are strongly entangled. However, there is a finite rate constant for topological reconnection of the flux lines. The viscosity η of the phase is discussed using a simplified model, following that recently developed by Marchetti and Nelson [Phys. Rev. B 42, 9938 (1990)]. We find, as did these authors, a result of the form $\eta = \eta_0 e^{U/kT}$. However, our estimate for U is one-third of their value; the difference could be significant when the activation energy for flux-line reconnection is large.

When high- T_c superconductors are placed in a magnetic field at low temperatures, a conventional Abrikosov flux lattice is formed. At higher temperatures, however, the lattice may melt¹ into a vortex liquid. As pointed out by Marchetti and Nelson,^{2,3} this phase has interesting hydrodynamics which can often be described by the following equation:

$$-\gamma \mathbf{v} + \eta \nabla^2 \mathbf{v} + \mathbf{f}_T = \mathbf{0} . \tag{1}$$

A simple geometry for which this applies is a slab of thickness L in which the applied field \mathbf{H} is in the z direction but the flux-line velocity \mathbf{v} lies everywhere in the xy plane (Fig. 1). The Lorentz force \mathbf{f}_T is the force per unit volume acting on the flux lines; it arises from any currents that are flowing in the xy plane. The parameter γ in Eq. (1) is the Bardeen-Stephen friction coefficient,⁴ which represents the dissipation when flux lines move relative to the underlying crystal. The fluid viscosity is η ; this opposes velocity *gradients*. It is likely to be very small for flux lines that are unentangled, but can become large if entanglements impede the free flow of flux lines past one another. Despite the two-dimensional geometry of this particular problem, η is defined in Eq. (1) with the dimensions of a three-dimensional viscosity.

Several parameters are useful in describing the behavior of the flux liquid phase. Each flux line is a "directed random walk" which points along the z direction but has transverse excursions in the xy plane. These are characterized by a certain persistence length l_p whose depen-

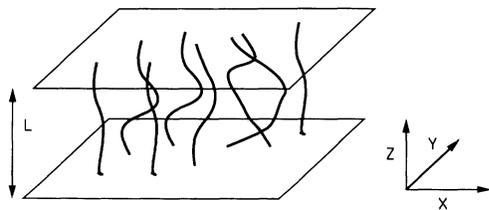


FIG. 1. Geometry of a shear flow in the xy plane for a thin slab. The applied field is in the z direction and flux lines are oriented accordingly.

dence on field strength and temperature is known.¹ The mean-square transverse displacement of a line (between the bottom and top faces of the sample) obeys

$$\langle R^2(L) \rangle \propto Ll_p , \tag{2}$$

and the threshold for entanglement arises when this quantity is of order

$$a^2 = \phi_0 / B , \tag{3}$$

where ϕ_0 is the flux quantum and B the field strength. The parameter a is of order the mean spacing between flux lines in the transverse direction. When the system is entangled, a characteristic length l_e can be defined via

$$l_e l_p = a^2 . \tag{4}$$

This l_e is the thickness of a marginally entangled sample, or equivalently the distance between entanglements when traced along a flux line.

In calculating the viscosity, η of the flux liquid phase one must consider the relation between stress and deformation of the flux lines. A fundamental result borrowed from polymer theory is that the shear stress (in the present geometry, this σ_{xy}) obeys⁵

$$\sigma_{xy} = C \frac{kT}{l_p} \int_0^L \left\langle \frac{\partial R_x}{\partial s} \frac{\partial R_y}{\partial s} \right\rangle , \tag{5}$$

where $C \approx 1/La^2$ is proportional to the number of chains per unit volume, $R(s)$ is the spatial trajectory of a polymer, and s its arc-length coordinate. In the present case, s can simply be replaced by z since the polymers are directed along the z axis. The fact that chains are directed does not alter the expression for the transverse (xy) components of shear stress.

In general, to calculate η one first defines a time-dependent modulus $G(t)$ by $\sigma_{xy}(t) = \Theta G(t)$, where Θ is a small imposed step strain applied at time $t=0^+$ and $\sigma_{xy}(t)$ the resulting stress. Then the viscosity is given by⁵

$$\eta = \int_0^\infty G(t) dt . \tag{6}$$

For entangled systems, one conventionally divides the

time integral into two parts: time scales shorter and longer, respectively, than the intrinsic relaxation time τ_e of a piece of chain of length l_e . In a short time of order τ_e , all conformational degrees of freedom within such a piece of chain relax, subject to the constraint that the overall end-to-end vector cannot do so (since at this scale the chain is entangled with its surroundings). For longer times, Eq. (5) can be replaced by

$$\sigma_{xy} \propto C \frac{LkT}{l_e} \langle u_x u_y \rangle, \quad (7)$$

where u_i is a unit vector along a “tube segment” or coarse-grained piece of chain of length l_e (This formula should still apply in the case of directed walks as studied here; the tube segment vectors can simply be projected onto the xy plane, defining a two-dimensional, undirected “polymer.”) The issue, then, is the time relaxation of the orientation distribution for tube segments over which the average in (7) is taken.

Assuming that the long-time part of the response ($t \geq \tau_e$) dominates the integral in Eq. (6), the viscosity can be estimated as^{2,3,7}

$$\eta = G_0 \tau, \quad (8)$$

where [from (7)] $G_0 \simeq kTCL / l_e \simeq (l_e a^2)^{-1}$ is a characteristic elastic modulus, and τ a characteristic relaxation time for stress. To estimate the viscosity we therefore need to predict τ .

Obukhov and Rubinstein⁶ have studied the case where the flux lines have an infinite energy barrier to topological reconnection. In this case, the only way to shear the liquid in the xy plane is by a complicated unwinding of the end points of the flux lines. (These end points are confined to the upper and lower faces of the sample.) These authors calculated a characteristic relaxation time $\tau(L) \propto \exp[(L/\xi_z)^3]$, which diverges with system size.

Marchetti and Nelson^{2,3} have pointed out that this divergence will disappear if flux lines are allowed to reconnect topologically (Fig. 2). Such reconnections have a certain local activation energy E . The magnitude of E was estimated by Obukhov and Rubinstein in terms of other material parameters and can be very large at low field strengths (of order $50kT$ at 77 K in Y-Ba-Cu-O according to Ref. 1). Marchetti and Nelson made the following estimate for the viscosity using an analogy with the reptation model of polymer physics:^{5,7}

$$\eta_{MN} \simeq \eta_0 e^{3E/kT}, \quad (9)$$

where the preexponential factor is

$$\eta_0 = G_0 \tau_e. \quad (10)$$

The basis of this estimate was as follows. Imagine each flux line subdivided into segments of length l_e , separated by “collision points” in the z direction at which the chain closely approaches a neighbor. At each of these collision points there is a probability of order $e^{-E/kT}$ that the flux line is “fused” with its neighbor (i.e., at the top of the energy barrier for reconnection). The spacing between such fusion points provides a characteristic length $\lambda = l_e e^{E/kT}$. Now suppose that for dynamical purposes we can imag-

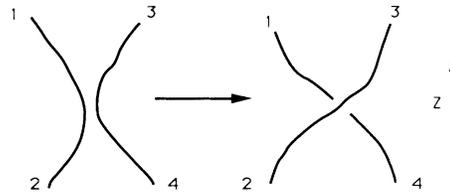


FIG. 2. Topological reconnection of two vortex lines by bond interchange.

ine the chains to be cut up into pieces of size λ . According to the reptation theory for chains with free ends,⁷ the characteristic relaxation time is then

$$\tau_{MN} = \tau_e (\lambda / l_e)^3 = \tau_e e^{3E/kT}, \quad (11)$$

which gives directly Eq. (9).

This estimate is interesting but clearly somewhat oversimplified. In the reptation picture, the existence of free chain ends is essential to controlling the relaxation time,^{5,8} whereas in the present system such ends are never actually present. Therefore it is unlikely that the flux lines behave as though broken into finite pieces of length λ . Moreover, it is known that polymers which can undergo reversible reactions (topological exchanges) do not obey the usual law of reptation,⁸ which is based on the entangled motion of “permanent” chains. In view of these difficulties, it is worth trying to find a more detailed estimate of the relaxation time.

The basic process to be considered is depicted in Fig. 2. In the language of polymerization kinetics, this is called a “bond interchange” reaction, in which two flux lines “swap halves” with each other.⁸ A reaction with the same intermediate state but different decay products is one where the original lines maintain their identities but simply pass through one another (we call this a “crossing” reaction).

An important observation is that, since stress is carried by tube segments, a reaction that merely reassigns tube segments from one chain to the other does not actually relax very much stress (Fig. 3). The stress is derived from the orientation distribution function of tube segments which, to the first approximation, is unaffected by a bond-interchange event. Of course, the interchange does alter the local constraints on the two tube segments immediately involved, but to a good approximation *only* these segments can relax orientation during the process. Correspondingly, if the stress is to relax by bond interchange, the characteristic time scale for relaxation is the waiting time for roughly one reaction to occur within each entanglement length l_e . Only after every tube segment has seen of order one reaction will a finite fraction of the stress have relaxed.

During this same time scale, of order one crossing reaction will also have occurred per tube segment. However, this does not change the result for the stress relaxation time: A system with crossing reactions only would also require (at least) of order one per tube segment before a reasonable amount of relaxation is achieved.⁸

Thus we are led to an estimate of the stress relaxation

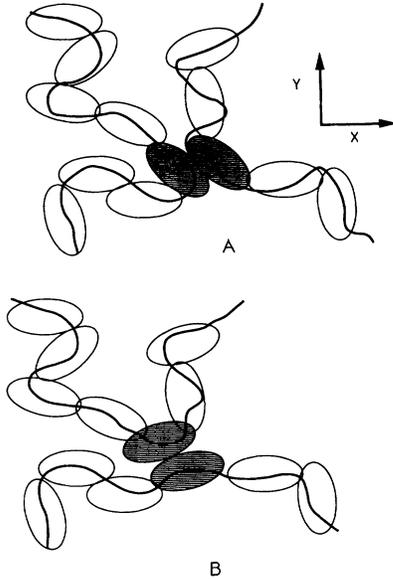


FIG. 3. Tube segments (portions of chain of length l_e) seen in projection in the xy plane, before (a) and after (b) a bond interchange event. Each ellipse represents a tube segment. Only those tube segments directly involved in the reaction event are able to relax orientation as a result of the process (shaded). Thus to relax the stress (orientation of tube segments) of order one such event per tube segment is required.

time τ as the waiting time for a bond-interchange event within one "tube segment" of length l_e . To calculate this, we note that the time scale for conformational change within such a segment is (by definition) τ_e . Bond interchange will occur when the random motions of neighboring sections of chain happen to have enough thermal energy to mount the activation barrier E for topological change. Using τ_e as an effective attempt frequency, we may then estimate the waiting time for a reaction within a tube segment as

$$\tau \simeq \tau_e e^{E/kT}. \quad (12)$$

According to our arguments, this is the characteristic time scale for stress relaxation.

Equation (12) is very similar to Eq. (11) except that the factor 3 is now missing from the exponent. Whether this is important depends obviously on the magnitude of E . If this is only of order kT the difference between (12) and (11) is minimal and hardly worth commenting upon (since both approaches have several untested assumptions). On the other hand, if E/kT is large, the difference between (11) and (12) is very substantial. (Even for $E/kT=5$,³ the results differ by a factor of 10^4 .) Combining the result (12) with Eq. (8), we find for the viscosity

$$\eta \simeq \frac{kT}{l_e a^2} \tau_e e^{E/kT} \quad (13a)$$

$$= \eta_0 e^{E/kT}. \quad (13b)$$

It is interesting to relate the preexponential factor $\eta_0 = CLkT\tau_e/l_e$ to other measurable quantities, especial-

ly the friction factor γ of Eq. (1). This is quite straightforward though it requires detailed consideration of τ_e , the conformational relaxation time of a piece of chain of length l_e . Consider first a straight vortex being dragged through a crystal at speed v . According to Eq. (1), this experiences a frictional force per unit length

$$F \simeq \gamma a^2 v. \quad (14)$$

From the total drag force on a piece of vortex of length l_p we can at once find the mobility μ_0 or diffusion constant $D_0 = kT\mu_0$ of such a section:

$$D_0 \simeq \frac{kT}{\gamma l_p a^2}. \quad (15)$$

The rotational relaxation time of a persistence length is of order

$$\tau_0 = l_p^2 / D_0 \simeq l_p^3 \gamma a^2 / kT, \quad (16)$$

whereas the relaxation time for a tube segment (piece of chain of length l_e) obeys

$$\tau_e \simeq \tau_0 (l_e / l_p)^2. \quad (17)$$

This last relation is a standard result from polymer theory (the relaxation time of a Rouse chain) and is neatly rederived in the appendix of Ref. 3. Finally we obtain

$$\eta_0 \simeq (kT / l_e a^2) (l_p^3 \gamma a^2 / kT) (l_e / l_p)^2 \quad (18)$$

$$= \gamma l_e l_p \simeq \gamma a^2, \quad (19)$$

where Eq. (4) is used to get the last form.⁹ Finally therefore we may write our estimate as

$$\eta \simeq \gamma a^2 e^{E/kT}. \quad (20)$$

Equation (20) is of interest since it leads to a very simple estimate of the characteristic "screening length" δ for vortex fluid flows obeying Eq. (1):

$$\delta = (\eta / \gamma)^{1/2}. \quad (21)$$

This is the characteristic length scale over which viscous effects are important. For example, in two-dimensional flow with fixed boundaries, which would give Poiseuille flow for a Newtonian fluid, the regions of high shear are confined to a thickness δ near the walls (with a plug flow in the central region).² From (20) we obtain

$$\delta \simeq a e^{E/2kT}, \quad (22)$$

which relates the screening length to the mean spacing between flux lines. Obviously for the viscous term in Eq. (1) to be important in the hydrodynamic range of length scales ($\gg a$) we must demand a large crossing energy E . For (say³) $E \simeq 5kT$ the screening length is not much more than a , the friction term γ dominates, and the hydrodynamic treatment based on viscosity is probably not useful. On the other hand, for (say³) $E \sim 50kT$ one has $\delta \simeq 10^{10}a$ and clearly the viscous term is in control for all reasonable sized samples.

The dependence on the activation energy E is obviously critical. According to Ref. 3, this may itself be temperature dependent since the actual crossing energy is a

function of the mutual orientation of vortex line segments and may be reduced at high temperatures where strongly transverse orientations can arise. More careful estimates of this quantity for different materials, and its temperature dependence, may be needed before Eq. (12) can be tested. In any case, we emphasize that the only real difference between the present estimates and those of Marchetti and Nelson is the factor-of-3 reduction found here in the effective activation energy for the relaxation

time [Eq. (12) vs Eq. (11)]. As mentioned already, this change could have a big numerical effect for large values of the crossing energy E . On the other hand, the difference is of no predictive value unless E can in fact be estimated (or measured) to accuracies better than a factor of 3.

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²M. C. Marchetti and D. R. Nelson, Phys. Rev. B **42**, 9938 (1990).

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⁴See, e.g., Y. B. Kim, M. J. Stephen, and W. F. Vinen, in *Superconductivity*, edited by R. D. Park (Marcel Dekker, New York, 1969), Vol. 2.

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⁹This argument follows closely that of Refs. 2 and 3 but our final result is simpler than Eq. (1.17) of Ref. 3, which is based on a wrong formula relating μ_0 to γ .