Mutual friction in superfluid ^{3}He: Effects of bound states in the vortex core

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The motion of singular quantized vortex lines in superfluid ^{3}He is considered for the A and B phases. Mutual friction is calculated within a microscopic quantum-mechanical Green's-function formalism, valid for dynamical processes. This enables us to include all the different physical phenomena in a unified approach. We consider axisymmetric vortices for temperatures considerably lower than T_c. In this regime, the main contribution to the force exerted on a moving vortex line originates from the localized Fermi excitations occupying quantized energy eigenstates in the vortex core. These ^{3}He quasiparticle states are similar to the quantized motion of charge in a magnetic field; thus vortex motion in ^{3}He resembles the Hall phenomenon in metals. The outcome is that the viscous drag cannot simply be expressed through the cross sections for ^{3}He quasiparticles scattering off the vortex, but is rather due to the mutual interactions between the localized quasiparticles and the normal excitations. Our calculations conform with the experimental values for the mutual-friction parameters. We also discuss vortex oscillations, and predict that strong dissipation should be observed at a resonant frequency of about 10 kHz, owing to transitions between the bound-state energy levels. This effect could be used for detecting and measuring the quantization of the quasiparticle bound-state spectrum in the vortex-core matter for superfluid ^{3}He.

I. INTRODUCTION

Physical phenomena determining the origin of mutual friction in quantum liquids are among the most fundamental features underlying superfluidity, since they involve intrinsic dynamical processes in superfluid systems. Mutual friction, i.e., the interaction between the superfluid and normal components of the liquid during their relative motion, is caused by the motion of quantized vortices[1] due to the Magnus force—owing to the superflow—and the force produced by the normal excitations. This problem has been studied in detail for superfluid ^{4}He (He II; see, for example, recent reviews,[2,3] and references therein). Vortex motion in superconductors has also been discussed extensively.[4,5] Mutual friction for these two "conventional" superfluid systems is now understood, at least qualitatively. In He II, the force exerted on a vortex by the normal component is due to normal excitations scattered off the moving vortex; it can be expressed through the corresponding scattering cross sections for the quasiparticles. However, vortex motion in superconductors is a viscous flow because the mean free path of the quasiparticles is considerably shorter than that in He II, in comparison with the vortex-core size.

Quantized superfluid vortices in ^{3}He are of great current interest, in particular, due to the many different experiments in the rotating state.[6] The first mutual-friction measurements have been done for both the B phase[7] and the A phase;[8–10] attempts have also been made towards Andronikashvili-type measurements of mutual friction in superfluid ^{3}He.[11] Theoretically, mutual friction has been considered for continuous vortices in the A phase.[12,13] It was suggested that vortices and the normal component interact through the orbital Cross-Anderson viscosity,[14] owing to the temporal variations of the local $\mathbf{l}$ texture produced by the moving vortex. Physical processes involved in the motion of singular A- and B-phase vortices are much more complicated. One possibility is, as for He II, that the force exerted by the excitations is governed by the quasiparticle scattering off the vortex, since the mean-free path in ^{3}He is larger than the vortex-core size. However, the physical situation should be more similar to that in superconductors, since the superfluid states in ^{3}He represent generalized BCS pairing. In particular, the coherence length ξ, which is associated with the hard vortex-core structure, is of the same order as for the BCS superconductors and is considerably longer than that in He II.

In this paper, we consider the motion of singular vortices using an approach based on the microscopic theory of nonstationary processes. The same method has been employed earlier for vortex motion in superconductors.[15,16] This scheme can be applied to both the scattering-dominated processes and the viscous flow of vortices. We consider axisymmetric vortices at temperatures considerably lower than the critical temperature, T_c. In this case, the main contribution to the force originates from quasiparticles localized in the vortex core and occupying quantized energy eigenstates with definite angular-momentum projections on the vortex axis. These quasiparticle states are similar to charge transport in a magnetic field; consequently, vortex motion is analogous to the Hall phenomenon in metals.

The relation between the components of the mutual-friction force parallel and perpendicular to the vortex velocity, $\mathbf{v}_L$, depends—as for the Hall effect—on the product of the quasiparticle mean-free time and the energy-level spacing. The parameters for superfluid ^{3}He at low temperatures are such that the vortices move principally with the superflow at a small angle, α, with respect to

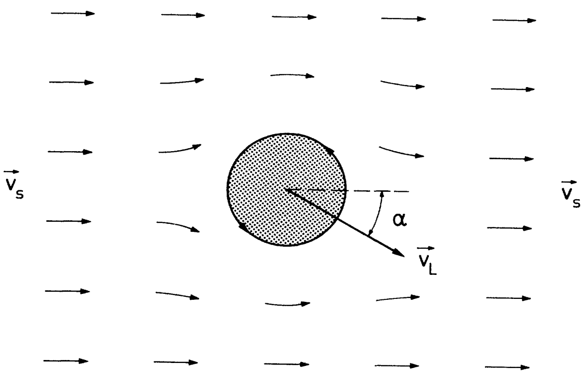

FIG. 1. Schematic illustration for a moving quantized vortex line in superfluid ^{3}He. The superflow velocity at infinity is denoted by $\mathbf{v}_s$; $\mathbf{v}_L$ is the vortex velocity. The "Hall angle" $\alpha_H \equiv \pi/2 - \alpha$, where α is the angle extended between the vectors $\mathbf{v}_s$ and $\mathbf{v}_L$.

the superfluid velocity, see Fig. 1. The transverse component of the vortex velocity, $(v_L)_\perp$, is determined by the viscous drag, which, however, cannot be expressed through cross sections for the ^{3}He quasiparticle scattering by the vortex, but is due to the mutual interactions between the localized quasiparticles and the normal excitations. Our results for the mutual-friction parameters agree in their order of magnitude with the experimental observations.[7–10]

We also discuss effects associated with vortex oscillations. We show that an additional dissipation emerges in the superfluid when the oscillation frequency is at resonance with the transition frequency between the energy levels (at about several tens of kHz). This effect can be used for an experimental investigation of the bound-state spectrum in the vortex-core matter.

Section II discusses the general expression for the forces acting on a moving quantized vortex in terms of the microscopic dynamical properties of the superfluid system. In Sec. III, we consider vortex motion in the collisionless (ballistic) regime for normal quasiparticles. In Sec. IV, we study the motion of singular vortices at low temperatures, taking into account interactions between the quantized Fermi quasiparticles localized in the vortex core and the normal continuum excitations (the scattering states). The processes of quasiparticle relaxation are modeled through an effective-relaxation-time (τ) approximation. The structures of the bound states are discussed in Sec. V for several examples of vortices in the two phases, A and B. We consider the results obtained for the forces in Sec. VI; in Sec. VII we estimate effects of vortex oscillations. Results are summarized in Sec. VIII.

II. GENERAL EXPRESSIONS

Vortex motion can be formulated in terms of the balance of forces acting on an isolated vortex:

$$\mathbf{F}^{(M)} + \mathbf{F}^{(\mathrm{exc})} = \mathbf{0}\ , \tag{1}$$

where $\mathbf{F}^{(M)}$ is the Magnus force exerted by the superflow $\mathbf{v}_s$, and $\mathbf{F}^{(\mathrm{exc})}$ is the force produced by the normal component. All the forces are per unit length of the vortex.

Here we only consider axisymmetric vortices (for the A phase, in particular, this means that the anisotropy vector $\boldsymbol{l}$ far from the vortex core should be parallel to the vortex axis). In this case, the Magnus force in Eq. (1) may be represented as

$$\mathbf{F}^{(M)} = \rho_s(\mathbf{v}_s - \mathbf{v}_L) \times \boldsymbol{\kappa}\ . \tag{2}$$

Here ρ_s and $\mathbf{v}_s$ are the superfluid density and velocity at distances far from the vortex core, $\mathbf{v}_L$ is the vortex velocity, and $\boldsymbol{\kappa}$ the circulation vector; for ^{3}He, $\kappa = \pi N/m$, where N is the number of circulation quanta and m the mass of a ^{3}He atom.

The force produced by the normal excitations is

$$\mathbf{F}^{(\mathrm{exc})} = \mathcal{D}\,(\mathbf{v}_n - \mathbf{v}_L) + \mathcal{D}'\hat{\boldsymbol{\kappa}} \times (\mathbf{v}_n - \mathbf{v}_L)\ , \tag{3}$$

where $\mathbf{v}_n$ is the velocity of the normal flow, and $\hat{\boldsymbol{\kappa}}$ is the unit vector along $\boldsymbol{\kappa}$. The task is to calculate the coefficients $\mathcal{D}$ and $\mathcal{D}'$ associated with mutual friction.

Calculation of the separate contributions to the force $\mathbf{F}^{(\mathrm{exc})}$ produce difficulties in just how to account correctly for all the physical phenomena involved.[3] In this paper, we use a unified approach developed earlier for similar situations in superconductors.[4,5,15,16] It provides a general expression, which connects the mass flow far from the vortex core—the so-called transport current $\mathbf{j}_s = \rho_s\mathbf{v}_s$—and the vortex velocity, $\mathbf{v}_L$, in terms of the fully microscopic dynamical characteristics of the superfluid system described by Green's functions. This approach has been described in detail;[15] it can be applied almost directly to superfluid ^{3}He in the same way as for superconductors. The result is the balance equation (1), where the force produced by the normal excitations is expressible as

$$\mathbf{F}^{(\mathrm{exc})} \cdot \mathbf{d} = \frac{1}{L}\int d^3r \int \frac{d\varepsilon}{8\pi i}\,\mathrm{Tr}\{\mathcal{H}_d(\mathbf{r})\ [\mathcal{G}^{(n)}_\varepsilon(\mathbf{r},\mathbf{r}) - \mathcal{G}^{(a)}_\varepsilon(\mathbf{r},\mathbf{r})]\}\ . \tag{4}$$

Here $\mathbf{d}$ is an arbitrary constant vector, and L denotes the length of the vortex. The Green's function $\mathcal{G}_\varepsilon$ and the "Hamiltonian" $\mathcal{H}$ are matrices both in the Nambu and in the spin spaces:

$$\mathcal{G}_\varepsilon \equiv \begin{pmatrix} \hat{G} & \hat{F} \\ -\hat{F}^\dagger & \hat{\bar{G}} \end{pmatrix}\ ;\ \ \mathcal{H} \equiv \begin{pmatrix} \hat{0} & -\hat{\Delta}_{\mathbf{p}} \\ \hat{\Delta}^\dagger_{\mathbf{p}} & \hat{0} \end{pmatrix}\ . \tag{5}$$

We denote such matrices with the capital script symbols [e.g., $\mathcal{G}$, $\mathcal{H}$ (however, the symbols $\mathcal{D}$ and $\mathcal{D}'$ are reserved for the coefficients associated with mutual friction)], and also by the check notation (e.g., $\check{1}$, $\check{\Sigma}$), while a caret $\hat{}$ means a matrix in the spin space only; Tr stands for a trace taken over both the Nambu and spin indices. The Green's functions $\mathcal{G}^{(n)}_\varepsilon$ and $\mathcal{G}^{(a)}_\varepsilon$ are defined in the frame of reference where the normal component is at rest as follows:

$$\mathcal{G}^{(n)}_\varepsilon(\mathbf{r}_1,\mathbf{r}_2) = -\frac{1}{2T}\cosh^{-2}\left(\frac{\varepsilon}{2T}\right)\ [\mathbf{v}_L\cdot\mathbf{p}^{+(2)}\mathcal{G}^R_\varepsilon(\mathbf{r}_1,\mathbf{r}_2) - \mathbf{v}_L\cdot\mathbf{p}^{(1)}\mathcal{G}^A_\varepsilon(\mathbf{r}_1,\mathbf{r}_2)]\ , \tag{6}$$

where $\mathbf{p}^{(1)} = -i\boldsymbol{\nabla}^{(1)}$, $\mathbf{p}^{+(2)} = +i\boldsymbol{\nabla}^{(2)}$, and $\mathcal{G}^{R(A)}$ denotes the retarded (advanced) Green's function. According to (Refs. 16 and 17) the anomalous Green's function is

$$\mathcal{G}_\varepsilon^{(a)}(\mathbf{r}_1,\mathbf{r}_2) = \frac{i}{2T}\cosh^{-2}\left(\frac{\varepsilon}{2T}\right)\int \mathcal{G}_\varepsilon^R(\mathbf{r}_1,\mathbf{r})\mathcal{H}_v(\mathbf{r})\mathcal{G}_\varepsilon^A(\mathbf{r},\mathbf{r}_2)\,d^3r + \int\int \mathcal{G}_\varepsilon^R(\mathbf{r}_1,\mathbf{r}')\,\check{\Sigma}_\varepsilon^{(a)}(\mathbf{r}',\mathbf{r}'')\mathcal{G}_\varepsilon^A(\mathbf{r}'',\mathbf{r}_2)\,d^3r'\,d^3r'' . \tag{7}$$

Here $\check{\Sigma}_\varepsilon^{(a)}$ is the self-energy matrix due to quasiparticle collisions. In Eqs. (4) and (7), we use the notations

$$\mathcal{H}_d = \mathbf{d}\cdot\boldsymbol{\nabla}\mathcal{H},\quad \mathcal{H}_v = \mathbf{v}_L\cdot\boldsymbol{\nabla}\mathcal{H} . \tag{8}$$

To derive Eq. (4), a Galilean transformation was made[15] for the regular Green's functions $\mathcal{G}_\varepsilon^{R\,(A)}$. This can also be done in the presence of quasiparticle collisions with each other, since these processes do not violate Galilean invariance.

The Green's function $\mathcal{G}_\varepsilon^{(n)}$ determines the normal density through the normal mass current in the rest frame of the vortex:

$$\begin{aligned}\mathbf{j}_n &\equiv -\rho_n\mathbf{v}_L\\ &= \int\frac{d\varepsilon}{16\pi i}\mathrm{Tr}[(\mathbf{p}^{(1)}+\mathbf{p}^{+(2)})\,\tau_3\,\mathcal{G}_\varepsilon^{(n)}(\mathbf{r}_1,\mathbf{r}_2)]_{\mathbf{r}_1=\mathbf{r}_2} ,\end{aligned} \tag{9}$$

where τ_3 is the Pauli matrix in the Nambu space. The superfluid density is $\rho_s = \rho - \rho_n$, with ρ denoting the total density of the liquid.

Equations (6) and (7) are written in the frame of reference where the normal component is at rest; therefore, we put $\mathbf{v}_n = 0$ in Eq. (3).

The balance equations (1) to (3), together with Eqs. (6) and (7), can as well be obtained by calculating the derivative of the thermodynamic potential. This has been performed also in the calculation[18] of the forces acting on the moving superfluid ^{3}He A-B interface.

III. VORTEX MOTION IN THE BALLISTIC REGIME

In the collisionless regime, where one can neglect quasiparticle relaxation, Eq. (4) for the force $\mathbf{F}^{(\mathrm{exc})}$ can be transformed into an expression that explicitly contains only the contributions from the scattering states. This can be done in a way similar to that for superconductors.[15] Here we briefly summarize the essentials of the principle.

The retarded and advanced Green's functions satisfy the equation

$$\mathcal{G}^{-1}\mathcal{G}_\varepsilon^{R\,(A)}(\mathbf{r}_1,\mathbf{r}_2) = \check{1}\,\delta(\mathbf{r}_1-\mathbf{r}_2) , \tag{10}$$

where

$$\mathcal{G}^{-1} = -\left(\frac{\nabla^2}{2m}+E_F\right)\check{1} - \varepsilon\,\tau_3 + \mathcal{H} . \tag{11}$$

From the identity $\mathcal{H}_v = \mathbf{v}_L\cdot\boldsymbol{\nabla}\mathcal{G}^{-1}$ and Eq. (10), one can evaluate expressions such as

$$\int d^3r\,\mathcal{G}_\varepsilon^R(\mathbf{r}_1,\mathbf{r})\mathcal{H}_v(\mathbf{r})\mathcal{G}^A(\mathbf{r},\mathbf{r}_2)$$

using integration by parts; we find[16]

$$\begin{aligned}&\frac{1}{2L}\mathrm{Tr}\int[\mathcal{H}_d\,\mathcal{G}_\varepsilon^{(a)}(\mathbf{r},\mathbf{r})]\,d^3r\\ &\quad=\frac{1}{2L}\mathrm{Tr}\int[\mathcal{H}_d\,\mathcal{G}_\varepsilon^{(n)}(\mathbf{r},\mathbf{r})]\,d^3r\;-4\pi i\,f^{(\mathrm{sc})}(\varepsilon) .\end{aligned} \tag{12}$$

Here $f^{(\mathrm{sc})}(\varepsilon)$ is the spectral density of the force produced by quasiparticles scattered off the vortex:

$$\mathbf{d}\cdot\mathbf{F}^{(\mathrm{exc})} = \int_{-\infty}^{+\infty} f^{(\mathrm{sc})}(\varepsilon)\,d\varepsilon , \tag{13}$$

where

$$\begin{aligned}f^{(\mathrm{sc})}(\varepsilon) = -\frac{1}{16\pi T}\cosh^{-2}\left(\frac{\varepsilon}{2T}\right)\int dS_i^{(1)}\int dS_k^{(2)}\mathrm{Tr}\Bigg[&\frac{p_k^{(2)}}{2m}(\mathbf{v}_L\cdot\mathbf{p}^{(2)})\mathcal{G}_\varepsilon^A(2,1)\frac{p_i^{(1)}}{2m}(\mathbf{d}\cdot\mathbf{p}^{(1)})\mathcal{G}_\varepsilon^R(1,2)\\ &+(\mathbf{v}_L\cdot\mathbf{p}^{(2)})\frac{p_i^{+(1)}}{2m}\mathcal{G}_\varepsilon^A(2,1)(\mathbf{d}\cdot\mathbf{p}^{(1)})\frac{p_k^{+(2)}}{2m}\mathcal{G}_\varepsilon^R(1,2)\\ &+\left(\frac{p_k^{(2)}p_i^{+(1)}}{(2m)^2}(\mathbf{v}_L\cdot\mathbf{p}^{(2)})\mathcal{G}_\varepsilon^A(2,1)\right)(\mathbf{d}\cdot\mathbf{p}^{(1)})\mathcal{G}_\varepsilon^R(1,2)\\ &+(\mathbf{v}_L\cdot\mathbf{p}^{(2)})\mathcal{G}_\varepsilon^A(2,1)\left(\frac{p_i^{(1)}p_k^{+(2)}}{(2m)^2}(\mathbf{d}\cdot\mathbf{p}^{(1)})\mathcal{G}_\varepsilon^R(1,2)\right)\Bigg] ;\end{aligned} \tag{14}$$

the integrations over $d\mathbf{S}^{(1)}$ and $d\mathbf{S}^{(2)}$ extend over remote cylindrical surfaces surrounding the vortex, with the surface $S^{(1)}$ enclosed by the surface $S^{(2)}$.

We see that the terms with $\mathcal{G}_\varepsilon^{(n)}$ drop out of the force $\mathbf{F}^{(\text{exc})}$, and there only remains the contribution from the scattering states. Indeed, the spectral density $f^{(\text{sc})}(\varepsilon)$ only contains the quasiparticle states that belong to the continuum spectrum, whose wave functions are finite at large distances from the vortex.

The spectral density $f^{(\text{sc})}(\varepsilon)$ can be expressed in terms of the transport (σ_{tr}) and transverse ($\sigma_\perp$) scattering cross sections. Here we will not present the corresponding calculations but only give the estimates indicating when the collisionless limit is realized. The cross sections σ_{tr} and $\sigma_\perp$ are determined by the differences in the scattering phase shifts for partial waves with azimuthal momenta n and $n \pm 1$: $\delta_n - \delta_{n\pm 1}$. Quasiparticle collisions can be neglected when the differences $\delta_n - \delta_{n\pm1}$ are considerably larger than the quasiparticle decay at distances on the order of the vortex-core size, R: i.e., when $\partial\delta_n/\partial n \gg R/\ell$. The change in the phase shifts δ_n is of order unity when the impact parameter $b = n/p_\perp$ varies on the scale of the vortex-core size, R; this yields

$$\frac{\partial \delta}{\partial n} \sim \frac{1}{p_F R} . \tag{15}$$

Therefore, the collisionless limit is realized when the quasiparticle mean free path ℓ satisfies the condition of a strong inequality:

$$\ell \gg (p_F R) R . \tag{16}$$

For singular vortices with core sizes $R \sim \xi_0 \sim v_F/T_c$, the above condition becomes

$$\ell \gg \xi_0 \left(\frac{E_F}{T_c}\right) . \tag{17}$$

The quasiparticle mean free path in superfluid ^{3}He is[19]

$$\ell(T) \sim \frac{\rho}{\rho_n} \ell_n(T_c) , \tag{18}$$

where $\ell_n(T_c) \sim \xi_0(E_F/T_c)$ is the mean free path in the normal state at T_c. One sees that Eq. (17) can only be satisfied at low temperatures, $T \ll T_c$, where $\rho_n \ll \rho$. For $T \sim T_c$, the scattering of quasiparticles by a vortex is substantially affected by quasiparticle relaxation due to mutual collisions with each other. Therefore, the force $\mathbf{F}^{(\text{exc})}$ cannot be expressed in terms of the scattering cross sections.

The collisionless condition in Eq. (16) becomes more restrictive for the motion of continuous vortices and vortices with core sizes considerably larger than ξ_0; it can be satisfied only at very low temperatures. At $T \sim T_c$, however, the motion of such vortices is "viscous flow" and corresponds to the "dirty limit" which can be treated hydrodynamically.[3]

We also estimate the friction coefficient $\mathcal{D}$ in Eq. (3) in the collisionless limit $\tau \to \infty$ for a singular vortex; $\mathcal{D}$ is proportional to the transport cross section. For example, in He II:[20]

$$\mathcal{D} = -\frac{1}{2} \int \left(\frac{\partial n_0}{\partial \varepsilon}\right) p_\perp^2 \, v_G \, \sigma_{\text{tr}}(\mathbf{p}) \frac{d^3 p}{(2\pi)^3} ,$$

where v_G is the group velocity. The transport cross section may be estimated as

$$\sigma_{\text{tr}} \sim \frac{1}{q} \sum_n \left(\frac{\partial \delta}{\partial n}\right)^2 .$$

According to Eq. (15), $\sigma_{\text{tr}} \sim 1/(p_F^2 \xi_0)$; this gives

$$\mathcal{D}^{(\text{sc})} \sim \frac{\kappa \rho_n}{p_F \xi_0} . \tag{19}$$

These estimates are comparable to those made for superconductors.[15]

To summarize, we conclude that for temperatures of order T_c, the mean free path for quasiparticles is not long enough to allow neglecting the quasiparticle relaxation and only treating the scattering of excitations off a moving vortex. However, for low enough temperatures where the concept of ballistic quasiparticles is justified, the number of normal excitations is exceedingly low, and they contribute very little to the drag force acting on the vortex. As we shall see later, the main contribution originates from the scattering of normal excitations by quasiparticles localized in the vortex core and thus moving together with the vortex. This scenario is discussed in the following section.

IV. SINGULAR VORTICES AT LOW TEMPERATURES

In this section we calculate the force $\mathbf{F}^{(\text{exc})}$ acting on a singular vortex at low temperatures, taking into account the finite relaxation time for quasiparticles. The mean free time, τ, in ^{3}He is much longer than T^{-1}; this will help us to simplify the ensuing calculations considerably.

A. Relaxation-time approximation

In order to take into account the quasiparticle relaxation processes exactly, one has to calculate the corresponding self-energies for quasiparticle collisions and to solve the resulting equations for the Green's functions in the case of the specific order-parameter distributions within the vortices of interest. Such a task seems to be quite difficult in its full generality, and can hardly be treated analytically. Therefore, here we begin with a considerably simplified approach, which in a sense is equivalent to an effective relaxation-time (τ) approximation.

In this approach, the mean free time, τ, for ^{3}He quasiparticles is introduced through the substitution $\varepsilon \to \varepsilon \pm i/2\tau$ in the retarded (advanced) Green's functions, respectively, the latter term with $\check{\Sigma}_\varepsilon^{(a)}$ in Eq. (7) being omitted. Naturally, this approach is not quite accurate; however, it seems physically reasonable and is believed to describe at least qualitatively correctly the main features of the problem in the limit of rare quasiparticle collisions, $\tau T \gg 1$. For superconductors, this has been demonstrated explicitly.[16] Note that the presence of quasiparticle collisions does not affect the derivation of Eq. (4) in the limit $\tau T \gg 1$.

At low temperatures, the main contribution to $\mathbf{F}^{(\mathrm{exc})}$ arises from the bound states in the vortex core with energies $E \ll T_c$. The quasiparticle states for an axisymmetric vortex can be specified[21] by the quantum numbers (k, n, s), where k is the momentum along the vortex axis (the z axis), n is the "azimuthal" quantum number, and s includes the radial quantum number and the quantum number describing the spin states of the ^{3}He quasiparticles.

We expand the regular Green's functions in terms of the Bogoliubov quasiparticle wave functions:

$$\mathcal{G}_\varepsilon^{R\,(A)}(\mathbf{r}_1, \mathbf{r}_2) = \sum_{n,s} \int \frac{dk}{2\pi} \frac{\mathcal{U}_n^{(s,k)}(\mathbf{r}_1)\mathcal{U}_n^{\dagger(s,k)}(\mathbf{r}_2)}{E_n^{(s,k)} - \varepsilon \mp i/2\tau} . \tag{20}$$

Here we separate the azimuthal quantum number n, which plays an important role in what follows. Above, the Bogoliubov wave functions $\mathcal{U}$ are written in the Nambu space, i.e.,

$$\mathcal{U} = \begin{pmatrix} \hat{u} \\ -\hat{v} \end{pmatrix}, \quad \mathcal{U}^\dagger = (\hat{u}^\dagger, \hat{v}^\dagger) . \tag{21}$$

Within the τ approximation, the wave functions $\mathcal{U}$ satisfy the collisionless equation

$$\left[-\left(\frac{\nabla^2}{2m} + E_F\right)\check{1} + \mathcal{H}\right] \mathcal{U}_n^{(s,k)} = E_n^{(s,k)} \tau_3 \mathcal{U}_n^{(s,k)} , \tag{22}$$

and they also obey the orthogonality conditions

$$\sum_{n,s} \int \frac{dk}{2\pi} \tau_3 \mathcal{U}_n^{(s,k)}(\mathbf{r}_1)\mathcal{U}_n^{\dagger(s,k)}(\mathbf{r}_2) = \check{1}\,\delta(\mathbf{r}_1 - \mathbf{r}_2) \tag{23}$$

and

$$\mathrm{Tr} \int d^3r\, \mathcal{U}_n^{\dagger(s,k)}(\mathbf{r})\, \tau_3 \mathcal{U}_m^{(s',k')}(\mathbf{r}) = \delta_{s,s'}\delta_{n,m} 2\pi\delta(k - k') . \tag{24}$$

The eigenfunctions of Eq. (22) have the form

$$\mathcal{U}_n^{(s,k)}(\mathbf{r}) = e^{ikz + in\varphi}\, \mathcal{W}_n^{(k,s)} \equiv e^{ikz+in\varphi} \begin{pmatrix} \hat{u}_n \\ -\hat{v}_n \end{pmatrix} , \tag{25}$$

for an axisymmetric vortex .

B. Force due to excitations

Let us start with calculating the first term in Eq. (4). Using Eqs. (6) and (20), we find

$$\begin{aligned}\frac{1}{L}\int d^3r \int \frac{d\varepsilon}{8\pi i} \mathrm{Tr}\,[\mathcal{H}_d(\mathbf{r})\mathcal{G}_\varepsilon^{(n)}(\mathbf{r},\mathbf{r})] = -\frac{1}{L}\int \frac{d\varepsilon}{16\pi i T}\cosh^{-2}\left(\frac{\varepsilon}{2T}\right) \\ \times \sum_{n,s}\int \frac{dk}{2\pi}\left(\frac{\langle(\mathbf{d}\cdot\nabla\mathcal{H})\mathcal{U}_n^{(s,k)}(\mathbf{r})\,[i\mathbf{v}_L\cdot\nabla\mathcal{U}_n^{\dagger(s,k)}(\mathbf{r})]\rangle}{E_n^{(s,k)} - \varepsilon - i/2\tau} \right. \\ \left. - \frac{\langle[-i\mathbf{v}_L\cdot\nabla\mathcal{U}_n^{(s,k)}(\mathbf{r})]\mathcal{U}_n^{\dagger(s,k)}(\mathbf{r})(\mathbf{d}\cdot\nabla\mathcal{H})\rangle}{E_n^{(s,k)} - \varepsilon + i/2\tau}\right),\end{aligned} \tag{26}$$

where

$$\langle\cdots\rangle \equiv \mathrm{Tr}\int d^3r(\cdots) .$$

In what follows, we need large values for the azimuthal number, i.e, $n \gg 1$. In this case, we obtain

$$\mathbf{v}_L\cdot\nabla\mathcal{U}_n^{(s,k)} = e^{ikz}\left[e^{i(n+1)\varphi}v_-\left(\frac{\partial\mathcal{W}_n}{\partial\rho} - \frac{n}{\rho}\mathcal{W}_n\right) + e^{i(n-1)\varphi}v_+\left(\frac{\partial\mathcal{W}_n}{\partial\rho} + \frac{n}{\rho}\mathcal{W}_n\right)\right] , \tag{27}$$

and a similar expression for $\mathcal{U}^\dagger$ above:

$$v_\pm \equiv v_{Lx} \pm i v_{Ly} . \tag{28}$$

The wave functions $\mathcal{W}_n$ are products of the rapidly varying (over distances on the order of p_F^{-1}) wave functions of the normal state, i.e., the Hankel functions $H_n^{(1)}(q\rho)$ and $H_n^{(2)}(q\rho)$, where

$$q^2 = p_F^2 - k^2 , \tag{29}$$

and slow functions varying over distances of order ξ_0 and containing the order parameter. To the leading approximation, the derivative in $\partial\mathcal{W}_n/\partial\rho$ is only taken of the functions $H_n^{(1,2)}(q\rho)$. According to the recurrence relations for the Hankel functions, one can put

$$\frac{\partial\mathcal{W}_n^{(s,k)}}{\partial\rho} \pm \frac{n}{\rho}\mathcal{W}_n^{(s,k)} = \pm q\,\mathcal{W}_{n\mp 1}^{(s,k)} , \tag{30}$$

since the changes of the slow functions in $\mathcal{W}_n$, produced by the substitutions $n \to n\pm 1$, are small. Using Eq. (30), we get

$$\mathbf{v}_L\cdot\nabla\mathcal{U}_n = qv_+\mathcal{U}_{n-1} - qv_-\mathcal{U}_{n+1} , \tag{31}$$

Another useful relation is the identity

$$\langle \mathcal{U}_m^{\dagger(s',k')}(\mathbf{d}\cdot\nabla\mathcal{H})\mathcal{U}_n^{(s,k)}\rangle = (E_n^{(s,k)} - E_m^{(s',k')})\langle \mathcal{U}_m^{\dagger(s',k')}\tau_3\,\mathbf{d}\cdot\nabla\mathcal{U}_n^{(s,k)}\rangle\ , \tag{32}$$

which follows from Eq. (22).

Since $\tau^{-1} \ll T \sim \varepsilon$, one may employ

$$\frac{1}{x-i\delta} - \frac{1}{x+i\delta} = 2\pi i\delta(x)\ ,$$

for the ε integration in Eq. (26), after which we now find

$$-\int \frac{q^2}{32\pi T}\,dk \sum_{n,s}\left(\frac{\partial E_n^{(s,k)}}{\partial n}\right) \times \cosh^{-2}\left(\frac{E_n^{(s,k)}}{2T}\right)[\mathbf{v}_L \times \hat{\kappa}]\cdot\mathbf{d}\ . \tag{33}$$

Here we have used Eqs. (24), (31), and (32).

Next we calculate the second term in Eq. (4). Within our approximation for $\mathcal{G}^{(a)}$, we have

$$\frac{1}{L}\int d^3r \int \frac{d\varepsilon}{8\pi i}\mathrm{Tr}[\mathcal{H}_d\,\mathcal{G}_\varepsilon^{(a)}(\mathbf{r},\mathbf{r})] = \int \frac{d\varepsilon}{16\pi T}\cosh^{-2}\left(\frac{\varepsilon}{2T}\right) \times \sum_{n,m,s,s'}\int\int \frac{dk\,dk'}{(2\pi)^2}\frac{\langle \mathcal{U}_m^{\dagger(s',k')}(\mathbf{d}\cdot\nabla\mathcal{H})\mathcal{U}_n^{(s,k)}\rangle\langle \mathcal{U}_n^{\dagger(s,k)}(\mathbf{v}_L\cdot\nabla\mathcal{H})\mathcal{U}_m^{(s',k')}\rangle}{(E_n^{(s,k)}-\varepsilon-i/2\tau)(E_m^{(s,k)}-\varepsilon+i/2\tau)}\ . \tag{34}$$

We transform this expression in a similar way using Eqs. (24), (31), and (32); in this case the result is

$$+i\sum_{n,s}\int \frac{q^2}{16\pi T}\,dk\,\cosh^{-2}\left(\frac{E_n^{(s,k)}}{2T}\right)\left(\frac{\partial E_n^{(s,k)}}{\partial n}\right)^2\left(\frac{d_-v_+}{\partial E_n^{(s,k)}/\partial n + i/\tau} - \frac{d_+v_-}{\partial E_n^{(s,k)}/\partial n - i/\tau}\right)\ . \tag{35}$$

Finally, the coefficients $\mathcal{D}$ and $\mathcal{D}'$ in Eq. (1) are obtained by combining Eqs. (33) and (35); we find

$$\mathcal{D} = m\kappa\int_0^{p_F}\frac{q^2}{16\pi^2 T}\,dk\sum_{n,s}\cosh^{-2}\left(\frac{E_n^{(s,k)}}{2T}\right)\frac{\tau(\partial E_n^{(s,k)}/\partial n)^2}{\tau^2(\partial E_n^{(s,k)}/\partial n)^2+1}\ , \tag{36}$$

$$\mathcal{D}' = -m\kappa\int_0^{p_F}\frac{q^2}{16\pi^2 T}\,dk\sum_{n,s}\cosh^{-2}\left(\frac{E_n^{(s,k)}}{2T}\right)\frac{\partial E_n^{(s,k)}/\partial n}{\tau^2(\partial E_n^{(s,k)}/\partial n)^2+1}\ . \tag{37}$$

Until now we have in no way exploited any specific property of the bound-state spectrum for the ^{3}He quasiparticles. The examples on the singular ^{3}He vortices, to be considered in the following section, show that the properties of the spectrum are similar to those for superconductors.[22] For a given ^{3}He quasiparticle spin state, there exists a branch characterized by only one value of the radial quantum number whose levels densely fill the energy interval from the negative to the positive gap energy, the separation between the levels being considerably less than the energy gap. The low-energy bound states possess an equidistant spectrum with respect to the azimuthal quantum number n. In what follows, we consider vortices that have low-energy excitation spectra with general properties such as those described above. We assume that the separation between the levels,

$$\frac{\partial E_n^{(s,k)}}{\partial n} = -\omega_0^{(s)}(q)\ , \tag{38}$$

is independent of n. In accordance with the results obtained in the following section for several examples of singular vortices, we assume also that $\omega_0^{(s)} > 0$ for vortices with the circulation $\boldsymbol{\kappa}$ along the positive $\hat{z}$ axis. Performing the integrations over n, instead of summations in Eqs. (36) and (37), we derive

$$\mathcal{D} = \kappa\rho\int_0^{\pi/2}\frac{3}{4}\sin^3\theta\,d\theta\sum_s\frac{\omega_0^{(s)}(\theta)\tau}{\omega_0^{(s)}(\theta)^2\tau^2+1}\ , \tag{39}$$

$$\mathcal{D}' = \kappa\rho\int_0^{\pi/2}\frac{3}{4}\sin^3\theta\,d\theta\sum_s\frac{1}{\omega_0^{(s)}(\theta)^2\tau^2+1}\ , \tag{40}$$

where θ is the angle between the quasiparticle momentum and the vortex axis: $q = p_F\sin\theta$. Now the quantum number s describes the two spin states of a quasiparticle and assumes two values, since there is only one value for the radial quantum number.

Before discussing the results in Eqs. (39) and (40), it is instructive to consider several examples of the bound-state spectra for both the A- and B-phase vortices. We turn to this in the following section.

V. BOUND STATES IN VORTEX CORES

Different types of vortices in superfluid ^{3}He have been classified according to their discrete internal symmetries, both in the B phase[23] and in the A phase;[24] here we consider only two of the simplest examples.

A. The most symmetric A-phase vortex

The most symmetric possible vortex in the A phase has two simultaneous orbital components in the order parameter.[24] We write

$$\hat{\Delta}_{\mathbf{p}} = i\hat{\sigma}^{(\alpha)}\hat{\sigma}^{(2)}A_{\alpha i}\hat{p}_i \ , \tag{41}$$

and put $A_{\alpha i} = d_\alpha a_i$, where d_α is a real unit vector in the spin space; it is assumed constant. Above, $\hat{\sigma}$ is the Pauli matrix in the spin space, and $\hat{p}_i = -i\nabla_i/p_F$. Here we consider a vortex that has the orbital part of the order parameter in the form

$$\mathbf{a} = a_1(\rho)\hat{\mathbf{e}}_\rho - ia_2(\rho)\hat{\mathbf{e}}_\varphi \ , \tag{42}$$

where $\hat{\mathbf{e}}_\rho$ and $\hat{\mathbf{e}}_\varphi$ are unit vectors in the cylindrical coordinate frame $(\rho,\ \varphi,\ z)$ with the z axis along the vortex core. At the center of the vortex $a_1 = a_2 = 0$. For large distances from the core $a_1 = a_2 = a = \text{const}$, which yields $\mathbf{a} = a(\hat{\mathbf{e}}_x - i\hat{\mathbf{e}}_y)\exp(i\varphi)$. This corresponds to a singly quantized vortex with $\boldsymbol{\kappa}$ along the positive z axis (i.e., $n = 0$, $m = 1$ in the notation of Ref. 24). The anisotropy vector $\boldsymbol{l}$ is fixed along $-\hat{\mathbf{z}}$.

In order to solve the Bogoliubov equations (22), let us express $\hat{u}_n$ and $\hat{v}_n$ in Eq. (25) as

$$\hat{u}_n = \hat{s}u_n, \quad \hat{v}_n = -i\hat{\sigma}^{\star(\nu)}\hat{\sigma}^{\star(2)}\hat{s}\, d_\nu v_n \ , \tag{43}$$

where $\hat{s}$ is an arbitrary normalized constant spinor. [Note that the above v_n (a superfluid ^{3}He coherence factor) is to be distinguished from $\mathbf{v}_n$ (the normal-fluid velocity).] There are two independent spinors according to the two projections for the quantum numbers. The azimuthal quantum number n assumes integer values for this type of A-phase vortex.

With Eq. (43), the spin part of the wave functions is separated. The orbital wave functions u_n and v_n only depend on ρ and satisfy the coupled set of equations

$$\frac{\partial^2 u_n}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial u_n}{\partial\rho} + \left(q^2 - \frac{n^2}{\rho^2}\right)u_n + \frac{2im}{p_F}\left(a_1\frac{\partial v_n}{\partial\rho} + \frac{a_2 n}{\rho}v_n\right) = -2mE\,u_n \ , \tag{44}$$

$$\frac{\partial^2 v_n}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial v_n}{\partial\rho} + \left(q^2 - \frac{n^2}{\rho^2}\right)v_n - \frac{2im}{p_F}\left(a_1\frac{\partial u_n}{\partial\rho} - \frac{a_2 n}{\rho}u_n\right) = +2mE\,v_n \ . \tag{45}$$

We consider states with $n \ll p_F\xi_0$ and $E \ll T_c$. Eqs. (44) and (45) can be solved in a way similar to that used[22] in superconductors. For distances $\rho \gg n/q$, the wave functions are

$$\begin{pmatrix} u_n \\ -v_n \end{pmatrix} = H_n^{(1)}(q\rho)\begin{pmatrix} f_1 \\ -g_1 \end{pmatrix} + H_n^{(2)}(q\rho)\begin{pmatrix} f_2 \\ -g_2 \end{pmatrix} , \tag{46}$$

where $f_{1,2}$ and $g_{1,2}$ are slow functions varying over distances of order ξ_0. They satisfy the coupled equations

$$\frac{\partial f}{\partial\rho} + \frac{ima_1}{p_F}g = \mp\left(\frac{ma_2 n}{qp_F\rho}g - \frac{imE}{q}f\right) , \tag{47}$$

$$\frac{\partial g}{\partial\rho} - \frac{ima_1}{p_F}f = \mp\left(\frac{ma_2 n}{qp_F\rho}f + \frac{imE}{q}g\right) . \tag{48}$$

The upper sign is for f_1 and g_1, while the lower sign is for f_2 and g_2.

The regular solutions, decreasing for $\rho \to \infty$, are

$$\begin{pmatrix} f_{1,2} \\ -g_{1,2} \end{pmatrix} = A_{1,2}\begin{pmatrix} e^{-\chi\pm i\psi} \\ ie^{-\chi\mp i\psi} \end{pmatrix} , \tag{49}$$

where

$$\chi(\rho) = \frac{m}{p_F}\int_0^\rho a_1\, d\rho \tag{50}$$

and

$$\psi = -\frac{m}{q}e^{2\chi(\rho)}\int_\rho^\infty e^{-2\chi(\rho')}\left(E + \frac{na_2}{p_F\rho'}\right)d\rho' \ . \tag{51}$$

The solutions in Eq. (46) must be regular for $\rho \to 0$. This requires $A_1 = A_2$ and $\psi(\rho = 0) = 0$. The latter condition implies $E_n = -\omega_0 n$, where

$$\omega_0 = \int_0^\infty \frac{a_2}{p_F\rho}e^{-2\chi(\rho)}\, d\rho \Big/ \int_0^\infty e^{-2\chi(\rho)}\, d\rho \ . \tag{52}$$

For this specific example of a vortex, the interlevel spacing ω_0 does not depend on the quasiparticle momentum k or on the spin state s. Moreover, there exists the zero-energy level, "zero mode", with $n = 0$.

In this case, Eqs. (39) and (40) become

$$\mathcal{D} = \kappa\rho\frac{\omega_0\tau}{\omega_0^2\tau^2 + 1}, \quad \mathcal{D}' = \kappa\rho\frac{1}{\omega_0^2\tau^2 + 1} \ . \tag{53}$$

Here we have considered a singly quantized vortex with $\boldsymbol{l}$ far from the core directed along $-\hat{\mathbf{z}}$. The vortex with the $\boldsymbol{l}$ orientation along $+\hat{\mathbf{z}}$ is also possible (this corresponds to $n = m+1$ in the notation of Ref. 24). For such a vortex, the separation between the levels $\omega_0^{(s)}(q)$ would not equal that in Eq. (52), and one would obtain values for $\mathcal{D}$ and $\mathcal{D}'$ different from those in Eq. (53). This is similar to the so-called "internal Magnus effect,"[25] suggested recently for ion motion in superfluid ^{3}He-A: the longitudinal and transverse components of the force $\mathbf{F}^{(\text{exc})}$ depend on the relative orientations of the vectors $\boldsymbol{l}$ and $\boldsymbol{\kappa}$.

B. The o vortex in the B phase

The most symmetric possible vortex in the B phase, the so-called o vortex,[23] has five coupled order-parameter components. Its structure was first calculated in the Ginzburg-Landau temperature regime by Ohmi, Tsuneto, and Fujita,[26] The set of Bogoliubov equations (22) contains four equations for the functions u_α and v_α. We shall not solve this general set of equations in the present paper. To get an overall understanding of the bound-state spectrum, we shall use a simpler ansatz for the order parameter in the vortex, i.e.:[27,28]

$$A_{\alpha i} = A(\rho) e^{i\varphi} \delta_{\alpha i} \ . \tag{54}$$

According to numerical calculations of Ref. 26, this approximation may be considered accurate to about 10%. With the unitary order parameter of Eq. (54), the spin degrees of the wave functions can again be separated. As in Eq. (43), we insert

$$\hat{u}_n = \hat{s} u_n, \ \ \hat{v}_n = -i\hat{\sigma}^{\star(\nu)}\hat{\sigma}^{\star(2)}\,\hat{s}\,\hat{p}_\nu v_n \ . \tag{55}$$

The functions u_n and v_n satisfy

$$(\nabla^2 + p_F^2) u_n - 2mA(\rho) e^{+i\varphi} v_n = -2mE\, u_n \ , \tag{56}$$

$$(\nabla^2 + p_F^2) v_n + 2mA(\rho) e^{-i\varphi} u_n = +2mE\, v_n \ , \tag{57}$$

where n now assumes half-integer values.

This set of equations is exactly the same as for vortices in superconductors. Using the results of Ref. 22, one immediately gets

$$E_n = -n\omega_0(q) \ , \tag{58}$$

where

$$\omega_0(q) = \int_0^\infty \frac{A(\rho)}{q\rho} e^{-2K(\rho)}\, d\rho \Bigg/ \int_0^\infty e^{-2K(\rho)}\, d\rho \tag{59}$$

and

$$K(\rho) = \frac{m}{q} \int_0^\rho A(\rho)\, d\rho \ . \tag{60}$$

The approximate result of Eqs. (58)–(60) for the low-energy excitation spectrum is in qualitative agreement with the quasiclassical calculations for the o vortex.[27] It also agrees with the statement of Refs. 21 and 28 that the energy spectrum for the o vortex does not possess anomalous zero-energy branches.

VI. DISCUSSION

Equations (39) and (40) for the coefficients $\mathcal{D}$ and $\mathcal{D}'$ in the force exerted on the moving vortex by normal excitations are similar to the corresponding expressions for the Ohmic and Hall components of conductivity for a normal metal with the simple electronic spectrum $\varepsilon = p^2/2m$. It is well known (see, for example, Kittel[29]) that in this case the conductivity tensor has the Drude form

$$\sigma_\parallel = \sigma_n \frac{1}{\omega_c^2\tau^2 + 1}, \ \ \sigma_\perp = \sigma_n \frac{\omega_c \tau}{\omega_c^2\tau^2 + 1} \ , \tag{61}$$

where $\sigma_n = ne^2\tau/m$ is the conductivity at zero magnetic field, and $\omega_c = eH/(mc)$ is the cyclotron frequency. In the dirty limit ($\omega_c\tau \ll 1$), electrons diffuse along the applied force, i.e., the electric field, and in the collisionless limit ($\omega_c\tau \gg 1$), they move perpendicular to the force. This behavior resembles our picture for vortex motion in superfluid ^{3}He.

This similarity results from the fact that a quasiparticle state on an energy level in the vortex core and possessing definite angular momentum resembles the motion of a charge along an orbit in an external magnetic field; this is illustrated in Fig. 2. Due to collisions with the normal excitations, the localized quasiparticles transfer momentum from the moving vortex to the normal component, thus producing the mutual friction. One may compare the force-balance equation (1), where $\mathbf{F}^{(M)}$ and $\mathbf{F}^{(\mathrm{exc})}$ are defined through Eqs. (2) and (3), with the expression for the electric current in the presence of the Hall effect, i.e., $j_i = \sigma_{ik} E_k$. Noting that the electric field is given by $\mathbf{E} = \frac{1}{c}[\mathbf{H} \times \mathbf{v}_L]$, where $\mathbf{v}_L$ now denotes the center-of-orbit drift velocity for the charge, we derive a correspondence between the coefficients $\mathcal{D}$ and $\mathcal{D}'$ and the components $\sigma_\parallel$ and $\sigma_\perp$ of the conductivity tensor:

$$\frac{\mathcal{D}}{\kappa} \to \frac{m\sigma_\parallel H}{ec} \ , \ \ \rho_s - \frac{\mathcal{D}'}{\kappa} \to \frac{m\sigma_\perp H}{ec} \ . \tag{62}$$

Hence, using Eq. (61), and also the fact that $\rho_s \to \rho$ for low temperatures, one immediately obtains Eqs. (53), but now with ω_0 instead of the cyclotron frequency ω_c. This comparison helps one to understand qualitatively the role that the bound states play on vortex motion in superfluid ^{3}He. The similarity to the Hall effect in metals has been exploited earlier[30] as the basis for simple phenomenological considerations on vortex motion in pure superconductors.

In a general situation, the vortex will move at an angle α with respect to the superflow velocity, $\mathbf{v}_s$, see Fig. 1. The angle is determined by

$$\tan\alpha = \frac{(v_L)_\perp}{(v_L)_\parallel} = \frac{\mathcal{D}}{\kappa\rho - \mathcal{D}'} \ , \tag{63}$$

where $(v_L)_\perp$ and $(v_L)_\parallel$ are the components of the vortex velocity perpendicular and parallel to the superflow velocity $\mathbf{v}_s$, respectively. The dissipation produced by one vortex is

$$\mathbf{F}^{(M)} \cdot \mathbf{v}_L = \rho_s [\mathbf{v}_s \times \boldsymbol{\kappa}] \cdot \mathbf{v}_L = \mathcal{D}(\mathbf{v}_L)^2 \ . \tag{64}$$

Equations (39) and (40) suggest that the force acting on the localized quasiparticles and, therefore, on the vortex depends strongly on the parameter

$$\omega_0^{(s)}\tau \sim \tau\frac{T_c^2}{E_F} \ . \tag{65}$$

If this parameter is small (frequent collisions), $\mathcal{D}' = \kappa\rho$ and $\mathcal{D} \sim \kappa\rho\omega_0\tau$. Since $\rho_s \to \rho$ at low enough temperatures, the balance equation (1) becomes

$$\rho\mathbf{v}_s \times \boldsymbol{\kappa} = \mathcal{D}\mathbf{v}_L \ . \tag{66}$$

The motion of the vortex would be a viscous flow along the external force $\rho\mathbf{v}_s \times \boldsymbol{\kappa}$, i.e., perpendicular to the superflow, with a velocity determined by the friction $\mathcal{D}$; the angle $\alpha \approx \pi/2$.

However, the mean free time in superfluid ^{3}He is quite long. For temperatures $T \sim T_c$, the parameter (65) is of order unity according to Eq. (18) and the situation is intermediate. Moreover, for lower temperatures, with $T \ll T_c$ (and, practically, starting already at $T \approx 0.5T_c$),

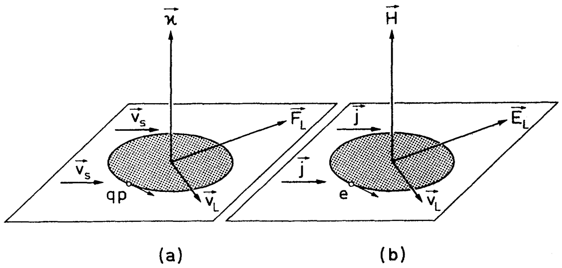

FIG. 2. Analogy between (a) the quasiparticle state qp at the localized energy levels in the vortex core and (b) charge e motion along an orbit in a magnetic field. Both have definite angular momenta and nearly equidistant energy levels. The force $\mathbf{F}_L = \rho_s[\boldsymbol{\kappa} \times \mathbf{v}_L]$ exerted by the moving vortex on the fluid is analogous to the electric field $\mathbf{E} = \frac{1}{c}[\mathbf{H}\times\mathbf{v}_L]$ expressed through the center-of-orbit drift velocity, denoted here also as $\mathbf{v}_L$.

the parameter $\omega_0\tau$ becomes large. At these temperatures, both $\mathcal{D}$ and $\mathcal{D}'$ are small in comparison with $\kappa\rho$:

$$\mathcal{D} \sim \frac{\kappa\rho}{\omega_0\tau}, \quad \mathcal{D}' \sim \frac{\kappa\rho}{\omega_0^2\tau^2} . \tag{67}$$

The vortex moves mainly with the superflow and perpendicular to the external force $\rho\mathbf{v}_s \times \boldsymbol{\kappa}$. The angle α and the dissipation given by Eq. (64) are small.

It is instructive to compare in the clean limit the friction coefficient due to the bound states, $\mathcal{D}$, with the friction $\mathcal{D}^{(\mathrm{sc})}$ produced by the excitations scattered off the vortex. According to Eqs. (67) and (18), the former is

$$\mathcal{D} \sim \frac{\kappa\rho_n}{\omega_0\tau_n(T_c)} \sim \kappa\rho_n , \tag{68}$$

where $\tau_n(T_c) \sim E_F/T_c^2$ is the mean free time in the normal state at $T = T_c$. This contribution is considerably larger than the friction from Eq. (19), produced by the scattering quasiparticles. Therefore, even in the ballistic regime at low temperatures, the friction coefficient is governed by the interaction of bound quasiparticles with the normal excitations, rather than by scattering of normal excitations by the vortex.

The coefficients $\mathcal{D}$ and $\mathcal{D}'$ define the mutual-friction parameters B and B', first introduced by Hall and Vinen.[1] Due to the large normal viscosity of ^{3}He, one has[3]

$$B - iB' = \frac{2\rho_n}{\kappa\rho_n\rho_s}\left(\frac{1}{\mathcal{D}+i\mathcal{D}'} - \frac{1}{i\rho_s\kappa}\right) . \tag{69}$$

Since, for low temperatures, $\mathcal{D}$ and $\mathcal{D}'$ are small in comparison with $\kappa\rho_s$, we find from Eqs. (68) and (69):

$$B = \frac{2\mathcal{D}}{\kappa\rho_n} \sim 1 . \tag{70}$$

This estimate for the mutual-friction coefficient B agrees in its order-of-magnitude with the experimental values.[7–10]

Using Eqs. (39) and (40), one can even draw some conclusions about the motion of continuous vortices and also other vortices having core sizes R considerably larger than ξ_0. For the latter vortices, the distance between the bound-state energy levels tends to be very small and the "clean-limit" condition, $\ell \gg v_F/\omega_0$, is again shifted down to very low temperatures, but even for $T \ll T_c$ the friction $\mathcal{D}$ should still be dominated by the bound states in the vortex core. Therefore, for temperatures that are not extremely low, the motion of vortices with large core sizes exhibit viscous flow.

VII. FREQUENCY-DEPENDENT RESPONSE

Heretofore, we have assumed that the superflow $\mathbf{v}_s$ near the vortex and its velocity $\mathbf{v}_L$ are time independent. One can also calculate the force exerted by the normal component on a vortex whose velocity $\mathbf{v}_L$ oscillates with frequency ω. We shall consider frequencies on the order of $\omega_0^{(s)} \ll T_c$. For superconductors, a similar problem has been solved previously.[31] The same method may be used in the present case as well. One obtains the frequency-dependent response $F^{(\mathrm{exc})}(\omega)$ for oscillation frequencies $\omega \ll T_c$ by substituting $\varepsilon \rightarrow \varepsilon \pm \omega/2$ in the retarded (advanced) Green's functions, respectively. According to our τ approximation, used for the quasiparticle relaxation, this reduces to the replacement of i/τ by $i/\tau{+}\omega$ in Eq. (35). As a result, we find for the coefficients $\mathcal{D}$ and $\mathcal{D}'$ in Eq. (3):

$$\mathcal{D}(\omega) = \kappa\rho \int_0^{\pi/2} \frac{3}{8}\sin^3\theta\, d\theta \sum_s \left(\frac{i\omega_0^{(s)}(\theta)}{\omega_0^{(s)}(\theta)+\omega+i/\tau} - \frac{i\omega_0^{(s)}(\theta)}{\omega_0^{(s)}(\theta)-\omega-i/\tau}\right) \tag{71}$$

and

$$\mathcal{D}'(\omega) = \kappa\rho \int_0^{\pi/2} \frac{3}{8}\sin^3\theta\, d\theta \sum_s \left(2 - \frac{\omega_0^{(s)}(\theta)}{\omega_0^{(s)}(\theta)+\omega+i/\tau} - \frac{\omega_0^{(s)}(\theta)}{\omega_0^{(s)}(\theta)-\omega-i/\tau}\right) , \tag{72}$$

respectively.

In the ballistic regime, $\tau \rightarrow \infty$, Eqs. (71) and (72) become completely independent of the model used to describe the quasiparticle relaxation; hence, in this sense they become exact in this limit.

A finite frequency, ω, of vortex oscillations affects both $\mathcal{D}$ and $\mathcal{D}'$. Let us consider the dissipation per one vortex, which is proportional to $\eta(\omega) \equiv \frac{1}{2}[\mathcal{D}(\omega) + \mathcal{D}(-\omega)]$. From Eq. (71), we obtain

$$\eta(\omega) = \kappa\rho \int_0^{\pi/2} \frac{3}{8} \sin^3\theta \, d\theta \sum_s \frac{\omega_0^{(s)}(\omega^2 + \omega_0^{(s)^2})}{\tau[(\omega_0^{(s)^2} + 1/\tau^2 - \omega^2)^2 + 4\omega^2/\tau^2]} . \quad (73)$$

Dissipation is seen to increase considerably for frequencies $\omega \sim \omega_0^{(s)}$. For $\tau \to \infty$, we have

$$\eta(\omega) = \pi\kappa\rho \int_0^{\pi/2} \frac{3}{8} \sin^3\theta \, d\theta \sum_s \omega_0^{(s)} \delta[\omega - \omega_0^{(s)}(\theta)] . \quad (74)$$

Provided that $\omega_0^{(s)}$ is independent of θ (as is the case for the A-phase vortex example considered in Sec. V), the dissipation will have a sharp peak at the matching frequency $\omega = \omega_0^{(s)}$. However, if $\omega_0^{(s)}(k)$ forms a "band," dissipation exists for frequencies in the range $\omega_{\rm min}^{(s)} < \omega < \omega_{\rm max}^{(s)}$; it is proportional to

$$\eta(\omega) = \frac{3}{8}\pi\kappa\rho\omega \sum_s \sin^3\theta_s \left| \frac{\partial \omega_0^{(s)}(\theta_s)}{\partial\theta} \right|^{-1} , \quad (75)$$

where $\omega_0^{(s)}(\theta_s) = \omega$. The characteristic oscillation frequencies are on the order of 10 kHz.

For oscillation frequencies $\omega \ll \omega_0^{(s)}$, and with $\tau \to \infty$, the coefficient $\mathcal{D}'$ becomes small. The balance equation (1), within the first-order expansion in $\omega/\omega_0^{(s)}$, now yields

$$\mathbf{F}^{(M)} = M \frac{\partial \mathbf{v}_L}{\partial t} , \quad (76)$$

where

$$M = \kappa\rho \int_0^{\pi/2} \left(\sum_s \frac{3\sin^3\theta}{4\omega_0^{(s)}(\theta)} \right) d\theta \quad (77)$$

is the effective mass per unit length associated with the vortex. For $\omega_0^{(s)} \sim 1/(mR^2)$, we get

$$M \sim \pi R^2 \rho ,$$

i.e., the vortex mass is on the order of the liquid mass inside the core of radius R. A similar estimate for the mass of a quantized vortex line in He II has been obtained earlier,[32] with the use of a different approach. Note that the vortex mass for superfluid ^{3}He is considerably larger than that for He II.

VIII. SUMMARY

We have shown that the force, $\mathbf{F}^{(\rm exc)}$, produced by the normal component on a vortex moving in superfluid ^{3}He is determined by processes that are far more complicated than just the quasiparticle scattering off the vortex: the quasiparticles interacting with the vortex simultaneously relax through their mutual collisions with each other. At temperatures on the order of T_c, the force $\mathbf{F}^{(\rm exc)}$ is produced by both delocalized quasiparticles and those localized in the vortex core. For low temperatures (i.e., in the ballistic regime), the contribution to the force owing to the scattering states is less than the contribution of the localized quasiparticles interacting with normal excitations; however, both of these contributions are small and decrease rapidly with diminishing temperature. A similar situation occurs for the motion of negative ions in superfluid ^{3}He-A, where a reactive force arises due to an "internal" Magnus effect, as was recently suggested theoretically.[25]

We have calculated the force for low temperatures when the low-energy bound states localized in the vortex core are important. A quasiparticle state on these energy levels is similar to the motion of a charge along an orbit in an applied magnetic field, and the motion of a vortex carrying the localized quasiparticles is similar to the Hall effect. The relative magnitudes of the longitudinal and transverse (with respect to the vortex velocity) components of the force depend on the product of the transition frequency between the levels and the mean-free time of quasiparticles. Two limiting cases are possible: (i) for a short mean-free time, or a small separation between the levels, vortex motion is a viscous flow perpendicular to the superflow, i.e., parallel to the applied force with the velocity determined by the viscous drag, and (ii) for infrequent collisions between the ^{3}He quasiparticles, a vortex moves mainly with the flow and its viscous drag is small.

A similar situation occurs for type-II superconductors. Experiments show that both in the dirty ($\ell \ll \xi_0$) and in the pure ($\ell > \xi_0$, but yet $\ell \ll \xi_0 E_F/T_c$) superconductors, the Hall angle α_H is small (note that $\alpha_H \equiv \pi/2 - \alpha$, where α is the angle in Fig. 1), and the vortex velocity is perpendicular to the current. The conductivity calculated using quasiclassical kinetic equations[5] coincides with the result of calculations based on the method[16] similar to that described above, and corresponds to the dirty limit. It has not yet been experimentally possible to realize the clean limit in superconductors.

The relations between the numerical values of the relevant parameters for superfluid ^{3}He are such that case (i) can take place for the continuous vortices and for vortices with core sizes considerably larger than ξ_0 (at least for not excessively low temperatures). Case (ii) is realized for singular vortices at low temperatures. Our results imply that the mutual-friction coefficient B is on the order of unity, which agrees with the experimental observations.[7–10]

In superfluid ^{3}He, interesting effects are predicted for vortices oscillating with frequencies on the order of the transition frequency between the bound-state energy levels (in the range of about 10 kHz). One should observe strong dissipation at the resonant frequency even for very low temperatures. Experimentally, it is, in principle, possible to excite the vortex resonances with the help of ions moving in an ac electric field oscillating transverse to the vortex axis, as in superfluid ^{4}He.[33] However, such a method is more difficult for ^{3}He, owing to the predicted weakness of the ion-vortex interaction.[34] Another possibility is to use second sound as in superfluid ^{4}He to study the frequency dependence of mutual friction.[35] The oscillation of vortices could be employed for detecting the bound states in the core.

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