Influence of weak localization on the threshold of parametric excitation of magnons in low-dimensional magnets

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The influence of multiple elastic scattering on the magnon parametric excitation in low-dimensional magnets is studied. The multiple-scattering results in the strong backscattering of magnons. This backscattering can interfere with the parametric excitation because both phenomena involve pairs of magnons with opposite momenta. It is shown that the influence of the backscattering on the threshold of parametric excitation in two-dimensional systems is proportional to the logarithm of the size, even if it exceeds the inelastic mean free path. The possibilities of experimental observation of the effect are discussed.

I. INTRODUCTION

It is well known that the backscattering of waves of various nature on defects is anomalously strong in low-dimensional media.[1] The strong backscattering means that the higher terms in the amplitude of the backscattering increase with the phase relaxation length l_ϕ. For a D-dimensional system the additional term in the backscattering amplitude is of order

$$(\Gamma_k/kv_k)(l_\phi/l)^{D-2}. \tag{1.1}$$

Here Γ_k is the frequency of elastic scattering and l is the mean free path due to the elastic scattering:

$$l = v_k/\Gamma_k. \tag{1.2}$$

For a two-dimensional (2D) system this additional term diverges logarithmically with l_ϕ:

$$(\Gamma_k/kv_k)\ln_{10}(l_\phi/l). \tag{1.3}$$

The phase relaxation length l_ϕ can be very large at low temperature, so the additional terms in the backscattering amplitude are important in spite of the small parameter $(\Gamma_k/v_k k)$ in 1D and 2D systems. The strong backscattering leads to a coupling of quasiparticles with opposite momenta and suppresses the quasiparticle propagation.[1] The localization is called weak if the terms (1.1) or (1.3) are relatively small.

There is another phenomenon in which pairs of quasiparticles with opposite momenta are involved, that is parametric excitation of quasiparticles by a homogeneous ac field. This phenomenon has been found in various media, and it has been studied in detail in magnets, where magnons are excited. The homogeneous ac field $h\exp(-i\omega_p t)$, called pumping, can produce pairs of quasiparticles with momenta $\mathbf{k}_1$ and $\mathbf{k}_2$ so that momentum and energy conservation conditions are satisfied:

$$\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{0}, \quad \omega_{\mathbf{k}_1} + \omega_{\mathbf{k}_2} = \omega_p. \tag{1.4}$$

It is clear that the quasiparticles have opposite momenta $\mathbf{k}_1 = -\mathbf{k}_2$ and equal frequencies: $\omega_{\mathbf{k}_1} = \omega_{\mathbf{k}_2} = \omega_p/2$. If the amplitude of the pumping field h exceeds a threshold value $h_{\rm th}$ the quasiparticles number grows exponentially until a nonlinear process fixes the growth. The threshold amplitude depends critically on the inelastic relaxation frequency. If there were no inelastic relaxation the quasiparticles created by the pumping would never disappear, so the threshold would be equal to zero. In general, for a 3D system with both inelastic and elastic scattering the threshold value $h_{\rm th}$ can be estimated as follows:[2]

$$|h_{\rm th}V_{\mathbf{k}}| \sim [\gamma_{\mathbf{k}}(\gamma_{\mathbf{k}} + \alpha\Gamma_{\mathbf{k}})]^{1/2}, \tag{1.5}$$

where $\gamma_{\mathbf{k}}$ is the inelastic scattering frequency, $V_{\mathbf{k}}$ is the coefficient of coupling of the pumping field with the quasiparticles, and α is a constant of order of 1.

It is obvious that both phenomena, localization and parametric excitation, interfere in low-dimensional magnets. In a homogeneous medium, without defects, the pumping provides full correlation between amplitudes and phases of parametrically excited magnons with opposite momenta, so one can introduce new quasiparticles, pairs, which are composed of the two initial magnons, by analogy with BCS theory of superconductivity. In particular, the threshold value is analogous to the critical temperature of a superconductor. However, there is an essential difference in the influence of defects on quasiparticles in a ferromagnet and in a superconductor. In a superconductor the amplitude of coupling is isotropic and therefore there is almost no influence of the scattering of quasiparticles on the critical temperature. In contrast, the amplitude of the magnon coupling is highly anisotropic. This results in the strong dependence of the threshold value on the frequency of scattering. In a medium with defects the correlation between magnons with opposite momenta is weaker than in a homogeneous one, but it is also very important.[2] *A priori* it is not even clear what the character of the divergence of the backscattering amplitude of the pairs in a low-dimensional medium is. The purpose of this paper is to study the influence of weak localization on the threshold of magnon parametric excitation.

The most interesting low-dimensional medium where magnons can be parametrically excited is a thin ferrite film. In some sense, such a film can be be considered a 2D system. It is found that the influence of weak localization on the threshold gives an additional logarithmic term to the threshold value:

$$(\Gamma_k^2/\gamma k v_k)\ln_{10}(L^2/\Lambda l), \tag{1.6}$$

where Λ is the inelastic mean free path: $\Lambda = v_{\mathbf{k}}/\gamma_{\mathbf{k}}$. In contrast to electronic systems, the logarithmic term grows with the size of a system L even if it exceeds the inelastic mean free path Λ. The reason for such dependence of the threshold on the size of a sample is that there are two characteristic inelastic scattering times of different natures in a system of parametrically excited magnons. The first one is the usual inelastic scattering time $\tau_{\mathbf{k}} = \gamma_{\mathbf{k}}^{-1}$ due to inelastic scattering of magnons on thermal magnons and other quasiparticles. The second inelastic scattering time, $\Theta_{\mathbf{k}} = \delta\omega_{\mathbf{k}}^{-1}$, is the correlation time in the system of parametrically excited magnons. The correlation frequency $\delta\omega_{\mathbf{k}}$ is due to scattering of parametrically excited magnons on other parametrically excited magnons.[3] The inelastic scattering frequency $\gamma_{\mathbf{k}}$ describes the damping of magnons in absence of parametric pumping. However, the parametric pumping creates new quasiparticles composed of magnons with opposite momenta, and the lifetime of these quasiparticles is $\Theta_{\mathbf{k}}$, but not $\tau_{\mathbf{k}}$. The correlation frequency $\delta\omega_{\mathbf{k}}$ tends to zero when the amplitude of the pumping field reaches the threshold value because the number of parametrically excited magnons tends to zero at that amplitude of the pumping field. There are two characteristic lengths in a system of parametrically excited magnons: the usual inelastic mean free path $\Lambda = v_{\mathbf{k}}\tau_{\mathbf{k}}$, and the correlation length $\lambda = v_{\mathbf{k}}\Theta_{\mathbf{k}}$. The latter is the spatial scale to cut off the divergent term caused by localization. The correlation length of parametically excited magnons tends to infinity at the threshold of parametric excitation because of exact compensation between magnon damping and their creation by the parametric pumping; therefore there is no inelastic cutoff at the threshold of parametric excitation for the localization effect.

The fact that the correlation length tends to infinity at the parametric excitation threshold can be easily understood for a homogeneous system that does not contain defects, i.e., $\Gamma_{\mathbf{k}} = 0$. The dynamical equations for the amplitudes of parametrically excited magnons $a_{\mathbf{k}}$ and $a_{-\mathbf{k}}^{\dagger}$ can be written in the linear approximation as follows:[4]

$$\begin{aligned} &\dot{a}_{\mathbf{k}} + (\gamma_{\mathbf{k}} + i\omega_{\mathbf{k}})a_{\mathbf{k}} + ih\exp(-i\omega_p t)V_{\mathbf{k}}a_{\mathbf{k}}^{\dagger} = 0, \\ &\dot{a}_{-\mathbf{k}}^{\dagger} + (\gamma_{\mathbf{k}} - i\omega_{\mathbf{k}})a_{-\mathbf{k}}^{\dagger} - ih\exp(i\omega_p t)V_{\mathbf{k}}^{*}a_{\mathbf{k}} = 0. \end{aligned} \tag{1.7}$$

The solution of this set of equations is

$$\begin{aligned} &a_{\mathbf{k}} \sim \exp(-i\omega_p t/2)\exp(\rho_{\mathbf{k}}t), \\ &a_{\mathbf{k}}^{\dagger} \sim \exp(i\omega_p t/2)\exp(\rho_{\mathbf{k}}t), \\ &\rho_{\mathbf{k}} = -\gamma_{\mathbf{k}} \pm [|hV_{\mathbf{k}}|^2 - (\omega_{\mathbf{k}} - \omega_p/2)^2]^{1/2}. \end{aligned} \tag{1.8}$$

The parametric instability takes place if the increment $\rho_{\mathbf{k}}$ is positive at least for one pair of magnons $(a_{\mathbf{k}}, a_{-\mathbf{k}}^{\dagger})$. The increment reaches its maximum for some wave vectors $\mathbf{k}$ situated on the resonant surface:

$$\omega_{\mathbf{k}} = \omega_p/2. \tag{1.9}$$

The threshold value h_{th} is the minimum value of the pumping field amplitude h provided that the maximum increment reaches the marginal value $\max\rho_{\mathbf{k}} = 0$. According to (1.8) the amplitudes of parametrically excited magnons, $a_{\mathbf{k}}$ and $a_{-\mathbf{k}}^{\dagger}$ are monochromatic in this case. Therefore the correlation time is infinitely long at the threshold. This statement remains correct for an inhomogeneous medium because the elastic scattering does not change the frequency of magnons. The most unstable modes are spatially inhomogeneous but monochromatic in that case.[2]

Probably, the threshold value h_{th} is the most interesting quantity to measure because the correlation length λ tends to infinity at the threshold; however it is finite beyond the threshold. Therefore the influence of localization is strongest at the threshold. The influence of the localization could be significant in a dirty magnet at low temperature and a small group velocity of magnons when the term before the logarithm in (1.6) is large enough.

A consistent way to describe a nonequilibrium system of parametrically excited quasiparticles scattered on randomly positioned defects is to use a diagram technique. Such a technique was developed by Zakharov and L'vov.[2] They used Wyld's diagram technique[5] originally proposed to describe the problem of turbulence. The technique is based on the dynamic equation for quasiparticle amplitude and a procedure of averaging over thermal fluctuations in a thermostat system interacting with the nonequilibrium system.

The weak localization in an equilibrium system without anomalous coupling can be described by means of the summation of the series of the maximally crossed diagrams. For a nonequilibrium system one should take two kinds of Green's functions, e.g., retarded and distributive, into account.[5,6] The former is the average response of a quasiparticle amplitude to a small external force; the latter is proportional to the pair-correlation function. For a system with anomalous correlations, such as the system of parametrically excited magnons, it is necessary to introduce the anomalous Green's functions in addition to the normal ones.[2]

In Sec. II the set of equations for the retarded Green's functions and the pair-correlation functions in the maximally crossed diagram approximation is obtained. The equation for the threshold amplitude as a condition for existence of a nontrivial correlation function appears. Section III is devoted to calculation of the threshold of parametric excitation in a normally magnetized ferrite film with defects. A possibility of experimental observation of the influence of the localization on the threshold of magnon parametric excitation is discussed in Sec. IV. In particular, it is shown that this influence can be observable in dirty or porous ferrite films at low temperature and small group velocity of magnons.

II. WEAK LOCALIZATION OF PARAMETRICALLY EXCITED MAGNONS

The dynamics of a system of parametrically excited magnons can be described by the Hamiltonian H:

$$H = H_0 + H_p + H_{sc}. \tag{2.1}$$

Here H_0 is the Hamiltonian of free magnons:

$$H_0 = \sum_{\mathbf{k}} \omega_{\mathbf{k}} a^\dagger_{\mathbf{k}} a_{\mathbf{k}}, \tag{2.2}$$

$\omega_{\mathbf{k}}$ is the energy spectrum, $a^\dagger_{\mathbf{k}}$ and $a_{\mathbf{k}}$ are the Bose operators of magnon creation and desruction. H_p is the term due to the pumping field:

$$H_p = 1/2 \sum_{\mathbf{k}} [h \exp(-i\omega_p t) V_{\mathbf{k}} a^\dagger_{\mathbf{k}} a^\dagger_{-\mathbf{k}} + \text{H.c.}], \tag{2.3}$$

$hV_{\mathbf{k}}$ is the coefficient of coupling of quasiparticles with opposite momenta. H_{sc} results in the elastic scattering of magnons on defects:

$$H_{sc} = \sum_{\mathbf{k},\mathbf{k}'} g_{\mathbf{k},\mathbf{k}'} \eta_{\mathbf{k}-\mathbf{k}'} a^\dagger_{\mathbf{k}} a_{\mathbf{k}'}, \tag{2.4}$$

where $g_{\mathbf{k},\mathbf{k}'}$ is the amplitude of scattering on an isolated defect and $\eta_{\mathbf{k}}$ is the Fourier transform of the defect density, which is a random function.

In order to study the nonequilibrium behavior of a magnon system one should introduce an interaction with an external thermostat, a big system that is in a state of the thermodynamic equilibrium. The Hamiltonian of this interaction is

$$H' = \sum_{\mathbf{k}} (a^\dagger_{\mathbf{k}} f_{\mathbf{k}} + f^\dagger_{\mathbf{k}} a_{\mathbf{k}}). \tag{2.5}$$

The force $f_{\mathbf{k}}$ is a result of the influence of all thermal magnons with energies far from $\omega_p/2$, which do not interact with the pumping field, phonons, and other quasiparticles. According to the fluctuation-dissipation theorem there is a dissipation with the rate $\gamma_{\mathbf{k}}$ in the magnon system. The value $\gamma_{\mathbf{k}}$ is proportional to the pair correlation function of the force $f_{\mathbf{k}}$:

$$\gamma_{\mathbf{k}} = (\pi/2) \tanh(\omega_{\mathbf{k}}/T)[\langle f^\dagger_{\mathbf{k}} f_{\mathbf{k}}\rangle + \langle f_{\mathbf{k}} f^\dagger_{\mathbf{k}}\rangle]. \tag{2.6}$$

Here T is the thermostat temperature. Thus, the dynamical equation for the magnon amplitude $a_{\mathbf{k}}$ is

$$(d/dt + \gamma_{\mathbf{k}} + i\omega_{\mathbf{k}}) a_{\mathbf{k}} + ih \exp(-i\omega_p t) V_{\mathbf{k}} a^\dagger_{-\mathbf{k}} + \sum_{\mathbf{k}'} g_{\mathbf{k},\mathbf{k}'} \eta_{\mathbf{k}-\mathbf{k}'} a_{\mathbf{k}'} = -if_{\mathbf{k}}. \tag{2.7}$$

In order to describe statistical characteristics of the parametrically excited magnon system one should introduce Green's functions. It is convenient to use Fourier transforms. The retarded Green's functions, which are called further Green's functions simply, are

$$\begin{aligned} &G_q^{++} = \overline{\langle \delta a_q/\delta f_q\rangle}, G_q^{-+} = \overline{\langle \delta a^\dagger_{\overline{q}}/\delta f_q\rangle}, \\ &G_q^{+-} = (G_{\overline{q}}^{-+})^* = \overline{\langle \delta a_q/\delta f^\dagger_{\overline{q}}\rangle}, \\ &G_q^{--} = (G_{\overline{q}}^{++})^* = \overline{\langle \delta a^\dagger_{\overline{q}}/\delta f^\dagger \overline{q}\rangle}, \\ &q = (\mathbf{k}, \omega),\ \overline{q} = (-\mathbf{k}, \omega_p - \omega). \end{aligned} \tag{2.8}$$

Here the angular brackets indicate averaging over the thermal fluctuations of the force f_q, and the overbar indicates averaging over the defect positions. The pumping field forces the coupling of magnon amplitudes a_q and $a^\dagger_{\overline{q}}$; therefore the anomalous Green's functions G_q^{+-} and G_q^{-+} appear. One can represent the solution to the equation (2.7) as a series in the powers of the scattering amplitude $g_{\mathbf{k},\mathbf{k}'}$ and the pumping field:

$$a_q = G_q^0 f_q + G_q^0 P_{\mathbf{k}} f^\dagger_{\overline{q}} + \sum_{\mathbf{k}'} G_q^0 g_{\mathbf{k},\mathbf{k}'} \eta_{\mathbf{k}-\mathbf{k}'} f_{\mathbf{k}',\omega} + \cdots, \tag{2.9}$$

where G_q^0 is the free magnon Green's function:

$$G_q^0 = (\omega - \omega_{\mathbf{k}} + i\gamma_{\mathbf{k}})^{-1}, \tag{2.10}$$

and $P_{\mathbf{k}}$ is the pumping amplitude:

$$P_{\mathbf{k}} = hV_{\mathbf{k}}. \tag{2.11}$$

Direct averaging of the variational derivatives (2.8) yields Dyson's equation for the matrix Green's function $\hat{G}_q = G_q^{\alpha\beta} (\alpha, \beta = \pm)$:

$$\hat{G}_q = \hat{G}_q^0 + \hat{G}_q^0 \hat{\Sigma}_q \hat{G}_q, \tag{2.12}$$

$$\begin{aligned} &\hat{G}_q^0 = \begin{pmatrix} G_q^0 & 0 \\ 0 & G_{\overline{q}}^{0*} \end{pmatrix}, \quad \hat{\Sigma}_q = \begin{pmatrix} \Sigma_q & \Pi_q \\ \Pi^*_{\overline{q}} & \Sigma^*_{\overline{q}} \end{pmatrix}, \\ &\hat{G}_q = \begin{pmatrix} G_q^{++} & G_q^{+-} \\ G_q^{-+} & G_q^{--} \end{pmatrix}. \end{aligned} \tag{2.13}$$

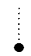
Here $\hat{\Sigma}_q$ is the mass operator matrix. It represents the sum of the disconnected parts of Green's functions. The set of Dyson's equations resembles that for a superconducting system. The reason for the similarity is in the coupling of quasiparticles with opposite momenta in both kinds of systems. In contrast to superconductivity, parametrically excited quasiparticles are bosons, there is no parametric excitation of fermions. It should be noted that the matrix retarded Green's function $\hat{G}_q$ has nothing in common with the matrix Green's function in the Keldysh diagram technique for nonequilibrium systems. The former is obtained from the normal and anomalous retarded Green's functions, the latter is obtained from the normal retarded and distributive Green's functions. The solution to Eq. (2.12) is

$$\begin{aligned} &G_q^{++} = -(\epsilon + \epsilon_q + i\Gamma_q)/\Delta_q, \\ &G_q^{-+} = \Pi^*_{\overline{q}}/\Delta_q, \end{aligned} \tag{2.14}$$

$$\epsilon = \omega - \omega_p/2, \epsilon_q = \omega_{\mathbf{k}} + \mathrm{Re}\Sigma_q - \omega_p/2,$$
$$\Gamma_q = -\mathrm{Im}\Sigma_q, \tag{2.15}$$
$$\Delta_q = \nu_q^2 + \epsilon_q^2 - 2\Gamma_q\epsilon - \epsilon^2,$$
$$\nu_q = (\gamma_q^2 - |\Pi_q|^2)^{1/2}.$$

In order to find the Green's functions it is necessary to express the mass operator matrix $\hat{\Sigma}_q$ in terms of the Green's functions. It is convenient to introduce a diagrammatic representation for the Green's functions and the amplitude of scattering in order to obtain the mass operator representation:

$\hat{G}_q =$, (2.16)

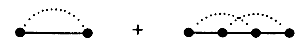
$g_{\mathbf{k},\mathbf{k}'}\eta_{\mathbf{k}-\mathbf{k}'}\hat{I} =$  ,

where $\hat{I}$ is the unit (2×2) matrix.

For small defect concentration one can neglect the higher correlation function of the defect density and take only the pair-correlation function into account:

$$\langle \eta_{\mathbf{k}}\eta^*_{\mathbf{k}'}\rangle = \eta^2_{\mathbf{k}}\delta(\mathbf{k}-\mathbf{k}'). \tag{2.17}$$

The mass operator for a system with defects can be drawn as follows:

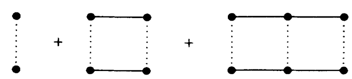
$\hat{\Sigma}_q = \hat{\Sigma}_q^0 +$  $+ \cdots$ (2.18)

$$\hat{\Sigma}_q^0 = \begin{pmatrix} 0 & P_{\mathbf{k}} \\ P^*_{\mathbf{k}} & 0 \end{pmatrix}. \tag{2.19}$$

In a 3D system the first diagram in (2.18) is the largest one. In low-dimensional systems the maximally crossed diagrams are the greatest ones. The method of summation of the maximally crossed diagram series is based on separation of the middle Green's funcion in a mass operator diagram and on transformation of the maximally crossed diagrams into ladder series.[1,7] Consequently, the mass operator can be represented as follows:

$$\Sigma_q^{\alpha\beta} = \Sigma_q^{0\alpha\beta} + \sum_{\mathbf{k}',\gamma,\delta} W^{\alpha\gamma\beta\delta}_{\mathbf{k}\mathbf{k}'\mathbf{k}\mathbf{k}'}G^{\gamma\delta}_{\mathbf{k}'}, \tag{2.20}$$

where $W^{\alpha_1\alpha_2\alpha_3\alpha_4}_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4}$ is the sum of the ladder series:

$\hat{W} =$  . (2.21)

Analytically Eq. (2.21) can be written as follows:[1]

$$W^{\alpha_1\alpha_2\alpha_3\alpha_4}_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4} = cg_{\mathbf{k}_1\mathbf{k}_3}g_{\mathbf{k}_2\mathbf{k}_4}\delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)\delta_{\alpha_1,\alpha_3}\delta_{\alpha_2,\alpha_4} + \sum_{\mathbf{k}_5\mathbf{k}_6\alpha_5\alpha_6} cg_{\mathbf{k}_1\mathbf{k}_5}g_{\mathbf{k}_2\mathbf{k}_6}G^{\alpha_1\alpha_5}_{\mathbf{k}_5}G^{\alpha_2\alpha_6}_{\mathbf{k}_6}W^{\alpha_5\alpha_6\alpha_3\alpha_4}_{\mathbf{k}_5\mathbf{k}_6\mathbf{k}_3\mathbf{k}_4}\delta(\mathbf{k}_5+\mathbf{k}_6-\mathbf{k}_3-\mathbf{k}_4). \tag{2.22}$$

Here c is the defect concentration. It is assumed that the defects are positioned independently and randomly. Consequently,

$$\eta^2_{\mathbf{k}} = c. \tag{2.23}$$

In the simplest case, when the defects are small compared with the wavelength and they scatter magnons isotropically the amplitude of scattering $g_{\mathbf{k}\mathbf{k}'}$ is a constant,

$$g_{\mathbf{k}\mathbf{k}'} = g, \tag{2.24}$$

and the equations for the vertices $W^{\alpha_1\alpha_2\alpha_3\alpha_4}_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4}$ can be solved analytically. There are 16 equations for various sets $(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$. They are grouped into four sets of four equations for each pair (α_3,α_4). In general, the vertex $W^{\alpha_1\alpha_2\alpha_3\alpha_4}_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4}$ can be represented as follows:

$$W^{\alpha_1\alpha_2\alpha_3\alpha_4}_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4} = w^{\alpha_1\alpha_2}(\mathbf{Q},\mathbf{p},\mathbf{s}|\alpha_3\alpha_4)\delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4), \tag{2.25}$$

where

$$\mathbf{Q} = (\mathbf{k}_1+\mathbf{k}_2)/2, \quad \mathbf{p} = \mathbf{k}_1-\mathbf{k}_2, \quad \mathbf{s} = \mathbf{k}_3-\mathbf{k}_4. \tag{2.26}$$

For $w^{\alpha\beta}(\mathbf{Q},\mathbf{p},\mathbf{s}|\gamma,\delta)$ one can obtain from (2.22)

$$w^{\alpha\beta}(\mathbf{Q},\mathbf{p},\mathbf{s}|\gamma,\delta) = c|g|^2\delta_{\alpha\gamma}\delta_{\beta\delta} + \frac{v_0}{(2\pi)^2}\int c|g|^2 G^{\alpha\sigma}_{\mathbf{Q}+\mathbf{p}'/2}G^{\beta\tau}_{\mathbf{Q}-\mathbf{p}'/2}w^{\sigma\tau}(\mathbf{Q}\mathbf{p}'\mathbf{s}|\gamma\delta)d\mathbf{p}'. \tag{2.27}$$

It is clear that for $g_{\mathbf{k},\mathbf{k}'}$=const the vertex $w^{\alpha\beta}$ depends only on the momentum $\mathbf{Q}$ but not on the momenta $\mathbf{p}$ and $\mathbf{s}$. The matrix elements

$$M^{\alpha\beta}(\mathbf{Q}) = \frac{v_0}{(2\pi)^2}\int c|g|^2 G^{\alpha\sigma}_{\mathbf{Q}+\mathbf{p}/2}G^{\beta\tau}_{\mathbf{Q}-\mathbf{p}/2}d^2p \tag{2.28}$$

can be found from the Green's functions (2.14). For a normally magnetized ferrite film the coefficient of coupling of magnons $V_{\mathbf{k}}$ takes the form

$$V_{\mathbf{k}} = V \exp(2i\varphi_{\mathbf{k}}), \quad V = g\mu\omega_m/(2\omega_p), \tag{2.29}$$

where $\varphi_{\mathbf{k}}$ is the polar angle of the wave vector $\mathbf{k}$, g is the gyromagnetic ratio, μ is the Bohr's magneton, and $\omega_m = 4\pi g\mu M$ (M is the ferrite magnetization).

Thus, the nonzero matrix elements $M^{\alpha\sigma\beta\tau}(\mathbf{Q})$ are

$$M^{++++} = M^{----} = 1 + \xi|\Pi|^2, \quad M^{-+-+} = M^{+-+-} = 1 - \xi(\Gamma^2 + \nu^2), \quad M^{+--+} = M^{-++-} = -\xi|\Pi|^2. \tag{2.30}$$

Here Γ_{el} is the frequency of elastic scattering

$$\Gamma_{\text{el}} = \frac{v_0}{(2\pi)^2} \int c|g|^2 \delta(\omega - \omega_{\mathbf{k}}) d^2k, \quad \Gamma = \gamma - \text{Im}\Sigma, \quad \nu = (\Gamma^2 - |\Pi|^2)^{1/2}, \quad \xi = \frac{\Gamma_{\text{el}}}{\nu^2(4\nu^2 + v^2Q^2)^{1/2}}, \tag{2.31}$$

and v is the group velocity. The nonzero solutions of the system of linear equations (2.27) have the form

$$\begin{aligned} w^{++}(Q|++) &= w^{--}(Q|--) = \frac{c|g|^2}{1 + \xi|\Pi|^2}, \\ w^{+-}(Q|+-) &= w^{-+}(Q|-+) = \frac{c|g|^2[1 - \xi(\Gamma^2 + \nu^2)]}{1 - 2\xi(\Gamma^2 + \nu^2) + 4\xi^2\Gamma^2\nu^2}, \\ w^{+-}(Q|-+) &= w^{-+}(Q|+-) = \frac{c|g|^2\xi|\Pi|^2}{1 - 2\xi(\Gamma^2 + \nu^2) + 4\xi^2\Gamma^2\nu^2}. \end{aligned} \tag{2.32}$$

Substituting the solution (2.32) into the equation for the mass operator (2.20) one can obtain the mass operator and consequently the Green's functions. The Green's functions are found in Sec. III.

In order to obtain the threshold of parametric excitation of magnons it is necessary to study the pair-correlation functions:

$$n_q = \overline{\langle a_q^\dagger a_q \rangle}, \quad \sigma_q = \overline{\langle a_q a_{\overline{q}} \rangle}. \tag{2.33}$$

The equations for the pair correlation functions can be derived by averaging the products of series for a_q and $a_q^\dagger$ or $a_{\overline{q}}$ (2.7) over the random force f_q and the random defect positions. Each term of the series for the normal correlation function n_q begins from G_q^0 and ends at G_q^{0*}. Each term for the anomalous correlation fnction σ_q begins from G_q^0 and ends at $G_{\overline{q}}^0$. It is convenient to introduce the matrix of the pair-correlation functions:

$$\hat{N}_q = \begin{pmatrix} n_q & \sigma_q \\ \sigma_{\overline{q}}^* & n_{\overline{q}} \end{pmatrix}. \tag{2.34}$$

Wyld's equations for the correlation functions can be written in the matrix form:

$$\hat{N}_q = \hat{G}_q \hat{\Phi}_q \hat{G}_q^\dagger, \tag{2.35}$$

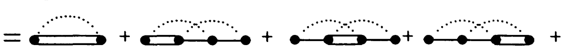
where $\hat{G}_q^\dagger$ is the hermitian conjugate Green's function matrix, and $\hat{\Phi}_q$ is the distributive mass operator matrix. It can be written as follows:

$$\hat{\Phi}_q = \begin{pmatrix} \Phi_q & \Psi_q \\ \Psi_{\overline{q}}^* & \Phi_{\overline{q}}^* \end{pmatrix}, \tag{2.36}$$

where Φ_q and Ψ_q are the sums of disconnected diagrams contained in the normal and anomalous correlation functions. Introducing a diagrammatic representation for the matrix $\hat{N}_q$,

$\hat{N}_q =$, (2.37)

one can obtain the diagram series for the distributive mass operator:

$\hat{\Phi}_q =$  (2.38)

The diagrams for the distributive mass operator Φ_q (2.38) differ from those for the mass operator Σ_q only by all possible substitutions of one of the Green's functions in each diagram by a pair-correlation function. The sum of maximally crossed diagrams for the distributive mass operator (2.38) consist of three groups of diagrams: (i) the diagrams with the double line (pair-correlation function) in the middle of a diagram, (ii) the diagrams with the double line in the left part of a diagram, and (iii) the diagrams with the double line in the right part of a diagram.

The sum of the first group of diagrams can be represented the same way as the sum for the mass operator Σ_q:

$$\hat{\Phi}_q^{(1)} = \frac{v_0}{(2\pi)^2} \int \hat{U}_{\mathbf{k}\mathbf{k}'} \hat{N}_{\mathbf{k}'} d^2k', \tag{2.39}$$

where $\hat{U}_{\mathbf{k}\mathbf{k}'}$ is the sum of a series of ladder diagrams that can be drawn using the same ladder diagram series as (2.21). The difference of the series for the vertex $\hat{U}$ in the distributive mass operator from that for the mass operator $\hat{W}$ consists in the substitution of the Green's function matrix $\hat{G}_q$ in the lower line by $\hat{G}_q^\dagger$. The vertices $U_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4}^{\alpha_1\alpha_2\alpha_3\alpha_4}$ can be obtained the same way as the vertices $W_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4}^{\alpha_1\alpha_2\alpha_3\alpha_4}$. The result is

$$U^{\alpha_1\alpha_2\alpha_3\alpha_4}_{\mathbf{k_1k_2k_3k_4}} = u^{\alpha_1\alpha_2}(\mathbf{Q}|\alpha_3\alpha_4)\delta(\mathbf{k_1}+\mathbf{k_2}-\mathbf{k_3}-\mathbf{k_4}), \tag{2.40}$$

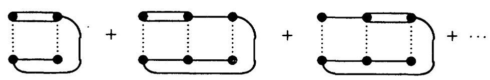
where the nonzero matrix elements $u^{\alpha_1\alpha_2}(\mathbf{Q}|\alpha_3\alpha_4)$ are

$$u^{++}(Q|++) = u^{--}(Q|--) = \frac{c|g|^2}{1-\xi(\Gamma^2+\nu^2)}, \tag{2.41}$$

$$u^{-+}(Q|+-) = u^{+-}(Q|-+) = \frac{c|g|^2\xi|\Pi|^2}{1-2\xi|\Pi|^2},$$

$$u^{+-}(Q|+-) = U^{-+}(Q|-+) = \frac{c|g|^2(1+\xi|\Pi|^2)}{1-2\xi|\Pi|^2}. \tag{2.42}$$

The second group of diagrams for the distributive mass operator $\hat{\Phi}_q$ can be represented as follows:

$\hat{\Phi}^{(2)}_q =$  (2.43)

Here the central Green's function connects the upper and lower lines of the Green's functions. So two vertices, $\hat{W}$ and $\hat{U}$, appear. The first results in summation of the diagrams with the Green's functions in both, the upper and the lower, lines. The second is a result of summation of the maximally crossed diagrams with the Green's functions in one line and the conjugate Green's functions in another. The diagrams for the third part of the distributive mass operator $\hat{\Phi}^{(3)}_q$ can be obtained from the series (2.42) by the exchange of lower and upper lines of the Green's and pair-correlation functions.

Thus, the full system of equations to describe the influence of the localization on the parametric excitation of magnons has been obtained. The solution of the equations (2.14), (2.20), and (2.32) yields the Green's function $\hat{G}_q$. If the Green's functions are known, one can find the vertices $U^{\alpha_1\alpha_2}(Q|\alpha_3\alpha_4)$ (2.40). Wyld's equation (2.35) and the expressions for the distributive mass operator (2.39) and (2.42), where

$$\hat{\Phi}_q = \hat{\Phi}^{(1)}_q + \hat{\Phi}^{(2)}_q + \hat{\Phi}^{(3)}_q \tag{2.44}$$

represent the full set of equations for the pair-correlation function $\hat{N}_q$.

III. THE THRESHOLD OF PARAMETRIC EXCITATION OF MAGNONS IN A FERRITE FILM

The first step to calculate the parametric excitation threshold is to find the normal and anomalous mass operators Σ and Π. This yields the retarded Green's function matrix $\hat{G}$. In the case of the strong elastic scattering, when the elastic scattering frequency $\Gamma_{\rm el}$ is greater than the frequency of inelastic scattering γ,

$$\Gamma_{\rm el} \gg \gamma,$$

the effective pumping amplitude is of order

$$|\Pi| \sim (\gamma\Gamma_{\rm el})^{1/2}. \tag{3.1}$$

According to (2.32) one can obtain from (2.20):

$$\Gamma_{\mathbf{k}} = -\Sigma_{\mathbf{k}} = \gamma + \frac{v_0}{(2\pi)^2}\int c|g|^2 \frac{1}{1+\xi|\Pi|^2}\frac{\Gamma}{\epsilon^2_{\mathbf{k-Q}}+\nu^2}d\mathbf{Q}, \tag{3.2}$$

$$\Pi_{\mathbf{k}} = hV_{\mathbf{k}} + \frac{v_0}{(2\pi)^2}\int c|g|^2 \frac{1-\xi(\Gamma^2+\nu^2)}{1-2\xi(\Gamma^2+\nu^2)+4\xi^2\Gamma^2\nu^2}\frac{\Pi_{\mathbf{k-Q}}}{\epsilon^2_{\mathbf{k-Q}}+\nu^2}d\mathbf{Q}. \tag{3.3}$$

The equation for the magnon distribution function $\hat{N}_{\mathbf{k}}$ can also be obtained. In order to derive that equation in a consistent form one has to express the distributive mass operators $\mathbf{\Phi_k}$ and $\mathbf{\Psi_k}$ in terms of the retarded Green's functions and pair-correlation functions. It was shown in the preceeding section that they can be represented as sums of three parts of different structure. For the symmetric parts of the distributive mass operators, where graphs contain the pair-correlation function in the center, one can obtain

$$\Phi^{(1)}_{\mathbf{k}} = \frac{v_0}{(2\pi)^2}\int c|g|^2\left(\frac{1}{1-\xi(\Gamma^2+\nu^2)} + \frac{\xi|\Pi|^2}{1-2\xi|\Pi|^2}\right)n_{-\mathbf{k}+\mathbf{Q}}d\mathbf{Q}, \tag{3.4}$$

$$\Psi^{(1)}_{\mathbf{k}} = \frac{v_0}{(2\pi)^2}\int c|g|^2\frac{1+\xi|\Pi|^2}{1-2\xi|\Pi|^2}\sigma_{-\mathbf{k}+\mathbf{Q}}d\mathbf{Q}. \tag{3.5}$$

The term $\Psi^{(1)}_{\mathbf{k}}$ is equal to zero because $\sigma_{\mathbf{k}} \sim \exp(2i\varphi_{\mathbf{k}})$. The terms $\Phi^{(2)}_{\mathbf{k}}$ and $\Psi^{(2)}_{\mathbf{k}}$ can be represented as follows:

$$\Phi^{(2)}_{\mathbf{k}} = [v_0/(2\pi)^2]^2\int d\mathbf{Q}d\mathbf{k}'(c|g|^2)^2 n_{\mathbf{k}'}G_{-\mathbf{k}'+\mathbf{Q}}\{G_{-\mathbf{k}+\mathbf{Q}}[1-\xi(\Gamma^2+\nu^2)]^{-1} + \xi|\Pi|^2G^*_{\mathbf{k}+\mathbf{Q}}[1-2\xi(\Gamma^2+\nu^2)+4\xi^2\Gamma^2\nu^2]^{-1}\}, \tag{3.6}$$

$$\Psi^{(2)}_{\mathbf{k}} = [v_0/(2\pi)^2]^2\int d\mathbf{Q}d\mathbf{k}'(c|g|^2)^2 L_{-\mathbf{k}+\mathbf{Q}}G_{-\mathbf{k}'+\mathbf{Q}}n_{\mathbf{k}'} \times\{[1-\xi(\Gamma^2+\nu^2)][1-2\xi(\Gamma^2+\nu^2)+4\xi^2\Gamma^2\nu^2]^{-1}(1+\xi|\Pi|^2)(1-2\xi|\Pi|^2)^{-1}\}. \tag{3.7}$$

As far as parts $\Phi^{(3)}_{\mathbf{k}}$ and $\Psi^{(3)}_{\mathbf{k}}$ are concerned, they are represented by symmetric diagrams to that for $\Phi^{(2)}_{\mathbf{k}}$ and $\Psi^{(2)}_{\mathbf{k}}$. Consequently,

$$\Phi^{(3)}_{\mathbf{k}} = \Phi^{(2)*}_{\mathbf{k}}, \; \Psi^{(3)}_{\mathbf{k}} = \Psi^{(2)}_{\mathbf{k}}. \tag{3.8}$$

Thus the full set of equations has been derived. In order to find the influence of the weak localization on the threshold it is necessary to obtain the threshold neglecting the localization. This means that all the processes of multiple scattering with small momentum difference Q should be neglected. Formally it can be done setting $\xi = 0$. In this approximation one can obtain, from (3.2)–(3.8):

$$\Gamma = \gamma + \Gamma_{\rm el}\Gamma/\nu, \tag{3.9}$$

$$\Pi_{\mathbf{k}} = hV_{\mathbf{k}}, \tag{3.10}$$

$$\Phi_{\mathbf{k}} = \frac{v_0}{(2\pi)^2}\int c|g|^2 n_{\mathbf{k}'} d\mathbf{k}', \tag{3.11}$$

$$\Psi_{\mathbf{k}} = 0. \tag{3.12}$$

The parametric magnon distribution function $n_{\mathbf{k}}$ is

$$n_{\mathbf{k}} = \frac{\Gamma^2 + |\Pi|^2 + \epsilon^2_{\mathbf{k}}}{(\epsilon^2_{\mathbf{k}} + \nu^2)^2}\Phi_{\mathbf{k}}. \tag{3.13}$$

This equation was obtained by Zakharov and L'vov.[2] In order to find the relation between the effective pumping Π and the elastic-scattering frequency one should make the integration in Eq. (3.13). For the integral number of parametric magnons n,

$$n = \frac{v_0}{(2\pi)^2}\int n_{\mathbf{k}} d\mathbf{k}, \tag{3.14}$$

the equation yields

$$n = (\Gamma_{\rm el}\Gamma^2/\nu^3)n. \tag{3.15}$$

A nontrivial solution of the equation exists if

$$\nu^3 = \Gamma_{\rm el}\Gamma^2. \tag{3.16}$$

This is the relation sought. The solution of the set of equations for the threshold in a 2D medium, (3.9), (3.10), and (3.16), in the case of strong elastic scattering ($\Gamma_{\rm el} \gg \gamma$) is

$$|h_{\rm th}V| = |\Pi| = (\gamma\Gamma_{\rm el})^{1/2}, \qquad \nu = \Gamma_{\rm el} + \gamma, \; \Gamma = \Gamma_{\rm el} + 3\gamma/2. \tag{3.17}$$

Substituting these relations (3.17) into vertices (2.32) and (2.40) one can estimate the influence of the backscattering. In the small Q limit, i.e., for almost exact backscattering, the denominators in the vertices are

$$1 - \xi(\Gamma^2 + \nu^2) = \frac{1}{2}\left(\frac{\gamma}{\Gamma_{\rm el}} + \frac{v^2Q^2}{4\Gamma^2_{\rm el}}\right), \tag{3.18}$$

$$1 - 2\xi(\Gamma^2 + \nu^2) + 4\xi^2\Gamma^2\nu^2 = \frac{1}{8}\frac{\gamma}{\Gamma_{\rm el}}\frac{v^2Q^2}{\Gamma^2_{\rm el}}. \tag{3.19}$$

The term independent of Q in the second expression equals zero. This provides the logarithmic divergence of the renormalized pumping Π (3.3) and some terms in the distributive mass operators. Only these most divergent terms should be taken into account among the terms of order of $(\Gamma_{\rm el}/kv)$. The equation for the pair-correlation function can be written in this approximation as follows:

$$n_{\mathbf{k}} = \frac{\Gamma^2 + |\Pi|^2 + \epsilon^2_{\mathbf{k}}}{(\epsilon^2_{\mathbf{k}} + \nu^2)^2}\left(\frac{v_0}{(2\pi)^2}\int c|g|^2 n_{\mathbf{k}'} d\mathbf{k}' - \frac{v_0}{(2\pi)^2} 2\,\mathrm{Re}\int_{L^{-1}<Q<Q_m} (c|g|^2)^2 \frac{-i\Gamma}{2\nu^2} n \frac{4\Gamma_{\rm el}|\Pi|^2}{\gamma v^2 Q^2} G^*_{-\mathbf{k}+\mathbf{Q}} d\mathbf{Q}\right). \tag{3.20}$$

Here L is the characteristic size of a sample and Q_m is the maximum value Q provided that the approximation (3.19) is valid. Comparing (3.19) with the terms of order of $(vQ/\nu)^4$ on can find that

$$(vQ_m)^2 \sim (\gamma\Gamma_{\rm el}) = |\Pi|^2. \tag{3.21}$$

The equation for the integral number of parametric magnons n can be obtained from (3.20):

$$n\left[1 - \left(\frac{\Gamma_{\rm el}\Gamma^2}{\nu^3} - \frac{2\Gamma_{\rm el}}{\pi kv}\ln(Q_m L)\right)\right] = 0. \tag{3.22}$$

The solution of the equation for the threshold (3.22) is

$$|\Pi|^2 = \Gamma_{\rm el}\left(\gamma - \frac{2\Gamma^2_{\rm el}}{\pi kv}\ln(Q_m L)\right). \tag{3.23}$$

Finally, the threshold can be found from the relation between renormalized pumping Π and the pumping field h (3.3):

$$|h_{\rm th}V|^2 = \left(1 - \frac{4\Gamma_{\rm el}}{\pi kv}\ln(Q_m L)\right)^2 \times \left(1 - \frac{2\Gamma^2_{\rm el}}{\pi\gamma kv}\ln(Q_m L)\right)(\gamma\Gamma_{\rm el}) \approx \left(1 - \frac{\Gamma^2_{\rm el}}{\pi\gamma kv}\ln(\gamma\Gamma_{\rm el}L^2/v^2)\right)(\gamma\Gamma_{\rm el}). \tag{3.24}$$

The additional term in (3.24) is due to the influence of weak localization. This term is essential at low temperature, when $\Gamma_{\rm el} \gg \gamma$, and for magnons with small group velocity v. It should be noted that the frequency of elastic scattering $\Gamma_{\rm el}$ is inversely proportional to the group velocity.

IV. CONCLUSION

Any ferrite film is, of course, three dimensional. In a film of thickness d there are many two-dimensional magnon modes with different transverse projections of the wave vector: $\kappa_n = \pi n/d$, where n is a natural number. The gap of the second transverse mode is of the order of

$$\omega_2 \approx \omega_1 + \omega_{\rm ex}(a/d)^2, \qquad (4.1)$$

where ω_1 is the gap of the first mode, $\omega_{\rm ex}$ is the magnon stiffness, and a is the lattice constant. If the frequency of the second mode is higher than the frequency of parametrically excited magnons

$$\omega_2 > \omega_p/2, \qquad (4.2)$$

then the parametric excitation of the second mode magnons is forbidden because of the energy and momentum conservation conditions (1.4). The second mode magnons cannot be excited as a result of the elastic scattering, either. Thus, if the condition

$$\omega_1 < \omega_p/2 < \omega_2 \qquad (4.3)$$

is satisfied, only first mode magnons are involved in the process of parametric excitation. This means that such a thin film can be considered as a 2D system.

Usually, high-quality films are in use for experimental studies. To observe the effect of the localization a dirty film in contrast is preferable. For a thin quality film a characteristic value of $\gamma/(kv)$ is of order of 10^{-4}. For grained and porous ferrite samples the parametric excitation threshold is 10–30 times higher than for homogeneous ones.[8,9] This means that the elastic scattering frequency is 10^2–10^3 greater than the damping γ according to the formula (1.5). Consequently, the term $\frac{2\Gamma_{\rm el}^2}{\pi\gamma kv}\ln(Q_m L)$ is of order of 0.03–0.3. So the influence of the localization can be observable.

As far as quasi-one-dimensional and quasi-two-dimensional ferromagnets that consist of well-separated chains or layers of magnetic ions are concerned, they are hardly treated as low-dimensional systems in the framework of problems related to parametric excitation. The reason is that the parametric excitation is caused by the long-ranged magnetic dipole interaction. This interaction makes the chains or layers of separated spins coupled, and the magnon specrum is highly anisotropic but three-dimensional in such magnets.

It should be noted that the localization effect is due to time reversibility of the elastic scattering process. Magnetic moment is a pseudovector: $\mathbf{M}(-t) = -\mathbf{M}(t)$. Therefore the influence of weak localization can be suppressed by any scalar or vector field. In particular, the localization must be sensitive to deformation. A rough estimate of the cutoff scale for the wave vector Q is $Q_{\rm min}/k \sim |\nabla \mathbf{u}|$, where $\mathbf{u}$ is the deformation.

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