Thermal conductivity near the superfluid transition in ^{3}He-^{4}He mixtures below T_λ

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(Received 22 October 1990)

We calculate the effective thermal conductivity $\kappa_{\rm eff}$ in the superfluid phase of ^{3}He-^{4}He mixtures below T_λ within the nonasymptotic field-theoretic version of the model of Siggia and Nelson. The nonuniversal parameters appearing in statics and dynamics are taken from an analysis above T_λ. The temperature-independent values of $\kappa_{\rm eff}$ for the concentrations analyzed (0.1%–37%) are in agreement with experiment.

I. INTRODUCTION

The nonasymptotic renormalization-group theory has been applied with great success to the superfluid transition in ^{4}He. The strategy of the method and the results obtained have been reviewed recently by Dohm.[1] The extension of the theory to the superfluid transition in ^{3}He-^{4}He mixtures[2,3] makes possible a quantitative comparison of the theoretical calculations with the large amount of measurements of the transport properties near this phase transition (see the recent review article by Meyer[4]). A first test of the theory has already been started by an analysis of the model of Siggia and Nelson[5] in its simpler version where the asymmetric couplings of the order parameter to the entropy density and the concentration have been neglected.[2,3] The neglect of these couplings lead to some inconsistencies connected with the temperature dependence of the static susceptibilities of entropy density and concentration. Strictly speaking these susceptibilities would be constants, but in the analysis of Refs. 2 and 3 they have been replaced by their experimental temperature-dependent values. This replacement introduced a weak temperature dependence in the asymptotic limit in the thermal conductivity above[2] (most noticeable at higher concentrations) as well as in the whole critical region below[3] T_λ. This temperature dependence is absent in the experimental results[6] as well as in the theoretical results by Onuki[7] for the complete model. The reason for this temperature independence is a compensation of static and dynamic terms in the thermal conductivity resulting from those couplings.

In this paper we apply a recent nonasymptotic model F' calculation for $T > T_\lambda$ (Ref. 8) to the superfluid phase. The aim is first to show within the nonasymptotic field-theoretic renormalization-group procedure the temperature independence of $\kappa_{\rm eff}$ and its continuity at T_λ. This allows us to predict the value of the thermal conductivity at and below T_λ since all dynamical nonuniversal parameters have been determined by a fit in a certain restricted temperature region of the transport coefficients above T_λ. We apply the theory to concentrations as low as 0.1% and as high as 37%. The theoretical model, however, is applicable along the whole λ line, which extends to concentrations of 67%. Crossover effects to the tricritical behavior at the tricritical point at 67%, however, are not contained in the theory and limit therefore the applicability of the results at higher concentrations. Furthermore, for a quantitative analysis static and dynamic measurements at the same concentration are necessary. These are most interesting further away from T_λ reaching deeper into the background ($t_X > 10^{-2}$).

II. MODEL EQUATIONS AND THERMAL CONDUCTIVITY

We start with the unrenormalized equations of motion of the Siggia-Nelson model for ^{3}He-^{4}He mixtures[3]

$$\frac{\partial \mathring{\psi}}{\partial t} = -2\mathring{\Gamma}\frac{\delta H}{\delta \mathring{\psi}^*} + i\mathring{\psi}\mathring{g}t\frac{\delta H}{\delta \mathring{q}} + \theta_\psi \,, \tag{1a}$$

$$\frac{\partial \mathring{\psi}^*}{\partial t} = -2\mathring{\Gamma}^*\frac{\partial H}{\partial \mathring{\psi}} - i\mathring{\psi}^*\mathring{g}t\frac{\delta H}{\delta \mathring{q}} + \theta_\psi^* \,, \tag{1b}$$

$$\frac{\partial \mathring{q}_1}{\partial t} = -\nabla \mathbf{J}_1 + \theta_{q_1} \,, \tag{2a}$$

$$\frac{\partial \mathring{q}_2}{\partial t} = -\nabla \mathbf{J}_2 + \theta_{q_2} \,, \tag{2b}$$

with the currents

$$\mathbf{J}_i = -\mathring{\underline{\Lambda}}_{ij}\nabla\frac{\delta H}{\delta \mathring{q}_j} - 2\mathring{g}_i\,\mathrm{Im}(\mathring{\psi}^*\nabla\mathring{\psi}) \,. \tag{3}$$

The Hamiltonian is given by $H = H_\psi + H_{\rm q}$

$$H_\psi\{\mathring{\psi}\} = \int d^dx \left[\tfrac{1}{2}\mathring{r}|\mathring{\psi}|^2 + \tfrac{1}{2}|\nabla\mathring{\psi}|^2 + \frac{\tilde{\mathring{u}}}{4!}|\mathring{\psi}|^4\right] , \tag{4}$$

$$H_{\rm q}\{\mathring{\psi},\mathring{q}\} = \int d^dx\,(\tfrac{1}{2}\mathring{q}^t\underline{\overline{A}}\mathring{q} + \tfrac{1}{2}\mathring{\gamma}^t\mathring{q}|\mathring{\psi}|^2 - \mathring{h}^t_{\rm ext}\mathring{q}) \,. \tag{5}$$

ψ denotes the complex order parameter. Below T_λ it has a finite expectation value and one may write $\mathring{\psi} = \langle\mathring{\psi}\rangle + \mathring{\psi}_L + i\mathring{\psi}_T$. The conserved densities

$$\mathring{q}(x) = \begin{bmatrix}\mathring{q}_1(x)\\ \mathring{q}_2(x)\end{bmatrix}$$

are the entropy per mass density $\mathring{q}_1(x) = \sqrt{N_A\rho}\,\delta\sigma(x)/R$ and $\mathring{q}_2(x) = \sqrt{N_A\rho}\,\delta c(x)$ the concentration fluctuation.

R is the gas constant, ρ the average mass density, and N_A the Avogadro number. The square root factor is introduced for convenience.

The matrix

$$\underline{\bar{A}}=\begin{pmatrix}\bar{a}_{11} & \bar{a}_{12}\\ \bar{a}_{12} & \bar{a}_{22}\end{pmatrix}$$

is given by the background susceptibilities for these densities. Only one of the $\bar{a}_{ij}$'s is independent.[9]

$$\mathring{\boldsymbol{\gamma}}=\begin{pmatrix}\mathring{\gamma}_1\\ \mathring{\gamma}_2\end{pmatrix}$$

are the couplings to the order parameter,

$$\mathring{\boldsymbol{h}}_{\text{ext}}=\begin{pmatrix}\mathring{h}_{1\,\text{ext}}\\ \mathring{h}_{2\,\text{ext}}\end{pmatrix}$$

the external fields. The conjugated fields[7] are

$$\mathring{h}_1=\sqrt{N_A\rho}\frac{\delta T}{T},\quad \mathring{h}_2=\sqrt{N_A\rho}\frac{\delta\Delta}{RT}\ , \tag{6}$$

and $\Delta=\mu_3/m_3-\mu_4/m_4$ is the difference in the chemical potentials of ^{3}He and ^{4}He. The functional derivatives read

$$\frac{\delta}{\delta\mathring{\boldsymbol{q}}}=\begin{pmatrix}\delta/\delta\mathring{q}_1\\ \delta/\delta\mathring{q}_2\end{pmatrix}$$

and $\delta/\delta\mathring{\psi}=\frac{1}{2}(\delta/\delta\mathring{\psi}_L-i\delta/\delta\mathring{\psi}_T)$.

The dynamical mode couplings are

$$\mathring{\boldsymbol{g}}=\begin{pmatrix}\mathring{g}_1\\ \mathring{g}_2\end{pmatrix}$$

with

$$\mathring{g}_1=\frac{m_4k_BT}{\rho\hbar}\sqrt{N_A\rho}\frac{\sigma}{R},\quad \mathring{g}_2=\frac{m_4k_BT}{\rho\hbar}\sqrt{N_A\rho}c \tag{7}$$

and the kinetic coefficients

$$\underline{\mathring{\Lambda}}_{\mathrm{q}}=\begin{pmatrix}\mathring{\lambda} & \mathring{L}\\ \mathring{L} & \mathring{\mu}\end{pmatrix}.$$

The relaxation rate of the order parameter is complex $\mathring{\Gamma}=\mathring{\Gamma}'+i\mathring{\Gamma}''$. The fluctuating forces $\theta_\psi,\theta_\psi^*$ and

$$\boldsymbol{\theta}_{\mathrm{q}}=\begin{pmatrix}\theta_{q_1}\\ \theta_{q_2}\end{pmatrix}$$

fulfill the Einstein relations

$$\langle\theta_\psi(x,t)\theta_\psi^*(x_1,t_1)\rangle=4\mathring{\Gamma}'\delta(t-t_1)\delta(x-x_1)\ , \tag{8}$$

$$\langle\boldsymbol{\theta}_{\mathrm{q}}(x,t)\otimes\boldsymbol{\theta}_{\mathrm{q}}(x_1,t_1)\rangle=-2\underline{\mathring{\Lambda}}_{\mathrm{q}}\nabla^2\delta(t-t_1)\delta(x-x_1)\ . \tag{9}$$

We follow Ref. 7 and identify the heat and mass currents by comparing with the hydrodynamic currents. The heat current is then given by

$$\mathbf{Q}=RT\sqrt{\rho/N_A}\mathbf{J}_1 \tag{10}$$

and the mass current by

$$\mathbf{I}=\sqrt{\rho/N_A}\mathbf{J}_2\ . \tag{11}$$

The thermal conductivity in the superfluid phase is measured under stationary conditions, requiring $\nabla\mu_4=0$ and $\dot{c}=0$. The first condition leads to a relation between the concentration and temperature gradient. We use the definitions of the conjugated fields and write

$$\nabla\mu_4=-\sigma\nabla T-c\nabla\Delta=0 \tag{12}$$

as

$$\nabla\mathring{h}_2=-\mathring{g}_1/\mathring{g}_2\nabla\mathring{h}_1\ . \tag{13}$$

On the other hand we have for the fluctuations the relation $\delta H/\delta\mathring{\boldsymbol{q}}=\mathring{\boldsymbol{h}}$,[7] and hence we get

$$\nabla\frac{\delta H}{\delta\mathring{q}_2}=-\frac{\mathring{g}_1}{\mathring{g}_2}\nabla\frac{\delta H}{\delta\mathring{q}_1}\ . \tag{14}$$

Using this relation and $\mathbf{I}=0$ we can write the heat current as

$$\mathbf{Q}=-\kappa_{\text{eff}}\nabla T \tag{15}$$

and

$$\kappa_{\text{eff}}=R\rho[\mathring{\lambda}-2(\mathring{g}_1/\mathring{g}_2)\mathring{L}+(\mathring{g}_1/\mathring{g}_2)^2\mathring{\mu}]\ . \tag{16}$$

The dynamical model will be treated within the field theoretic Lagrangian formalism[10] where we also have to introduce the auxiliary variables $\tilde{\mathring{\psi}}$ and $\tilde{\mathring{\boldsymbol{m}}}$. From the Lagrangian for the dynamical model we then can calculate the relevant vertex functions, which contain the effects of the critical fluctuations in the desired perturbational order. Thus, we obtain the complete expression for the thermal conductivity including the critical effects, when we identify the bare dynamical parameters by the appropriate vertex functions. From a dissipation fluctuation relation we have

$$\frac{\partial}{\partial k^2}\mathring{\Gamma}_{q\bar{q}}=\mathring{\Gamma}_{qq}\frac{\partial}{\partial k^2}\mathring{\Gamma}^{(d)}_{q\bar{q}}\ . \tag{17}$$

Comparing both sides with the lowest order by considering the quadratic terms in the Lagrangian[8] we find that the dynamical parameters $\underline{\bar{\Lambda}}_q$ have to be replaced by the pure dynamic vertex functions $(\partial/\partial k^2)\mathring{\Gamma}^{(d)}_{q\bar{q}}$ only, since $\underline{\bar{A}}\,\underline{\bar{\Lambda}}_q$ is identified with $(\partial/\partial k^2)\mathring{\Gamma}_{q\bar{q}}$ and $\underline{\bar{A}}$ with $\mathring{\Gamma}_{qq}$. Then we come to

$$\kappa_{\text{eff}}=R\rho\left[\frac{\partial}{\partial k^2}\mathring{\Gamma}^{(d)}_{q_1\bar{q}_1}-2\left[\frac{\mathring{g}_1}{\mathring{g}_2}\right]\frac{\partial}{\partial k^2}\mathring{\Gamma}^{(d)}_{q_1\bar{q}_2}+\left[\frac{\mathring{g}_1}{\mathring{g}_2}\right]^2\frac{\partial}{\partial k^2}\mathring{\Gamma}^{(d)}_{q_2\bar{q}_2}\right]. \tag{18}$$

The temperature dependence can be found by expressing the unrenormalized thermal conductivity through its re-

normalized counterparts. The expressions can be simplified by a change of the variables. We scale and rotate the vector $\mathbf{q}$, so that the static part is diagonalized and one asymmetric coupling is eliminated. This is achieved by the transformations[8,9] $\mathring{\boldsymbol{m}} = \underline{M}\mathring{\boldsymbol{q}}$ with

$$\underline{M} = \begin{bmatrix} \sqrt{a_{11}} & \sqrt{a_{11}}\frac{1}{R}\left[\frac{\partial\Delta}{\partial T}\right]_{P\lambda} \\ -\sqrt{a_{22}}\frac{RT}{C_\lambda}\left[\frac{\partial c}{\partial T}\right]_{P\lambda} & \sqrt{a_{22}}\frac{T}{C_\lambda}\left[\frac{\partial\sigma}{\partial T}\right]_{P\lambda} \end{bmatrix}, \tag{19}$$

consisting of the λ-line derivatives and

$$C_\lambda = T\left[\left[\frac{\partial\sigma}{\partial T}\right]_{P\lambda} + \left[\frac{\partial\Delta}{\partial T}\right]_{P\lambda}\left[\frac{\partial c}{\partial T}\right]_{P\lambda}\right]. \tag{20}$$

The parameters a_{ii} are

$$a_{11} = R/C_\lambda, \quad a_{22} = \frac{C_\lambda}{RT^2}\left[\frac{\partial T}{\partial c}\right]^2_{P\lambda}(\bar{a}_{11}C_\lambda/R - 1). \tag{21}$$

The mode couplings transform into

$$\mathring{g}_1 = \frac{m_4\sqrt{a_{11}}T_\lambda}{\hbar(N_A\rho)^{1/2}}\left[\sigma + \left[\frac{\partial\Delta}{\partial T}\right]_{P\lambda}c\right], \tag{22}$$

$$\mathring{g}_2 = \frac{m_4\sqrt{a_{22}}T_\lambda}{\hbar(N_A\rho)^{1/2}}\frac{RT_\lambda}{C_\lambda}\left[-\left[\frac{\partial c}{\partial T}\right]_{P\lambda}\sigma + \left[\frac{\partial\sigma}{\partial T}\right]_{P\lambda}c\right], \tag{23}$$

leading to

$$H_{\mathrm{m}} = \int d^dx\,(\tfrac{1}{2}\mathring{\boldsymbol{m}}^t\mathring{\boldsymbol{m}} + \tfrac{1}{2}\mathring{\gamma}\mathring{m}_2|\mathring{\psi}|^2 - \mathring{\boldsymbol{h}}^t\mathring{\boldsymbol{m}}), \tag{24}$$

with the new couplings

$$\mathring{\gamma}_1 = 0, \quad \mathring{\gamma}_2 \equiv \mathring{\gamma} = \left[\mathring{\gamma}_2 - \frac{1}{R}\left[\frac{\partial\Delta}{\partial T}\right]_{P\lambda}\mathring{\gamma}_1\right]\Big/\sqrt{a_{22}}. \tag{25}$$

The new kinetic coefficients read

$$\underline{\mathring{\Lambda}} = \begin{bmatrix} \mathring{\lambda} & \mathring{L} \\ \mathring{L} & \mathring{\mu} \end{bmatrix} = \underline{M}\,\underline{\mathring{\Lambda}}_q\underline{M}^t. \tag{26}$$

Taking all together κ_{eff} may be written

$$\kappa_{\mathrm{eff}} = \kappa_0[\mathring{\lambda} - 2(\mathring{g}_1/\mathring{g}_2)\mathring{L} + (\mathring{g}_1/\mathring{g}_2)^2\mathring{\mu}] \tag{27}$$

with

$$\kappa_0 = \rho R\left[\frac{\hbar\mathring{g}_2}{m_4RTc}\right]^2\frac{N_A\rho}{a_{22}a_{11}}. \tag{28}$$

The expression (18) now contains the vertex functions of the new variables in the form

$$\kappa_{\mathrm{eff}} = \kappa_0\left[\frac{\partial}{\partial k^2}\mathring{\Gamma}^{(d)}_{m_1\bar{m}_1} - 2\left[\frac{\mathring{g}_1}{\mathring{g}_2}\right]\frac{\partial}{\partial k^2}\mathring{\Gamma}^{(d)}_{m_1\bar{m}_2} + \left[\frac{\mathring{g}_1}{\mathring{g}_2}\right]^2\frac{\partial}{\partial k^2}\mathring{\Gamma}^{(d)}_{m_2\bar{m}_2}\right]. \tag{29}$$

III. RENORMALIZATION

Now we introduce the static and dynamic renormalization. These can be performed in the standard way, as it is done in model F for pure ^{4}He,[11] since only one asymmetric coupling appears. Thus we have ($\epsilon = 4-d$)

$$\mathring{\psi} = Z_\psi^{1/2}\psi, \quad \mathring{\tau} - \mathring{\tau}_c = Z_\psi^{-1}Z_\tau\tau, \quad \tilde{\mathring{u}} = \kappa^\epsilon Z_\psi^{-2}Z_u\tilde{u}A_d^{-1}, \tag{30a}$$

$$A_d = S_d\Gamma\left[1+\frac{\epsilon}{2}\right]\Gamma\left[1-\frac{\epsilon}{2}\right], \qquad S_d = \left[2^{d-1}\pi^{d/2}\Gamma\left[\frac{d}{2}\right]\right]^{-1}, \tag{30b}$$

$$\mathring{m}_2 = Z_m m_2, \quad \mathring{\gamma} = \kappa^{\epsilon/2}Z_\psi^{-1}Z_rZ_m\gamma A_d^{-1/2}. \tag{30c}$$

The auxiliary variables $\tilde{\mathring{\psi}}$ and $\tilde{\mathring{\boldsymbol{m}}}$ are normalized as

$$\tilde{\mathring{\psi}} = Z_{\tilde{\psi}}^{1/2}\tilde{\psi}, \quad \tilde{\mathring{m}}_2 = Z_m^{-1}\tilde{m}_2. \tag{30d}$$

We have made use of relations between different Z factors.[12,13] Z_r is the renormalization factor of the quadratic term in the usual Ψ^4 theory, which is obtained when one integrates out the variables $\mathbf{m}$. Since only one coupling γ is left after the transformation (19), m_1 and g_1 have not to be renormalized. Only the dynamic coupling $\mathring{g}_2$ renormalizes as

$$\mathring{g}_2 = \kappa^{\epsilon/2}Z_m g_2 A_d^{-1/2}. \tag{30e}$$

Finally, we renormalize the kinetic coefficients as

$$\mathring{\Gamma} = Z_\Gamma\Gamma, \quad \mathring{\Gamma}^* = Z_\Gamma^*\Gamma^*, \quad \mathring{\lambda}_i = Z_{\lambda_i}\lambda_i, \quad \lambda_i = \lambda, L, \mu. \tag{30f}$$

The ζ functions are defined by the renormalization factors accordingly to

$$\zeta_i = \kappa\frac{d}{d\kappa}Z_i^{-1}. \tag{31}$$

It is useful to introduce the following dimensionless dynamical parameters

$$w_1' = \Gamma'/\lambda, \quad w_2' = \Gamma'/\mu, \quad w_3 = L/(\lambda\mu)^{1/2}, \tag{32}$$

and the couplings

$$f_1 = g_1/(\lambda\Gamma')^{1/2}, \quad f_2 = g_2/(\mu\Gamma')^{1/2}. \tag{33}$$

The ζ functions and the renormalized vertex functions will then become functions of these parameters. We are now in a position to express the unrenormalized κ_{eff} through its renormalized counterpart, using the relations which hold for the vertex functions above as well as below T_λ:[8]

$$\frac{\partial}{\partial k^2}\mathring{\Gamma}^{(d)}_{m_1\bar{m}_1} = \frac{\partial}{\partial k^2}\Gamma^{(d)}_{m_1\bar{m}_1}, \tag{34a}$$

$$\frac{\partial}{\partial k^2}\mathring{\Gamma}^{(d)}_{m_1\bar{m}_2} = Z_m\exp\left[-\int_1^l\frac{dx}{x}\zeta_m\right]\frac{\partial}{\partial k^2}\Gamma^{(d)}_{m_1\bar{m}_2}, \tag{34b}$$

$$\frac{\partial}{\partial k^2}\mathring{\Gamma}^{(d)}_{m_2\bar{m}_2} = Z_m^2\exp\left[-2\int_1^l\frac{dx}{x}\zeta_m\right]\frac{\partial}{\partial k^2}\Gamma^{(d)}_{m_2\bar{m}_2}. \tag{34c}$$

All vertex functions depend on the parameters $\tilde{u}(l)$, $\gamma(l)$, $g_i(l)$ $(i=1,2)$, $\lambda(l)$, $\mu(l)$, $L(l)$, and $\Gamma(l)$. Their l dependence is found from the solutions of the flow equations

$$l\frac{d}{dl}\lambda(l)=\lambda(l)\zeta_\lambda ,$$
$$l\frac{d}{dl}\mu(l)=\mu(l)\zeta_\mu , \tag{35}$$
$$l\frac{d}{dl}L(l)=L(l)\zeta_L ,$$

with suitable initial conditions. For $g_i(l)$ we have

$$l\frac{d}{dl}g_1(l)=-\frac{\epsilon}{2}g_1(l),\quad l\frac{d}{dl}g_2(l)=-g_2(l)\left[\frac{\epsilon}{2}-\zeta_m\right] . \tag{36}$$

The perturbational dynamical contributions to the renormalized vertex functions have the following particular structure above as well as below T_λ:

$$\frac{\partial}{\partial k^2}\Gamma^{(d)}_{m_i\bar{m}_j}=\begin{bmatrix}\lambda(l) & L(l)\\ L(l) & \mu(l)\end{bmatrix} -\frac{1}{4\Gamma'(l)}\begin{bmatrix}g_1^2(l) & g_1(l)g_2(l)\\ g_1(l)g_2(l) & g_2^2(l)\end{bmatrix}(1+P) . \tag{37}$$

The expression P contains all higher than one loop terms. It is this structure which leads to a cancellation of all higher than zero loop terms in κ_{eff}. Inserting the vertex function we can write for the thermal conductivity the *exact* expression within this model,

$$\kappa_{\text{eff}}(l(t_X))=\kappa_0\{\lambda(l)-2[g_1(l)/g_2(l)]L(l) +[g_1(l)/g_2(l)]^2\mu(l)\} . \tag{38}$$

To connect the flow parameter l with the temperature t_Δ $(t_\Delta=[T_\lambda(\Delta)-T]/T_\lambda(\Delta))$, we impose the relation $l=[2t_\Delta(t_X)]^\nu$ [for the relation of t_Δ to the experimental value of t_X X molar concentration, $t_X=[T_\lambda(X)-T]/T_\lambda(X)$ see, e.g., Ref. 14]. The flow of $\tilde{u}(l)$ and $\gamma(l)$ is determined by the specific heat above T_λ.[8,12]

Equation (38) is the central result of our paper. All parameters appearing in this expression for the thermal conductivity are known from our analysis *above* T_λ, namely the initial conditions of the flow equations for the l-dependent dynamical parameters and the static quantities.

IV. DISCUSSION OF THE THERMAL CONDUCTIVITY

Before we evaluate the thermal conductivity we show (i) its temperature independence and (ii) its continuity at T_λ.

(i) The temperature derivative of $(d/dt_X)\kappa_{\text{eff}}(l(t_X))$ is proportional to $(d/dl)\kappa_{\text{eff}}(l)$. Performing the derivative one uses the flow equations and takes into account the structure of the ζ functions. Let us first separate the dynamical parts in the ζ functions:

$$\zeta_\mu=\zeta_\mu^{(d)}+2\zeta_m,\quad \zeta_L=\zeta_L^{(d)}+\zeta_m,\quad \zeta_\lambda=\zeta_\lambda^{(d)} . \tag{39}$$

Note that ζ_λ has only dynamic terms. The dynamical parts then have the structure

$$\zeta_\lambda^{(d)}=-\{g_1^2(l)/[2\Gamma'(l)\lambda(l)]\}(1+Q) , \tag{40a}$$
$$\zeta_\mu^{(d)}=-\{g_2^2(l)/[2\Gamma'(l)\mu(l)]\}(1+Q) , \tag{40b}$$
$$\zeta_L^{(d)}=-\{g_1(l)g_2(l)/[2\Gamma'(l)L(l)]\}(1+Q) , \tag{40c}$$

where Q contains the higher than one loop contributions. With Eqs. (35), (36), and (39) one finds

$$l\frac{d}{dl}\left[\lambda(l)-\frac{g_1(l)}{g_2(l)}L(l)\right]=0 \tag{41}$$

and

$$l\frac{d}{dl}\frac{g_1(l)}{g_2(l)}\left[\frac{g_1(l)}{g_2(l)}\mu(l)-L(l)\right]=0 , \tag{42}$$

from which immediately follows

$$\frac{d}{dt_X}\kappa_{\text{eff}}=0 . \tag{43}$$

(ii) In order to show the continuity of the thermal conductivity at T_λ we rewrite κ_{eff} in the dynamical variables $w_i(l)$ and $f_i(l)$

$$\kappa_{\text{eff}}=\kappa_0\frac{g_2(l)}{f_2(l)w_2^{1/2}(l)}\frac{g_1^2(l)}{g_2^2(l)}\left[1-\frac{2f_2(l)w_3(l)}{f_1(l)}+\left[\frac{f_2(l)}{f_1(l)}\right]^2\right] . \tag{44}$$

This has to be compared with the expression $\kappa_T(l(t_X))$ at T_λ, taken from the calculation above T_λ:[8]

$$\kappa_T=\rho C_\lambda\frac{c_+^2(l)g_2(l)}{f_2(l)w_2^{1/2}(l)}A_\kappa(l) , \tag{45}$$

where

$$c_+^2=[1+\gamma^2(l)F^+(u(l))]^{-1} , \tag{46}$$

and $l(t_X)$ is connected to t_X by $l(t_X)=(t_\Delta(t_X))^\nu$, and the function $t_\Delta(t_X)=[T-T_\lambda(\Delta)]/T_\lambda(\Delta)$ has to be taken above T_λ. F^+ is given by the amplitude function of the specific heat and $u(l)=\tilde{u}(l)-3\gamma^2(l)$. The amplitude reads

$$A_\kappa=\frac{[(1-\frac{1}{4}f_1^2)(1-\frac{1}{4}f_2^2)-(w_3-\frac{1}{4}f_1f_2)^2]}{(C_\lambda/C_{Pc}^+-1)(1-\frac{1}{4}f_1^2)+\frac{w_1'}{w_2'}c_+^2(1-\frac{1}{4}f_2^2)-2c_+\left[\left[\frac{C_\lambda}{C_{Pc}^+}-1\right]\frac{w_1'}{w_2'}\right]^{1/2}(w_3-\frac{1}{4}f_1f_2)} . \tag{47}$$

Although we have only calculated the one-loop expression for the thermal conductivity above T_λ, in the limit $T \to T_\lambda$, and only in this limit, the higher loop contributions drop out. The reason is again the structure of the vertex functions; see Eq. (37) (further details will be given in Ref. 8). Now let us perform the limit $T \to T_\lambda$, meaning $l \to 0$, for the one-loop expression explicitly. Within these limits $\gamma(l) \to 0$, $c_\pm^2(l) \to 1$, and the parameters w_3 and f_1/f_2 behave like[2]

$$w_3(l) = -1 + A_3 l^{1/2} \tag{48a}$$

and

$$f_1(l)/f_2(l) = -1 + A_f l^{1/2} . \tag{48b}$$

We want to note that the sign of the fixed point value $f_1(0)/f_2(0) \equiv f_1^*/f_2^*$ is determined by the sign of $\mathring{g}_1/\mathring{g}_2$.[8] As $\mathring{g}_1/\mathring{g}_2$ is negative[7] in the region of concentrations smaller than 37% we have taken $f_1^*/f_2^* = -1$. In consequence the fixed point value of w_3^* is $w_3(0) = -1$. For concentrations higher than 37% one has to take the fixed point values $f_1^*/f_2^* = 1$ and $w_3^* = 1$ instead. The ratio of the couplings g_1 and g_2 is finite in the limit $l \to 0$

$$\frac{g_1^2(l)}{g_2^2(l)} = \frac{f_1^2(l)}{f_2^2(l)} \frac{w_2(l)}{w_1(l)} \to w_2(0)/w_1(0) . \tag{49}$$

Even if the nonscaling fixed point is stable, for which $w_1(0) \equiv w_1^* = 0$ and $w_2(0) \equiv w_2^* = 0$, the ratio $w_2(0)/w_1(0)$ is finite. The same is true for $A_3(0)/w_2^{*1/2}$.[2] Inserting Eq. (45) into κ_T we obtain, after cancelling the A_f terms ($\epsilon = 1$),

$$\kappa_T = \rho C_\lambda \frac{g_2(l)}{f_2^* w_2^{*1/2}} \frac{2A_3 l^{1/2}}{[\sqrt{C_\lambda/C_{Pc}^+ - 1} - g_2(l)/g_1(l)]^2} . \tag{50}$$

From the flow equations for $g_i(l)$ (36) follows [we have chosen $\kappa = \xi_0^{-1}$ (Ref. 2)]

$$g_1(l) = \mathring{g}_1 (\xi_0^{-1} l)^{-1/2} A_d^{1/2} , \tag{51a}$$

$$g_2(l) = \mathring{g}_2 (\xi_0^{-1} l)^{-1/2} A_d^{1/2} Z_m^{-1} \exp\left[\int_1^l \frac{dx}{x} \zeta_m \right] . \tag{51b}$$

We further use the static relation

$$\mathring{\Gamma}^{\pm}_{m_2 m_2} = a_{22} c_\pm^2 Z_m^{-2} \exp\left[2 \int_1^l \frac{dx}{x} \zeta_m \right] = \frac{C_\lambda}{RT^2} \left[\frac{\partial T}{\partial c} \right]_{P\lambda}^2 (C_\lambda/C_{Pc}^\pm - 1) \tag{52}$$

in order to eliminate the exponential in Eq. (51b). Then we get

$$\frac{g_1(l)}{g_2(l)} = \frac{\mathring{g}_1}{\mathring{g}_2} a_{11}^{1/2} c_+(l) \times \left[\frac{C_\lambda}{RT^2} \left[\frac{\partial T}{\partial c} \right]_{P\lambda}^2 (C_\lambda/C_{Pc}^+ - 1) \right]^{-1/2} . \tag{53}$$

Inserting this into Eq. (50) and using (20) and (21) we find

$$\kappa_T = \rho R \left[\frac{\hbar N_A (N_A \rho)^{1/2} \mathring{g}_1}{m_4 R T c a_{11}^{1/2}} \right]^2 \frac{2A_3}{f_2^* w_2^{*1/2}} \times \frac{\mathring{g}_2 (a_{22})^{-1/2} A_d^{1/2} \xi_0^{-1/2}}{\left[\frac{C_\lambda}{RT^2} \left[\frac{\partial T}{\partial c} \right]_{P\lambda}^2 (C_\lambda/C_{Pc}^+ - 1) \right]^{1/2}} . \tag{54}$$

Thus the asymptotic value can be reduced to

$$\kappa_T = \kappa_0 \frac{\mathring{g}_1^2}{\mathring{g}_2} \left[\frac{a_{22}}{a_{11}} \right]^{1/2} \frac{2A_3}{f_2^* w_2^{*1/2}} \times \frac{A_d^{1/2} \xi_0^{-1/2}}{\left[\frac{C_\lambda}{RT^2} \left[\frac{\partial T}{\partial c} \right]_{P\lambda}^2 (C_\lambda/C_{Pc}^+ - 1) \right]^{1/2}} . \tag{55}$$

On the other hand inserting the limiting behavior of the dynamical parameters, Eqs. (48), into Eq. (44) for κ_{eff}, leads to

$$\kappa_{\text{eff}} = \kappa_0 \frac{2A_3 l^{1/2}}{f_2(l) w_2(l)^{1/2}} \frac{g_1^2(l)}{g_2(l)} . \tag{56}$$

Then from Eq. (53) follows immediately that $\kappa_{\text{eff}} = \kappa(T_\lambda)$. Note that we equally well might have used Eq. (53) in its form below T_λ, where c_+ and C_{Pc}^+ are replaced by c_- and C_{Pc}^-, respectively. From the theoretical point of view both expressions should lead to the same flow, as also follows from Eq. (51). However, in practice there will be a difference, first because the $c_\pm$ are not known exactly, but only in some perturbational order, and second because the experimental $C_{Pc}^\pm$ may differ in accuracy.

V. CALCULATION OF $\kappa_{\text{eff}}(T_\lambda)$

We are now in a position to calculate κ_{eff} in two ways. On the one hand we can extrapolate the thermal conductivity above T_λ, $\kappa_T(t_X)$, to T_λ. This is done by using the flow equations for the dynamical parameters in the expression for $\kappa_T(t_X)$ and letting t_X become so small that $\kappa_T(t_X)$ reaches its plateau value. This value is then identified with κ_{eff} below T_λ. On the other hand one may directly calculate κ_{eff} at some value of t_X below T_λ [e.g., at the background value $l(t_{X0})$ from above].

Of course both calculations lead to the same result once the initial conditions of the dynamical parameters above T_λ have been chosen. We already have mentioned that the asymptotic λ value of the thermal conductivity reached from the normal fluid side is independent of the loop order.[8] The initial parameters are taken from a fit of the transport coefficients over a certain region of t_X. It is this procedure where both the low order (one-loop) approximation for the expressions of transport coefficients and the low order approximation (one loop for model F' terms and two loop for model E' terms) for the flow equations enter the result for the initial parameters found from the fit. In this respect the λ value of the thermal conductivity is a test of the approximations used in the theoretical calculations. One might, however, eliminate one dynamic parameter by imposing the condition

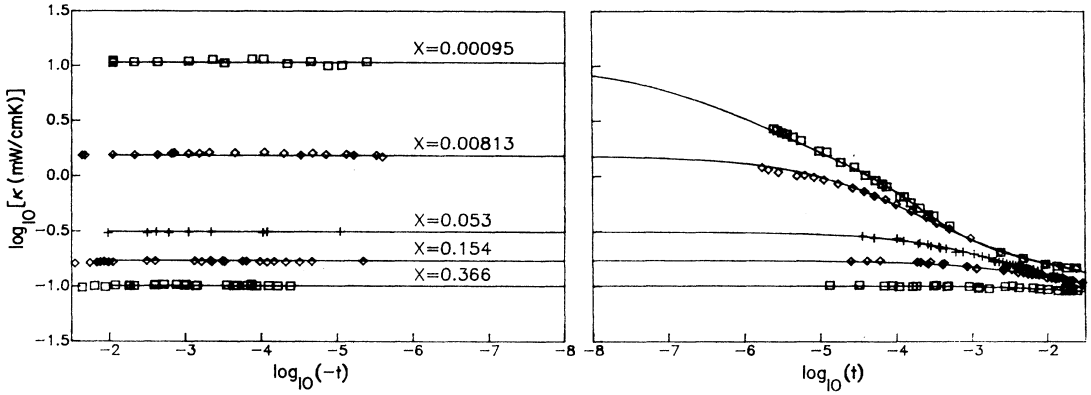

FIG. 1. Thermal conductivity above and below T_λ for different concentrations as a function of temperature $[t=|T-T_\lambda(X)|/T_\lambda(X)]$. The solid curves are obtained from a fit in a restricted temperature region (see text) above T_λ of the thermal conductivity and the thermal diffusion ratio (Ref. 8). The experimental data are taken from Refs. 6, 15, 16, and 17.

$\kappa_{\rm eff}$(theor)$=\kappa_{\rm eff}$(expt).

In the extrapolation we also use the experimental determined flow of the static coupling γ outside the region where experimental specific heat data are available by using an extrapolation. However, the value of the specific heat also does not actually enter the λ value of κ_T.

In Fig. 1 we display the fit and the result of $\kappa_{\rm eff}$ for various concentrations. We fitted[8] κ_T and k_T over the temperature region $10^{-3}<t_X<10^{-1.5}$ for the molar concentrations $X=0.053$, 0.154, and 0.366.[15] For the smaller concentrations[16] we fitted over the region $10^{-5}<t_X<10^{-1.5}$ because otherwise there are not enough k_T data in the smaller temperature region. No fit was made for D since taking D into account reduces the excellent agreement of κ_T essentially. One may attribute this quantitative disagreement with the experimental values to the neglect of the model-F two-loop terms (see Ref. 8 for a discussion).

VI. DISCUSSION

Beside the thermal conductivity an interesting quantity is the diffusive relaxation rate Γ_0. From Khalatnikov's hydrodynamics one finds the relation[18]

$$\Gamma_0=\frac{\kappa_{\rm eff}}{\rho C_{Pc}(1+B)}\ , \tag{57}$$

with $B=\mathring{g}_1/\mathring{g}_2$. We have calculated Γ_0 from the determinant of the coefficient matrix of the linearized equations (1) and (2), leading to

$$\Gamma_0=\frac{\kappa_{\rm eff}}{\rho R}\frac{\mathring{\underline{g}}\,{}^2_2\det\underline{\overline{A}}}{\mathring{\underline{g}}\,{}^t\underline{\overline{A}}\,\mathring{\underline{g}}}\frac{1-\mathring{\underline{\gamma}}\,{}^t\underline{\overline{A}}\,{}^{-1}\mathring{\underline{\gamma}}\langle\mathring{\psi}\rangle^2/\mathring{\tau}_L}{1-(\mathring{\underline{g}}\,{}^t\mathring{\underline{\gamma}})^2\langle\mathring{\psi}\rangle^2/(\mathring{\tau}_L\mathring{\underline{g}}\,{}^t\underline{\overline{A}}\,\mathring{\underline{g}})} \tag{58}$$

in the zero-loop approximation, with $\mathring{\tau}_L=\mathring{\tau}+\langle\mathring{\psi}\rangle^2\tilde{\mathring{u}}/2$. In performing the transformation (19) the second ratio can be reduced to

$$\frac{\mathring{\underline{g}}\,{}^2_2\det\underline{\overline{A}}}{\mathring{\underline{g}}\,{}^t\underline{\overline{A}}\,\mathring{\underline{g}}}=\frac{R}{C_{Pc}(1+B)}\ . \tag{59}$$

Thus the two expressions in the unrenormalized theory are different and the difference will be zero when $\mathring{\gamma}=0$. In the critical region we may introduce the appropriate vertex functions for the bare quantities in (58) and express them through the renormalized vertex functions.[19] Although we have not calculated this complicated expression explicitly we expect that $\gamma(l)$ appears at the place of $\mathring{\gamma}$. Thus relation (57) is only fulfilled in the asymptotic region near T_λ or deep in the background region where the asymmetric coupling $\gamma(l)$ goes to zero.[7] In the nonasymptotic region corrections of the order $\gamma(l)$ appear.

For concentrations lower than 0.1% the thermal conductivity below T_λ deviates[20] from the c^{-1} law found by inserting the concentration dependence of the static and dynamic parameters into the expression (44).[2,8] This discrepancy is an unsolved problem of the background theory, rather than the critical theory, which is based on the hydrodynamic equations. We want to notice that the dynamic background parameters $\lambda(l_0)$, $\mu(l_0)$, and $L(l_0)$ are considered to be equal above *and* below T_λ. Therefore we expect the same concentration dependence of those dynamic parameters in the expressions for the transport coefficients above *and* below T_λ leading to the c^{-1} behavior (for another point of view see Ref. 21).

ACKNOWLEDGMENTS

We thank H. Meyer for sending us the data of his measurements in numerical form. This work was supported by the Fonds zur Förderung der wissenschaftlichen Forschung.

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