Frequency pulling in Josephson radiation

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It is shown that a theoretical result of the so-called pseudo-angular-momentum theory of Josephson tunneling predicting a small correction to the Josephson frequency relation $\omega=2eV/\hbar$ rests on a physically incorrect account of the voltage across the junction and can therefore not be upheld. A previous argument against the prediction is shown to be incorrect.

I. INTRODUCTION

Ever since Josephson published his famous results on two weakly coupled superconductors ("Josephson junction"),[1] there have been numerous investigations into the nature of these effects. From a fundamental point of view, those treatments which deduce the relations from the microscopic interactions within and between the two superconductors are of considerable interest. In particular, they are called for by the fact that the junction frequency relation

$$\Delta\dot{\varphi}=\frac{2eV}{\hbar}\equiv\omega \tag{1}$$

represents one of the most accurate ways to measure the fine-structure constant α,[2] a precise experimental value of which is of fundamental importance for the QED; hence, the question is[3] whether the microscopic theory fully confirms (1), or whether it reveals any corrections to this famous formula.

Indeed, some time ago the so-called pseudo-angular-momentum (PAM) approach to Josephson tunneling has predicted a correction to the relation (1),[4,5] whose magnitude, however, is found to be about 1 order of magnitude below present experimental accuracy. Nevertheless, for the theoretical picture of the effects as well as in light of the ever-increasing ingenuity of the experimenters, it is an interesting question whether this PAM prediction is true—in particular, whether it is vitiated by criticisms that were raised against the PAM several years ago.[6,7] In fact, in Ref. 6 it is claimed that the PAM argument is unfounded.

In this work, we hope to resolve this issue by showing two different things: (a) the argument in Ref. 6 against the PAM prediction is not correct, and (b) the prediction rests on a physically incomplete account of the charge imbalance (and thereby of the voltage) across the Josephson junction and can therefore not be maintained. Both points are basically straightforward consequences of our recent work,[8] to which we refer for background information on our approach to Josephson tunneling. To make the paper self-contained, we give an account of the line of reasoning of the PAM in Sec. II, concentrating on the essentials and trying to be as concise as possible. Our arguments for (a) and (b) are then developed in Sec. III; there, we also show that the microscopic theory does, in fact, predict a shift of the Josephson frequency, which is, however, of a different physical origin and whose magnitude is completely negligible.

II. THE ARGUMENT OF THE PAM

We shall develop the argument of the PAM in a framework (set up in Ref. 8) which differs slightly from the original treatments[4,5] but is completely equivalent to them. The PAM rests on the quasispin description of the strong-coupling BCS model. One has as the basic operators for the left superconductor R,

$$r_{\Lambda}^{\pm}\equiv\frac{1}{|\Lambda|}\sum_{\mathbf{k}\in\Lambda}\sigma_{\mathbf{k}}^{\pm}\ , \tag{2a}$$

$$r_{\Lambda z}\equiv\frac{1}{|\Lambda|}\sum_{\mathbf{k}\in\Lambda}\sigma_{\mathbf{k}z}\ , \tag{2b}$$

where the σ operators are Anderson's quasispin operators which create, annihilate, and count the electrons in pairs $(\mathbf{k},\uparrow;-\mathbf{k},\downarrow)$ and obey spin commutation relations. The $\mathbf{k}$ run over a set Λ containing momenta whose associated energies lie in a finite region around the Fermi level μ_R; their number is denoted by $|\Lambda|$. There is a similar set $s_{\Lambda}^{\pm},s_{\Lambda z}$ for the right superconductor S. (For more information on these definitions, see, e.g., Ref. 7.)

The Hamiltonian of the junction alone is then taken to be[9]

$$H_J=H_T+H_C\ ,$$

where

$$H_T=|\Lambda|\lambda(r_{\Lambda}^{+}s_{\Lambda}^{-}+s_{\Lambda}^{+}r_{\Lambda}^{-}) \tag{3}$$

is the Hamiltonian describing the tunneling of condensed pairs, and

$$H_C=\frac{Q^2}{2C}=|\Lambda|2Kz_{\Lambda}^2 \tag{4}$$

with

$$z_{\Lambda}\equiv\tfrac{1}{2}(r_{\Lambda z}-s_{\Lambda z}),\quad K=e^2\frac{|\Lambda|}{C}\ , \tag{5}$$

the electrostatic energy due to the capacity C of the junction. The operator z_{Λ} is interpreted to measure the (density of) the charge imbalance between R and S (it is

$\Delta N = N_R - N_S = 4|\Lambda| z_\Lambda$), and accordingly the PAM sets (z denotes the expectation value of z_Λ)

$$V = \frac{Q}{C} = \frac{2e}{C}|\Lambda| z = \frac{2K}{e} z \tag{6}$$

for the total voltage (dc and ac) across the junction. The PAM does not include the free BCS Hamiltonians of R,

$$H^R_{\mathrm{BCS}} = |\Lambda| [\varepsilon_R (2 r_{\Lambda z} + 1) - g r^+_\Lambda r^-_\Lambda] \tag{7}$$

and, similarly, of S into the description, which has been criticized in Ref. 8 but shall be followed in this section for simplicity. We shall come back to this point in Sec. III.

Of crucial importance for the frequency pulling is the idea that there is a coupling of the Josephson junction to the electromagnetic field in the junction cavity, which is represented for simplicity by a single mode $a^\#$, with $[a, a^*] = 1$. The coupling Hamiltonian can be written as[9]

$$H_I = (-i)|\Lambda|^{1/2} T (a^* s^+_\Lambda r^-_\Lambda - a r^+_\Lambda s^-_\Lambda) \tag{8}$$

and describes the process where a pair tunnels from R to S emitting a photon, and vice versa (assuming $V^{\mathrm{dc}} \sim \Delta N > 0$). The Hamiltonian of the total system is then

$$H = H_T + H_C + H_I + H_F , \tag{9}$$

$H_F = \hbar \Omega a^* a$ being the free Hamiltonian of the field.

It is then straightforward to calculate the Heisenberg equations of motion $[\hbar (d/dt) r_{\Lambda z} = i[H, r_{\Lambda z}]$, etc.]:

$$\hbar \frac{d}{dt} r_{\Lambda z} = \frac{\hbar}{2e} j_\Lambda + \frac{\hbar}{2e} j_{F\Lambda} , \tag{10a}$$

$$\hbar \frac{d}{dt} s_{\Lambda z} = -\frac{\hbar}{2e} j_\Lambda - \frac{\hbar}{2e} j_{F\Lambda} , \tag{10b}$$

$$\hbar \frac{d}{dt} r^+_\Lambda = -2i\lambda s^+_\Lambda r_{\Lambda z} + iK (r_{\Lambda z} - s_{\Lambda z}) r^+_\Lambda - 2T a^*_\Lambda s^+_\Lambda r_{\Lambda z} = \hbar \left[\frac{d}{dt} r^-_\Lambda \right]^* , \tag{10c}$$

$$\hbar \frac{d}{dt} s^+_\Lambda = -2i\lambda r^+_\Lambda s_{\Lambda z} - iK (r_{\Lambda z} - s_{\Lambda z}) s^+_\Lambda + 2T a_\Lambda r^+_\Lambda s_{\Lambda z} = \hbar \left[\frac{d}{dt} s^-_\Lambda \right]^* , \tag{10d}$$

$$\hbar \frac{d}{dt} a_\Lambda = \hbar (-\gamma - i\Omega) a_\Lambda - T s^+_\Lambda r^-_\Lambda . \tag{10e}$$

Here, $a_\Lambda \equiv a / |\Lambda|^{1/2}$, and we have introduced the operators

$$j_\Lambda \equiv \frac{2e}{\hbar} (-i) \lambda (r^+_\Lambda s^-_\Lambda - s^+_\Lambda r^-_\Lambda) \tag{11}$$

for the Josephson tunnel current and

$$j_{F\Lambda} \equiv \frac{2e}{\hbar} T (a r^+_\Lambda s^-_\Lambda + a^* s^+_\Lambda r^-_\Lambda) \tag{12}$$

for the dc Fiske current.[10] Furthermore, a linear cavity loss λ has been included, which can be accounted for with a reservoir model.[4,5,9]

Taking $|\mathrm{BCS}\rangle \otimes |\alpha\rangle$ as the initial state, where $|\alpha\rangle$ is some photon state and $|\mathrm{BCS}\rangle \equiv |\varphi_R\rangle \otimes |\varphi_S\rangle$ is the BCS ground state (see Ref. 7), and assuming $|\Lambda|$ large enough (e.g., $|\Lambda| \sim 10^{10}$), one can go over from (10) to the corresponding equations for expectation values with only a very small error;[8] we denote these expectation values by dropping the Λ index and by $\alpha(t)$ for a_Λ. Under these conditions, we have

$$r_z + s_z = 0, \quad r^+ r^- = s^+ s^- ,$$

for all times, and the system (10) can be expressed in the physical variables $\alpha(t)$ and (see Ref. 8)

$$c(t) \equiv \sqrt{r^+(t) r^-(t)} = \sqrt{s^+(t) s^-(t)} ,$$

Cooper-pair density

$$z(t) \equiv \tfrac{1}{2}[r_z(t) - s_z(t)] ,$$

difference of particle densities (within Λ) (13)

$$\Delta\varphi(t) \equiv \varphi_R(t) - \varphi_S(t) ,$$

phase difference between R and S .

We then have as the equations of motion describing the macroscopic behavior of the Josephson junction:

$$[c(t) + z^2(t)] = 0 , \tag{14a}$$

$$2e\dot{z}(t) = j(t) + j_F(t) - j_{\mathrm{ex}}(t) , \tag{14b}$$

$$\Delta\dot{\varphi}(t) = \frac{4K}{\hbar} z(t) - \frac{4\lambda}{\hbar} \cos\Delta\varphi(t) z(t) + i \frac{2T}{\hbar} [\alpha^*(t) e^{-i\Delta\varphi(t)} - \alpha(t) e^{i\Delta\varphi(t)}] z(t) , \tag{14c}$$

$$\dot{\alpha}(t) = (-\gamma - i\Omega) \alpha(t) - \frac{T}{\hbar} c(t) e^{-i\Delta\varphi(t)} . \tag{14d}$$

In (14), we have included a term $-j_{\mathrm{ex}}$ accounting for the external dc current which is carried by the wires to and from the junction; again, this term can be deduced from a reservoir ansatz.[4,5,9] The current (densities) j, j_F can be expressed with (13) as

$$j(t) = \frac{4e}{\hbar} \lambda c(t) \sin\Delta\varphi(t) , \tag{15a}$$

$$j_F(t) = \frac{2e}{\hbar} T c(t) [\alpha^*(t) e^{-i\Delta\varphi(t)} + \alpha(t) e^{i\Delta\varphi(t)}] . \tag{15b}$$

In Refs. 4 and 5 an even simpler system is used: setting $c(t) \approx \mathrm{const} = c_0$ (which is a good approximation[8]) and throwing out those terms which oscillate in the ac effect (their time mean is zero), one obtains

$$2e\dot{z}(t) = j_F(t) - j_{\mathrm{ex}}(t) , \tag{16a}$$

$$\Delta\dot{\varphi}(t) = \frac{4K}{\hbar} z(t) + i \frac{2T}{\hbar} [\alpha^*(t) e^{-i\Delta\varphi(t)} - \alpha(t) e^{i\Delta\varphi(t)}] z(t) , \tag{16b}$$

$$\dot{\alpha}(t) = (-\gamma - i\Omega) \alpha(t) - \frac{T}{\hbar} c_0 e^{-i\Delta\varphi(t)} . \tag{16c}$$

In the steady state, $\dot{z}(t) = 0$ (compare Ref. 7), so that (6) becomes

$$z = \text{const} = \frac{eV^{\text{dc}}}{2K} , \tag{17}$$

with this, we get, for (16b),

$$\Delta\dot{\varphi} = \left[1 + i\frac{T}{2K}(\alpha^* e^{-i\Delta\varphi} - \alpha e^{i\Delta\varphi}) \right] \frac{2eV}{\hbar} , \tag{18}$$

so that there is clearly a correction to the classical Josephson relation (1). With the ansatz $\Delta\varphi(t) = \nu t$, $\alpha(t) = \alpha_0 e^{-i\nu t}$, it is straightforward to calculate the corrected frequency to be

$$\nu = \frac{\gamma\omega - \Omega\Gamma}{\gamma - \Gamma} \approx \omega + \Gamma \quad [\text{if } (\omega - \Omega) \approx 1/\gamma,\ \Gamma \ll \gamma] \tag{19}$$

with

$$\Gamma = \frac{1}{e} j_F c_0 z . \tag{20}$$

The quantity Γ depends mainly on the film thickness x of the superconductors R and S ($\Gamma \sim 1/x^2$), and is typically (for $x \approx 1000$ Å) on the order of 5×10^4 Hz at $V = 10^{-2}$ V, thus the relative error $\Gamma/\omega \approx 10^{-9}$; this accuracy seems not to have been attained yet in the experiment (Ref. 11 quotes 2×10^{-8} as the best present value of accuracy, obtained at $V \approx 10^{-2}$ V).

III. DISCUSSION OF THE PAM ARGUMENT

In Ref. 6 it is claimed that the whole reasoning of the PAM sketched above is not well founded since the expectation value of the fundamental operator z_Λ, z, describing the charge imbalance, vanishes if taken in the BCS ground state $|\text{BCS}\rangle$. Hence, there could be no frequency pulling ($z = 0 \overset{(20)}{\Longrightarrow} \Gamma = 0$). But this argument obviously assumes that $|\text{BCS}\rangle$ is the state of the junction *for all times* since, in other states, z can, of course, be $\neq 0$; however, $|\text{BCS}\rangle$ is not invariant under the dynamics ($e^{iH_T t}$), so that the argument is not conclusive. Thus, a more thorough investigation seems to be in order. In Ref. 8 we have presented a somewhat detailed discussion of this problem, i.e., the status of the operator z_Λ. The result was that Ref. 6 (and, referring thereto, Ref. 7) does point to an important problem in that z cannot measure *permanent* charge imbalances and, hence, cannot be related to the dc voltage across the junction as done in (6) and explicitly in (17). The main reason is that z is a measure for the deviation from equilibrium in the two superconductors, and it is physically not correct to assume a permanent such deviation. The dc voltage should rather be related to the difference of the chemical potentials $\Delta\mu = \mu_R - \mu_S$: $V^{\text{dc}} = (1/e)\Delta\mu$. However, z does account for the fast ac voltages in the system [$V^{\text{ac}} = (2K/e)z$], so that it is *not* identically zero. This means that (6) needs to be replaced by

$$V = V^{\text{dc}} + V^{\text{ac}} = \frac{1}{e}(\Delta\mu + 2Kz) , \tag{21}$$

leading to equations of motion[8]

$$(c + z^2)^{\cdot} = 0 , \tag{22a}$$

$$2e\dot{z} = j + j_F - j_{\text{ex}} , \tag{22b}$$

$$\Delta\dot{\varphi} = \frac{2eV^{\text{dc}}}{\hbar} + \frac{4K}{\hbar}z - \frac{4\lambda}{\hbar}\cos\Delta\varphi z + i\frac{2T}{\hbar}(\alpha^* e^{-i\Delta\varphi} - \alpha e^{i\Delta\varphi})z , \tag{22c}$$

$$\dot{\alpha} = (-\gamma - i\Omega)\alpha - \frac{T}{\hbar}c_0 e^{-i\Delta\varphi} , \tag{22d}$$

instead of (14). If one now follows the same procedure as before and throws out the oscillating terms, $z(t)$ is one of them, one gets, instead of (16),

$$0 = j_F(t) - j_{\text{ex}}(t) , \tag{23a}$$

$$\Delta\dot{\varphi}(t) = \frac{2eV^{\text{dc}}}{\hbar} , \tag{23b}$$

$$\dot{\alpha}(t) = (-\gamma - i\Omega)\alpha(t) - \frac{T}{\hbar}c_0 e^{-i\Delta\varphi(t)} , \tag{23c}$$

so that the old relation (1) is reestablished. Thus, we see that the incorrect interpretation, (6) and (17), of z is the reason for the frequency pulling as predicted by the PAM; it vanishes if one uses the improved relation (21).

It is interesting to note that one can turn the argument around, observing that (6) is, in fact, experimentally disproven: the relation (17) means that R and S are individually far away from equilibrium, which makes it imperative to include the full BCS Hamiltonians of R and S into the description (compare Ref. 8). This, however, gives

$$\Delta\dot{\varphi} = \frac{2eV^{\text{dc}}}{\hbar} \left[1 + \frac{g}{K} + i\frac{T}{2K}(\cdots) \right] \tag{24}$$

instead of (18), i.e., a correction of $g/K \approx 10^{-2}$ to the Josephson relation. This clearly has not been found in the experiment.

Finally, we remark that (23b) is, in fact, not entirely exact; the term $(4\lambda/\hbar)\cos\Delta\varphi(t)z(t)$ in (22c) oscillates, but with a nonzero time mean: setting $j_F = j_{\text{ex}}$, $c \approx c_0$, and making the perturbation ansatz $\Delta\varphi(T) = (\omega + \Delta\omega)t = \nu t$, we get, in (22b),

$$\dot{z}(t) = \frac{2\lambda}{\hbar}c_0 \sin\nu t \Longrightarrow z(t) = -\frac{2\lambda}{\hbar\nu}c_0 \cos\nu t$$

and therefore, in (23b),

$$\Delta\dot{\varphi}(t) = \omega - \frac{4\lambda}{\hbar}\left[\frac{2\lambda}{\hbar\nu}\right] c_0 \cos^2\nu t \overset{!}{=} \nu .$$

With the time mean of $\cos^2\nu t$ being $\frac{1}{2}$, one easily calculates ($c_0 = \frac{1}{4}$)

$$\frac{\Delta\omega}{\omega} \approx \left[\frac{2\lambda}{\omega\hbar}\right]^2 = \left[\frac{\lambda}{eV}\right]^2 .$$

Typically, $\lambda = 10^{-9}$ eV,[8] so that, at $V = 10^{-2}$, we have a correction of the Josephson frequency on the order of 10^{-14}, which is completely negligible.[12]

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