Jastrow-Slater trial wave functions for the fractional quantum Hall effect: Results for few-particle systems

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We report results of a numerical study of the Jastrow-Slater trial wave functions for the fractional quantum Hall effect proposed by one of us. We study systems of up to eight electrons and find that these trial states are an extremely good approximation to the true Coulomb states.

I. INTRODUCTION

The purpose of this paper is to test the validity of the Jastrow-Slater trial wave functions for the fractional quantum Hall effect (FQHE) proposed by Jain.[1,2] These are given by

$$\chi_p = \mathcal{P} \prod_{j<k} (z_j - z_k)^{m-1} \Phi_n \ , \tag{1}$$

where the filling factors p and n (the subscript of χ or Φ will always denote the filling factor) are related by

$$p = \frac{n}{(m-1)n+1} \ , \tag{2}$$

Φ_n is the known wave function for the incompressible integer quantum Hall (IQHE) system[3] (i.e., for noninteracting electrons) with n filled Landau levels (LL's), χ_p is the trial wave function for the FQHE system at filling factor p, $z_j = x_j + iy_j$ denotes the position of the jth electron, $\mathcal{P}$ is an operator that projects the state onto the lowest Landau level, and m is an odd integer. These trial wave functions clearly satisfy some of the basic properties, such as translational invariance (see Appendix A).

In this scheme the quasiparticle excitations of the FQHE state are also related to the quasiparticle of the IQHE states in the same manner.[1,2] That is, quasielectron and quasihole excitations of χ_p are given by

$$\chi_p^{\rm qh} = \mathcal{P} \prod_{j<k} (z_j - z_k)^{m-1} \Phi_n^{\rm qh} \ , \tag{3}$$

$$\chi_p^{\rm qe} = \mathcal{P} \prod_{j<k} (z_j - z_k)^{m-1} \Phi_n^{\rm qe} \ , \tag{4}$$

respectively, where $\Phi_n^{\rm qh}$ is the state with n filled LL's except for a hole in the nth LL, and $\Phi_n^{\rm qe}$ is the state with n filled LL's and an electron in the $(n+1)$th LL.

Subsequently, Jain, Kivelson, and Trivedi[4] generalized these trial wave functions to describe the FQHE states at arbitrary filling factors in the presence of disorder. The generalized trial states are given by

$$\chi_\nu = \mathcal{P} \prod_{j<k} (z_j - z_k)^{m-1} \Phi_{\nu^*} \ , \tag{5}$$

where the filling factors ν and ν^* are related by

$$\nu = \frac{\nu^*}{(m-1)\nu^*+1} \ . \tag{6}$$

Here, Φ_{ν^*} is the Slater determinant ground state for noninteracting electrons at filling factor ν^*, which can be in principle calculated for arbitrary disorder. The trial wave function in Eq. (5) asserts that if the IQHE state at ν^* is uniquely determined (which is always the case with finite disorder), then a trial wave function for the FQHE state at ν^*, related to ν by Eq. (6), can be constructed by application of the Jastrow factor. When ν^* is an integer (say $\nu^* = n$) i.e., Φ_n corresponds essentially to n filled LL's even in the presence of (sufficiently weak) disorder and one recovers the earlier trial states of Eq. (1). (We will reserve n and p to denote the filling factors of *incompressible* IQHE and FQHE states, respectively.)

Those trial states of the present scheme, which are entirely in the lowest LL even without the projection operator, have been considered before. These are $\chi_{1/m}$ and their quasiholes $\chi_{1/m}^{\rm qh}$. These states are identical to the Jastrow states proposed by Laughlin,[5] because

$$\Phi_1 = \prod_{j<k} (z_j - z_k) \exp\left[-\tfrac{1}{4} \sum_j |z_j|^2 \right] \tag{7}$$

and

$$\Phi_1^{\rm qh} = \prod_j (z_j - z_0) \Phi_1 \ . \tag{8}$$

However, all other trial states make use of the higher LL's, and are new. It is worthwhile to point out here that our $\chi_{1/m}$ state ascribes the physics of the FQHE at $\nu = 1/m$ to *Jastrow-Slater* correlations rather than to Jastrow correlations alone, as was originally proposed by Laughlin. In the case of $\nu = 1/m$, these two viewpoints are equivalent, since the Slater-determinant state Φ_1 itself happens to be of Jastrow form. However, while pure Jastrow correlations are capable of describing only the states at $\nu = 1/m$, Jastrow-Slater correlations are generally valid and describe the physics of all FQHE states.

How does one know if these trial wave functions have anything to do with the reality? One very fortunate feature of the FQHE is that it is possible to compute the exact eigenstates of finite systems in the $B \to \infty$ limit, because then the Hilbert space is restricted to the lowest LL, and there are only a finite number of many-body basis states. It is appropriate to call these studies "nu-

merical experiments" because (i) *they represent the exact solutions of a realistic Hamiltonian* (say Coulomb interactions), and (ii) *they provide a faithful representation of the phenomenon of the FQHE,* in the sense that they show incompressibility at precisely those filling factors at which the FQHE is experimentally observed. Therefore, a comparison with numerical experiments has come to serve as the essential test of any trial wave function in the field of the FQHE. The most sensitive comparison is provided by the overlap of the trial wave function with the numerical experimental state. A large overlap not only guarantees that all the correlation functions of the numerical state are very well approximated by those of the trial state, but also ensures that the trial state has very good energies even in the thermodynamic limit. This is the case because the energy of state is mainly determined by its short-distance behavior, and good overlap for sufficiently large (in this case large compared to the magnetic length) systems signifies good short-distance correlations.

The Laughlin states, which are also a part of our scheme, have remarkably large overlaps with the true Coulomb state for small systems.[5–7] For example, Laughlin's $\frac{1}{3}$ state has an impressive overlap of about 0.995 with the exact Coulomb ground states for a system of nine electrons on the surface of a sphere.[6] The appealing simplicity of the Laughlin states and their extremely accurate representation of the numerical experiments make a very convincing case in favor of the Laughlin states, and as a result it is widely believed that they provide a satisfactory microscopic understanding of the FQHE at $\nu=1/m$.

Such an understanding was earlier not possible for the FQHE at a large number of other fractions. In this paper we study the simplest new trial states of our theory—namely, the $\frac{1}{3}$ state with one or more quasielectrons, and the incompressible states at $\frac{2}{5}$ and $\frac{2}{9}$. We find that these states also have extremely good overlaps with the true Coulomb states. This lends strong support to this class of trial states, and gives us confidence that they provide a valid microscopic theory of the *general* FQHE states.

The paper is organized as follows. In Sec. II, III, and IV we study (i) the $\frac{1}{3}$ states with a single quasielectron, (ii) the $\frac{1}{3}$ state with more than one quasielectron, and (iii) the incompressible states at $\frac{2}{5}$ and $\frac{2}{9}$, respectively. In all these cases, we compare our trial states with the true Coulomb state, and find that our trial states have an excellent overlap with the true state, better than any other trial states proposed in the literature. The effect of the projection operator is discussed in Sec. V and the paper is concluded in Sec. VI.

TABLE I. This table gives the squares of the overlaps of our quasielectron trial state $\chi^{qe}_{1/3}$ [Eq. (9)] and Laughlin's quasielectron trial state $\chi^{qe,L}_{1/3}$ [Eq. (10)] with the corresponding true Coulomb ground state. Disk geometry is used for the calculations. M is the total number of many-body basis states with the "correct" total angular momentum (i.e., the dimensionality of the matrix that needs to be diagonalized to obtain the Coulomb ground state). (a) uses the "open" boundary conditions and (b) uses "closed" boundary conditions (see the text for a definition of these boundary conditions).

(a)

N	M	$\lvert\langle\chi_{\text{true}}\lvert\chi^{qe}_{1/3}\rangle\rvert^2$	$\lvert\langle\chi_{\text{true}}\lvert\chi^{qe,L}_{1/3}\rangle\rvert^2$
3	3	1	1
4	15	0.9928	0.9966
5	84	0.9845	0.9922
6	532	0.9862	0.9742

(b)

N	M	$\lvert\langle\chi_{\text{true}}\lvert\chi^{qe}_{1/3}\rangle\rvert^2$	$\lvert\langle\chi_{\text{true}}\lvert\chi^{qe,L}_{1/3}\rangle\rvert^2$
3	3	1	1
4	11	0.9938	0.9974
5	46	0.9860	0.9934
6	217	0.9882	0.9771
7	1069	0.9659	0.9314
8	5529	0.9353	0.8770

II. SINGLE QUASIELECTRON

The wave function for a quasielectron at the origin is given by

$$\chi^{qe}_{1/3}=\mathcal{P}\begin{vmatrix} z_1^* & z_2^* & z_3^* & \cdots \\ 1 & 1 & 1 & \cdots \\ z_1 & z_2 & z_3 & \cdots \\ z_1^2 & z_2^2 & z_3^2 & \cdots \\ \vdots & \vdots & \vdots & \cdots \\ & & & \cdots \end{vmatrix} \times \prod_{j<k}(z_j-z_k)^2 \exp\left[-\tfrac{1}{4}\sum_j |z_j|^2\right] . \tag{9}$$

This wave function is different from various other wave functions proposed in the literature for the quasielectron. For concreteness, we will compare it with the trial wave function proposed by Laughlin,[5] which we denote by $\chi^{qe,L}_{1/3}$,

$$\chi^{qe,L}_{q/e}=\left[\prod_j \frac{\partial}{\partial z_j}\right]\chi_{1/3} . \tag{10}$$

The derivatives in this state do not act on the exponential factor.

We discuss in Appendix B how we obtain the true Coulomb ground states and in Appendix C how we obtain the lowest LL projection of various trial wave functions. In our calculation of the true Coulomb ground state we employ two types of boundary conditions: (i) "closed" boundary condition, in which we restrict the individual angular momentum of each electron to be less than or equal to the maximum individual angular momentum that an electron attains in the trial state; (ii) "open" boundary condition, in which we make no restriction on the value of the angular momenta of individual electrons. It is clear from our results that both boundary conditions give similar answers. The advantage of the closed boundary condition is that the total number of many-body basis states is much smaller compared to that with open boundary condition, thus making it possible to compute the true Coulomb ground state for larger systems.

In Tables I(a) and I(b) we give the overlaps of both of

these trial states with the Coulomb ground states calculated for the appropriate total angular momenta with open and closed boundary conditions. The largest system studied consists of eight electrons. (For the eight-electron system, the true Coulomb state was obtained by employing a modified Lanczos technique.[8] The initial vector was chosen to be one of the trial state.) Clearly our trial state for the quasielectron has near unity overlap with the true state. Furthermore, it is a better approximation for the true quasielectron state that the Laughlin quasielectron trial state, although both seem to do well. These results are consistent with our assertion that quasielectrons of the FQHE state are closely related to quasielectrons of the IQHE state.

We close this section with the following remarks.

(i) It is intuitively quite clear that in our trial state the quasielectron is created in such a manner that the incompressible $\frac{1}{3}$ state far from the quasielectron position (say, z_0) does not get perturbed. The heuristic argument that leads to this conclusion is that (a) the IQHE state Φ_1^{qe} with one electron in the second LL at z_0 has filling factor $\nu=1$ everywhere except in the vicinity of z_0; (b) therefore, when the Jastrow factor acts on this state, it produces $\frac{1}{3}$ state everywhere except near z_0; (c) the projection operator again disturbs the state only near z_0—it does not affect the $\frac{1}{3}$ state far from z_0 as it is already in the lowest LL. It is not very clear to us why the Laughlin quasielectron trial state should have this crucial property. This may be related to why the Laughlin quasielectron trial state becomes increasingly worse than ours as the system size increases.

(ii) Neglecting the projection operator, our trial state is, by construction, also valid for the $\nu=1$ state. The Laughlin trial state can also be written as the lowest LL projection of some state as follows:

$$\chi_{1/m}^{qe,L}=\mathcal{P}\prod_j z_j^* \chi_{1/m} \ . \tag{11}$$

However, if one removes the projection operator for $m=1$, it can be shown that this state does *not* describe Φ_1^{qe} because it has a logarithmically diverging energy $E_{qe}\sim(\ln N)\hbar\omega_c$, where N is the total number of electrons.

TABLE II. This table shows squares of the overlaps of trial states $[s,t]$ of Eq. (12) with the corresponding true Coulomb states. Disk geometry with closed boundary conditions is used.

N	State	M	(Overlap)^2
5	[5,0]	73	0.976
	[4,1]	46	0.986
	[3,2]	27	0.996
6	[6,0]	338	0.972
	[5,1]	188	0.988
	[4,2]	134	0.982
	[3,3]	88	0.055
7	[7,0]	1658	0.931
	[6,1]	1069	0.966
	[5,2]	663	0.970
	[4,3]	424	0.966
8	[8,0]	8512	0.939
	[7,1]	5529	0.935
	[6,2]	3436	0.974
	[5,3]	2174	0.980
	[4,4]	1430	0.024

III. MANY QUASIELECTRONS

One feature of our scheme is that it is straightforward to write trial wave functions for states with more than one quasielectron; these are simply related to the IQHE states in which the lowest LL is completely filled and there is more than one electron in the higher LL. For reasons given below, we consider only the specific class of trial states given by

$$[s;t]=\mathcal{P}\begin{vmatrix} z_1^* & z_2^* & \cdots \\ z_1^* z_1 & z_2^* z_2 & \cdots \\ \vdots & \vdots & \cdots \\ & & \cdots \\ z_1^* z_1^{t-1} & z_2^* z_2^{t-1} & \cdots \\ 1 & 1 & \cdots \\ z_1 & z_2 & \cdots \\ \vdots & \vdots & \cdots \\ & & \cdots \\ z_1^{s-1} & z_2^{s-1} & \cdots \end{vmatrix} \times\prod_{j<k}(z_j-z_k)^2\exp\left[-\tfrac{1}{4}\sum_j |z_j|^2\right] . \tag{12}$$

The Slater determinant in this state corresponds to the IQHE state in which there are s electrons in the lowest LL, occupying the single-particle state of angular momenta $0,1,\ldots,s-1$, and t electrons ($t\le s$) in the second LL, occupying the states of angular momenta $-1,0,1,\ldots,t-2$ (see Appendix A). [The form of the Slater determinant given in Eq. (12) is obtained after some row manipulation.] In other words, the FQHE trial state in Eq. (12) is related to IQHE state in which both the lowest LL and the higher LL are completely filled up to their respective boundaries. The state $[s;t]$ can be thought of as the $\frac{1}{3}$ state with t quasielectrons.

The reason for the above specific choice is that these trial states are translationally invariant. That is, the substitution $z_j\to z_j-z_0$ alters merely the center-of-mass coordinate (this substitution leaves the polynomial part of the states unchanged, as can be explicitly verified, and changes only the center-of-mass coordinate in the exponential part). For other choices of the Slater determinant, the trial state, in general will *not* be translationally invariant.

We again compare these trial states with the true Coulomb ground states calculated for the appropriate total angular momenta. The overlaps are given in Table II. These results show that the above many quasielectron trial states $[s;t]$ are a very accurate representation of the true states for $t<s$. (The states $[s;t]$ with $s=t$, which

one would like to identify with filling factor $\frac{2}{5}$, seem to do rather poorly; these will be discussed in greater detail in the next section.)

We would like to point out that our results are also consistent with the generalized states in Eq. (5). As mentioned earlier, the basic assertion of the states in Eq. (5) is that if the IQHE state can be determined uniquely, a reasonable approximation for the FQHE state [at a filling factor given by Eq. (6)] can be obtained by multiplication by the Jastrow factor. In the present case the IQHE state if uniquely determined because of the total angular momentum restriction; the IQHE state described by the Slater determinant of $[s,t]$ has the property that it is the unique ground state for $s+t$ electrons with total angular momentum equal to $s(s-1)+t(t-3)$.

To summarize the results of this section, the Jastrow-Slater trial wave functions appear to be valid even for more than one quasiparticle.

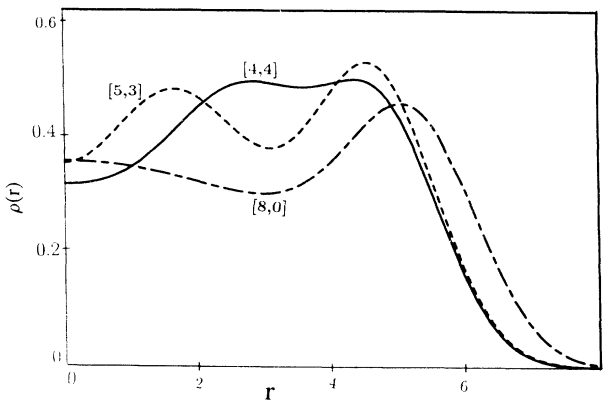

FIG. 1. This figure shows the one-electron densities of various trial states of the type $[s,t]$ [Eq. (12)] where the total number of electrons $(s+t)$ is eight, and the number of quasielectrons is t.

IV. INCOMPRESSIBLE STATES

Equation (1) implies that the trial wave function for the incompressible state at $\nu=\frac{2}{5}$ can be obtained by multiplying the IQHE state Φ_2 by the Jastrow factor $\prod_{j<k}(z_j-z_k)^2$ (and, of course, projecting it onto the lowest LL). Even though there is no truly unambiguous definition of two filled LL's in the disk geometry, it seems natural to define Φ_2 to be the state in which there is an equal number of electrons in the lowest two LL's. With this choice the states [3;3] and [4;4] of the preceding section would describe a system at $\nu=\frac{2}{5}$. As shown in Table II, these have very poor overlaps with the true ground states.

We show below that these poor overlaps are a result of boundary effects, which are overemphasized in our few-electron calculations, and that our trial states indeed describe the *(bulk)* physics of the incompressible FQHE states accurately.

The clue to the importance of boundary effects lies in the observation that the overlaps are rather good for the states of the form $[s,s-1]$ (which also correspond to a filling factor of $\frac{2}{5}$ in the thermodynamic limit). Since $[s,s-1]$ and $[s,s]$ differ only at the boundary, it is reasonable to conclude that boundary effects become important in $[s,s]$. This raises the question of why the existence of a boundary affects only states of the type $[s,s]$ and not the Laughlin state or the state $[s,t]$ with $s<t$. In order to answer this question, we show in Fig. 1 the one-electron density $\rho(r)$ of some trial states of the type $[s,t]$ for an eight-electron system [which is essentially also the $\rho(r)$ of the true Coulomb state for $t<s$]. The $\rho(r)$ of the Laughlin $\frac{1}{3}$ state [8,0] has one peak near the boundary, while the $\rho(r)$ of other states has two peaks. This might have been surprising, but, given our trial states, interpretation of these peaks is quite straightforward: there is one peak corresponding to the boundary of the lowest LL and one to the boundary of the higher LL. It is reasonable to assume that so long as the two peaks are far enough that the energy associated with their Coulomb interaction is small compared to the typical energy differences between the low-lying eigenstates of the system, they do not influence the physics of the true Coulomb ground state in any significant manner. However, as t approaches s, they come closer to each other (just as the boundaries come closer in the related IQHE state) and at some point their interaction starts affecting the ground state significantly. Since they are closest at $t=s$, we expect these boundary effects to be most severe at $\nu=\frac{2}{5}$, which, we believe, is the reason for the failure of our trial states of the type $[s,s]$.

In order to investigate if our trial states adequately describe the *bulk* properties of actual states, we have repeated our calculations for $\chi_{2/5}=\prod_{j<k}(z_j-z_k)^2\Phi_2$ and $\chi_{2/9}=\prod_{j<k}(z_j-z_k)^4\Phi_2$ in the spherical geometry,[6,7,9] in which there are no boundaries. (For completeness, a discussion of wave functions in the spherical geometry is included in Appendix A. For Coulomb matrix elements see Fano, Ortolani, and Colombo.[6]) Another advantage of the spherical geometry is that, unlike in disk geometry, the IQHE state with n filled LL's is uniquely defined, since each LL has only a finite degeneracy. Consequently, the states χ_p are also unambiguously determined.

The first thing to note is that on sphere our $\frac{2}{5}$ and $\frac{2}{9}$ states occur when the lowest LL degeneracy (N_L) is given by (Appendix A)

$$N_L=\tfrac{5}{2}N-3 \tag{13}$$

and

$$N_L=\tfrac{9}{2}N-5\ , \tag{14}$$

respectively, where N is the total number of electrons. This is consistent with the results of the *exact* diagonalization studies of few-particle systems which find cusps in the ground-state energy at these values of N_L.

TABLE III. This table shows squares of the overlaps of our $\frac{2}{5}$ and $\frac{2}{9}$ trial states with the corresponding Coulomb ground state. It also shows the relative energy difference between the trial state and the ground state, defined by $r_E \equiv (E_{tr} - E_0)/E_0$, where E_{tr} is the energy of the trial state and E_0 is the energy of the ground state. Spherical geometry is used. D is the number of independent uniform (zero angular momentum) states. For comparison, we also show some results for the Laughlin $\frac{1}{3}$ state taken from Ref. 6(b).

ν	N	N_L	D	M	(Overlap)^2	r_E (%)
$\frac{2}{5}$	4	7	1	5	1	0
	6	12	3	58	0.999 64	0.0040
	8	17	8	910	0.999 28	0.0055
$\frac{2}{9}$	4	13	2	43	0.999 88	0.0021
$\frac{1}{3}$[a]	5	13	2	73	0.998 12	0.024
	6	16	6	338	0.992 89	0.049
	7	19	10	1656	0.992 73	0.048

[a] Fano *et al.*, Phys. Rev. B **34**, 2670 (1986).

The overlaps of our trial states with the true Coulomb states are shown in Table III. For four electrons, there is only one uniform state at $\nu = \frac{2}{5}$, and the overlap of unity tells us nothing more than that our trial state is uniform. For the six- and eight-electron $\frac{2}{5}$ state and the four-electron $\frac{2}{9}$ state, however, our results are meaningful, since a random uniform trial state would have an overlap of approximately $1/\sqrt{D}$ with the true ground state, where D is the number of independent uniform states. The near-unity overlaps in these cases leave little doubt that our trial states are indeed an excellent approximation of the true Coulomb ground states.[10] The overlaps for our states are better than the overlaps of other trial states at $\frac{2}{5}$ and $\frac{2}{9}$ considered in the literature.[6,11]

It is now clear that the poor overlaps on the disk geometry were due to boundary effects. This shows the importance of using spherical geometry in the study of incompressible FQHE states. For Laughlin states both geometries seem reasonable, even though the overlaps of the Laughlin states are somewhat worse in the disk geometry than in the spherical geometry, presumably again because of boundary effects.

V. THE UNPROJECTED STATES

It is worth pointing out here that in typical experimental situations the interaction energy per electron is comparable to the cyclotron energy, and consequently significant LL mixing must occur in the ground state. Why are our trial states, which are strictly in the lowest LL, valid in such situations? The answer is that the finite-B FQHE state is in the same phase as the infinite-B FQHE state, and therefore a trial state valid in the $B \to \infty$ limit also describes the physics at finite magnetic fields. (Remember that we are considering here only completely spin-polarized ground states. In reality, it is possible that some finite-B FQHE states may not be in the same phase as the infinite-B state due to their different spin configuration.)

We now wish to make the important point that the unprojected trial states (i.e., our trial states without the projection operator $\mathcal{P}$) also provide a valid description of the FQHE. This is very useful since the unprojected states are much simpler to understand and work with, and provide a feel for the physics represented by the projected states. This follows from the two points discussed below.

(i) Recall that the lowest-LL trial states studied in this paper are obtained by simply projecting out the higher-LL component of our highly correlated unprojected states. Clearly, our study of the projected trial states is in essence a test of the validity of the correlations built into the unprojected states, and shows that they contain the essential correlations of the FQHE.

(ii) The unprojected states are not very far from the projected states in the phase diagram. This follows because even the unprojected states are, despite their use of higher-LL wave functions, predominantly in the lowest LL, as shown by Monte Carlo calculations.[12] For example, even though Φ_2 has only half of the electrons in the lowest LL, the unprojected $\frac{2}{5}$ state has approximately 96% electrons in the lowest LL and the unprojected $\frac{2}{9}$ state has 98% electrons in the lowest LL (in the thermodynamic limit). Further, Monte Carlo calculations[12] show that the interaction energy of the unprojected states is less than that of the projected states. For typical experimental parameters, the increase in the kinetic energy is compensated by the reduction in the interaction energy, and the projected and the unprojected states have comparable energies.

We stress that both the projected and the unprojected trial states make a special choice of the occupation of various LL's, but this choice is dictated by where a simple description is possible, and has no experimental significance. In particular, the unprojected states do not become "more correct" than the projected states for the FQHE at relatively small magnetic fields. In other words, the unprojected states *do not* address the physics of LL mixing. We would also like to mention, as should be clear to everybody by now, that our approach does not imply that higher LL's are necessary for the FQHE.

VI. CONCLUSION

In this paper, we have studied the simplest new trial states of Refs. 1 and 2, namely the quasielectron of the $\frac{1}{3}$ state and the incompressible $\frac{2}{5}$ and $\frac{2}{9}$ states, and find that these have very large overlaps with the corresponding true Coulomb states.

In addition to large overlaps, our incompressible states share another property with the Laughlin states: they are the exact ground states of some model Hamiltonians with hard-core interactions.[2] Exploiting this feature, Rezayi and MacDonald[13] have shown, through explicit numerical calculations, that our unprojected trial wave functions are adiabatically connected to the true $B \to \infty$ states.

Finally, our scheme fits various experiments nicely. In particular, it provides a unified framework for the entire phenomenon of the QHE, consistent with the experimental fact that there is no essential distinction between the observations of the IQHE and the FQHE plateaus. Another noteworthy feature or our theory is that it clearly identifies the prominently observed fractions.[1,2] For example, with $m=3$, Eq. (2) yields the FQHE sequence $\frac{1}{3},\frac{2}{5},\frac{3}{7},\frac{4}{9},\ldots$, which are the only (unambiguously) observed fractions in the range $\frac{1}{2} \geq \nu \geq \frac{1}{3}$.

We believe that a combination of all these factors makes a compelling case for our theory of the FQHE.

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APPENDIX A

We provide here a brief discussion of wave functions in both disk and spherical geometry.

1. Disk geometry

The Hamiltonian of an electron in magnetic field

$$H_0=\frac{1}{2m_e}\left[\mathbf{p}+\frac{e}{c}\mathbf{A}\right]^2 \tag{A1}$$

can be solved exactly for the eigenvalues and eigenfunctions. In the circular gauge, $\mathbf{A}=(B/2)(y,-x,0)$, the eigenstates are given by

$$n_{r,q}=(2\pi 2^{q+r}q!r!)^{-1/2}e^{1/4(x^2+y^2)}D^qD^{*r}e^{-(1/2)(x^2+y^2)}, \tag{A2}$$

where $r=0,1,2,\ldots$ is the LL index, $q=-r,-r+1,\ldots,0,1,2,\ldots$ is the angular momentum index, $D=\partial/\partial x+i\partial/\partial y$, and x and y are expressed in units of the magnetic length $l=(\hbar c/eB)^{1/2}$. In particular, the (unnormalized) single-particle states in the lowest LL are given by

$$\eta_{0,q}=z^qe^{-|z|^2/4} \tag{A3}$$

and those in the second LL are given by

$$\eta_{1,q}=z^q(2q+2+zz^*)e^{-|z|^2/4}. \tag{A4}$$

The IQHE wave functions Φ_n, and consequently the FQHE states χ_p, can be constructed with the help of these single-particle states. For example, $\chi_{2/5}$, which is related to Φ_2, is given by

$$\mathcal{P}\begin{vmatrix} z_1^* & z_2^* & \cdots \\ z_1^*z_1 & z_2^*z_2 & \cdots \\ \vdots & \vdots & \cdots \\ & & \cdots \\ z_1^*z_1^{N/2-1} & z_2^*z_2^{N/2-1} & \cdots \\ 1 & 1 & \cdots \\ z_1 & z_2 & \cdots \\ \vdots & \vdots & \cdots \\ & & \cdots \\ z_1^{N/2-1} & z_2^{N/2-1} & \cdots \end{vmatrix} \prod_{j<k}(z_j-z_k)^2\exp\left[-\tfrac{1}{4}\sum_j|z_j|^2\right]. \tag{A5}$$

The Slater determinant in this state corresponds to the state in which there are $N/2$ electrons in the lowest LL, occupying the single-particle states of angular momenta $0,1,\ldots,N/2-1$, and $N/2$ electrons in the second LL, occupying the states of angular momenta $-1,0,1,\ldots,N/2-2$. (The form of the Slater determinant is obtained after some straightforward row manipulation.)

It is clear that without the projection operator $\chi_{2/5}$ is translationally invariant; replacing all z_i by $z_i-\zeta$ leaves the polynomial part of $\chi_{2/5}$ unchanged. The projected state is obtained by replacing all z_i^* by $2\partial/\partial z_i$, with the

understanding that the derivatives do not act on the exponential.[2] It is easy to see that this state is also translationally invariant. Similarly all other states of Eq. (1) can be shown to be translationally invariant.

2. Spherical geometry

In spherical geometry, one considers a magnetic monopole of appropriate strength located at the center of the sphere. The eigenstates of an electron on the surface of the sphere are spherical harmonics, first obtained by Wu and Yang.[9] In the first two LL's, these are given by (without normalization)

$$Y_{q,q,m}=\left[\cos\frac{\theta}{2}\right]^{q+m}\left[\sin\frac{\theta}{2}\right]^{q-m}e^{i(m-q)\phi} \tag{A6}$$

and

$$Y_{q,q+1,m}=\left[\cos\frac{\theta}{2}\right]^{q+m}\left[\sin\frac{\theta}{2}\right]^{q-m}\times\left[-2(q+1)\sin^2\frac{\theta}{2}+(q+1-m)\right]e^{i(m-q)\phi}, \tag{A7}$$

where, for the rth LL ($r=0,1,\ldots$) $m=-(q+r),-(q+r-1),\ldots,(q+r)$. The magnetic field is such that the total number of flux quanta passing through the surface of the sphere is $2q$, which must be an integer according to Dirac's monopole quantization condition.

The degeneracy of the rth LL is $2(q+r)+1$. The state with the lowest LL completely filled has $N=2q+1$ electrons, and can be written in the usual form:

$$\Phi_1=\prod_{i<j}(u_iv_j-v_iu_j)\prod_k e^{-iq\phi_k}, \tag{A8}$$

where $u_j=\cos(\theta_j/2)e^{i\phi_j/2}$, and $v_j=\sin(\theta_j/2)e^{-\phi_j/2}$. In spherical geometry, the trial states of Eq. (1) are given by

$$\chi_p=\mathcal{P}\Phi_1^{m-1}\Phi_n . \tag{A9}$$

The lowest LL degeneracy of χ_p can be read off easily as follows. The q of a state can be obtained by determining the largest power of $e^{-i\phi_j}$ in the wave function. The q of χ_p is thus the sum of the q's of the individual states in the product. The value of q for Φ_n is given by

$$q_n=\frac{N-n^2}{2n}, \tag{A10}$$

where N is the number of electrons. In particular, for Φ_1,

$$q_1=\frac{N-1}{2} . \tag{A11}$$

Therefore, for χ_p we have

$$q_p=(m-1)\frac{N-1}{2}+\frac{N-n^2}{2n} \tag{A12}$$

and the lowest LL degeneracy of χ_p is given by $N_L=2q_p+1$. The particular choices $m=3,n=2$ and $m=5,n=2$ lead to Eqs. (13) and (14) for $\chi_{2/5}$ and $\chi_{2/9}$, respectively.

APPENDIX B

In this appendix we provide the Coulomb matrix elements that we use in our calculations. The interaction Hamiltonian is given by

$$H=\tfrac{1}{2}\sum_{r,s,t,u}a_r^\dagger a_s^\dagger a_u a_t\langle r,s|V|t,u\rangle , \tag{B1}$$

where $V=r^{-1}$, and

$$\langle r,s|V|t,u\rangle=[2^{r+s+t+u}(2\pi)^4 r!s!t!u!]^{-1/2}\int d^2z_1d^2z_2\frac{z_1^{*r}z_2^{*s}z_2^u z_1^t}{|z_1-z_2|}e^{-(|z_1|^2+|z_2|^2)/2} . \tag{B2}$$

This integral can be performed by transforming to the center of mass and relative coordinates. The result is

$$\langle r,s|V|t,u\rangle=\delta_{r+s,t+u}(-1)^{s+u}\frac{\pi r!s!t!u!}{4^{r+s}}\sum_{\alpha=0}^{r}\sum_{\beta=0}^{s}\sum_{\gamma=0}^{t}\sum_{\delta=0}^{u}\delta_{\alpha+\beta,\gamma+\delta}(-1)^{\beta+\delta}\frac{2^{\alpha+\beta-1}(\alpha+\beta)![2(r+s-\alpha-\beta)-1]!!}{\alpha!\beta!\gamma!\delta!(r-\alpha)!(s-\beta)!(t-\gamma)!(u-\delta)!} . \tag{B3}$$

For the Coulomb matrix elements in the spherical geometry, we refer the readers to Fano *et al.*[6]

APPENDIX C

We wish to determine the lowest LL projection of states of the type

$$\Phi[\{z_i\},\{z_i^*\}]\prod_{j<k}(z_j-z_k)^2\exp\left[-\tfrac{1}{4}\sum_i|z_i|^2\right] . \tag{C1}$$

Formally, it can be obtained by making the substitution[2] $z_j^*\to 2\partial/\partial z_j$ (where all z^*'s must be brought to the left of the z's prior to the substitution), with the understanding that the derivatives do not act on the exponential. However, in order to carry out our calculations, we need to determine the coefficient of each term of the type

$$z_1^{m_1}\cdots z_N^{m_N}\exp\left[-\tfrac{1}{4}\sum_i|z_1|^2\right] \tag{C2}$$

(where it is sufficient to consider only the case $m_1<m_2<\cdots<m_N$) in the expansion of this projected state. The coefficient is given by the overlap of this term with either the projected or the unprojected trial state.

There are a number of ways of determining the

coefficient of any such given term. The most straightforward way is to simply expand the polynomial, bring all z^*'s to the left of z's in each term, make the replacement $z^* \rightarrow 2\partial/\partial z$, and carry out the derivatives, possibly using symbolic manipulation computer techniques. However, this allows treatment of only very small systems ($N \leq 4$). The other technique is to calculate the overlap of Eq. (C2) with the unprojected trial state using Monte Carlo techniques. The disadvantage of this method is that it converges extremely slowly and therefore limits the accuracy tremendously. Yet another technique depends on the ability to find a model Hamiltonian for which the unprojected trial state is exact. Then the trial state can be generated and easily projected on to the lowest LL by exact numerical diagonalization of the Hamiltonian. This technique can be applied only to a small subset of states (e.g., $\chi_{2/5}$, see Ref. 10) and is again restricted to a small number of particles. We use the technique described below, which is fairly general and allows us to study systems of up to eight electrons without difficulty.

The terms in the expansion of the polynomial $\prod_{j<k}(z_j - z_k)^2\Phi$ which have a finite overlap with Eq. (C2) are of the form

$$(z_1^{*\alpha_1} \cdots a_N^{*\alpha_N}) z_1^{m_1+\alpha_1} \cdots z_N^{m_N+\alpha_N} \tag{C3}$$

and the overlap is the same as that of the state obtained from the above state with the replacement $z_j^* \rightarrow 2\partial/\partial z_j$ (the derivatives acting only on the polynomial part).

We go though each term of the Slater determinant and determine what term of the Jastrow factor would give us the desired powers. Suppose the term of Φ under consideration is of the form

$$(z_1^{*\alpha_1} \cdots a_N^{*\alpha_N}) z_1^{m_1-i_1+\alpha_1} \cdots z_N^{m_N-i_N+\alpha_N} . \tag{C4}$$

Then we need to determine the coefficient of the term

$$z_1^{i_1} z_2^{i_2} \cdots z_N^{i_N} \tag{C5}$$

in the Jastrow factor $\prod_{j<k}(z_j - z_k)^2$. It is given by

$$\sum_{j_1,\ldots,j_N=1}^{N} P(j_1,\ldots,j_N) P(i_1 - j_1,\ldots,i_N - j_N) , \tag{C6}$$

where $P(j_1,\ldots,j_N)$ is the permutation of $(j_1,\ldots,j_N)$ [i.e., $P(j_1,\ldots,j_N)$ is $+1$ or -1 depending on whether $(j_1,\ldots,j_N)$ is an even or odd permutation of $(1,2,\ldots,N)$, and is zero if any two of the arguments are equal].

This can be straightforwardly extended to states in spherical geometry.

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