Diverging strains in the phase-deformation model of sliding charge-density waves

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We show that when a spatially uniform, time-independent electric field is applied to a charge-density wave (CDW), the time-averaged local CDW velocity is spatially inhomogeneous, with nonzero spatial average, because models that do not allow for the coexistence of regions with different time-averaged mean velocities have places with infinite energy density. To establish this result we present a heuristic scaling argument, rigorous results on a simplified lattice model, and a study of the continuum phase equation of motion for the CDW. Flux-line motion in type-II superconductors is also considered; similarities and differences between flux-line motion and CDW motion are discussed.

I. INTRODUCTION

Understanding the dynamics of sliding charge-density waves (CDW's) requires insight into the competition between interactions and randomness in nonequilibrium systems. Interest in CDW's arises partly because the ideas needed to understand them may have applicability to a large class of systems with randomness.

Below the CDW transition temperature in the absence of impurities, charged-density wave systems exhibit a modulation in the charge density $\rho(\mathbf{r})$ of the form $\rho(\mathbf{r})=\rho_0+\psi\cos(\mathbf{Q}\cdot\mathbf{r}+\phi)$, where $\mathbf{Q}$ is not necessarily simply related to the spacing of the underlying lattice. In the absence of impurities, even an arbitrarily small electric field F induces a nonzero CDW current I_{CDW}.[1] In the presence of impurities, it is energetically favorable for the CDW to distort, and both the amplitude $\psi(\mathbf{r})$ and phase $\phi(\mathbf{r})$ of the CDW depend on the position $\mathbf{r}$. When an electric field F is applied to a CDW, the field must overcome this pinning energy before the CDW can move. Thus these deformations caused by the impurities lead to a nonlinear current-voltage characteristic, where CDW motion appears to set in at a nonzero threshold field F_T. Much theoretical effort has been devoted to understanding the dynamics of the CDW in the presence of electric fields, focusing on how the interactions between the many degrees of freedom affect the behavior. Some CDW's have threshold voltages that are small compared to the CDW condensation energy, implying that the impurity pinning strength is small compared to the energy cost of distorting the CDW significantly on the scale of the impurity spacing.[2] Since long-wavelength distortions of the phase can cost arbitrarily little energy, whereas amplitude fluctuations of any wavelength must cost an energy of the same order as the gap energy, many investigators have studied the Fukuyama-Lee-Rice (FLR) model, which has only phase degrees of freedom.[3]

One aspect of the FLR model that has attracted attention is the behavior near the threshold field F_T where the nonlinearity in the current-voltage characteristic sets in. For this model, if $F<F_T$, the system is entirely stationary, and for $F>F_T$ the time-averaged velocity of every local region is identical and nonzero. The depinning transition exhibits critical phenomena; for $F>F_T$ the velocity v obeys $v\propto(F-F_T)^{\zeta}$, where ζ depends on d, the number of dimensions. A diverging correlation length can be defined.[4]

In this paper we present in detail arguments showing that the critical phenomena exhibited by the FLR model do not survive if one considers the dynamics of real CDW's with both phase and amplitude degrees of freedom in the presence of an external electric field.[5] We show that when a uniform force is applied to a model with phase degrees of freedom only, regions of unbounded strain occur. These regions arise because velocity uniformity in the presence of random pinning is imposed by elastic forces. However, a region with an anomalously small pinning strength needs a force scaling with the volume of the region to keep it from moving, while the elastic force acts only on its surface, so that the strain per unit area scales as the linear dimension of the region. In any dimension, arbitrarily large regions with arbitrarily low pinning strength will occur in an infinite sample with bounded fluctuations in the pinning potential. Thus the reduction of more realistic models to phase-only models breaks down at the boundaries of these regions. In more realistic models phase slips will occur; we show that these lead to a finite current for any $F>0$ and discuss their effect on the transition that occurs within the FLR model. In addition we show that in one and possibly two dimensions, typical $\sqrt{N}$ fluctuations, which tend to zero as the size of the region diverges, lead to diverging strains.

First we consider a discretization of the FLR equations of motion for the phase degrees of freedom,[6] with equations of motion

$$\dot{x}_i=k\sum_{\delta,nn}(x_{i+\delta}-x_i)-V_i\sin(x_i-\beta_i)+F\ . \tag{1}$$

These equations describe a d-dimensional system of overdamped particles, each connected by springs obeying Hooke's law to its nearest neighbors, in the presence of a random pinning force described by V_i and β_i. Every spring has a spring constant of k. The variables x_i are the positions of the particles. The motion is constrained to be along the direction of the force F, which is independent of both position and time. In terms of the original FLR equations of motion, x_i can be viewed as the CDW phase at the ith impurity. The sinusoidal potential for each particle follows if one assumes the impurity contribution to the energy is proportional to the product of the random impurity potential and the local CDW charge. The absence of phase slips is enforced by the fact that the spring force $k(x_{i,i+\delta}-x_i)$ is linear in the particle separation for all $\delta x_{i,\delta} \equiv x_{i+\delta}-x_i$. Physically, one expects that the spring force has a maximum value, so that for large enough $\delta x_{i,\delta}$ the spring breaks and Eq. (1) is no longer valid. In terms of the original CDW equations, when the phase-deformation energy becomes too large, a phase slip occurs and a CDW wavelength is added. This process renders Eq. (1) inapplicable. Thus, for consistency, every nearest-neighbor pair separation $\delta x_{i,\delta} \equiv x_{i+\delta}-x_i$ must obey $|\delta x_{i,\delta}| < S_{\max}$ for some finite $S_{\max}$, so that the local elastic energy density is less than $\frac{1}{2}kS_{\max}^2$.

This paper shows that when F is made nonzero in an infinite system described by Eqs. (1), regions exist for which $\delta x_{i,\delta}$ is arbitrarily large. Thus Eqs. (1) are inadequate for a description of real CDW's.

Therefore, in order to model a real physical system, Eqs. (1) must be modified. Although Eqs. (1) have been applied not only to charge-density waves but also to flux lattices in type-II superconductors,[7] fundamental differences between the two systems become apparent when defects are accounted for. The modifications needed to model sliding charge-density waves are different from those needed for flux lattices because CDW wavelengths can be created freely at phase slips while magnetic flux lines are not created freely at grain boundaries of the flux lattice. Equations (1) can be generalized straightforwardly for the CDW case so that the pathology of regions with arbitrarily large energy density does not occur. When this is done, one can show that the time-averaged velocity is spatially inhomogeneous for any nonzero F and that the current-voltage characteristic is either smooth or has a jump discontinuity. Therefore the critical behavior present in the phase-only model does not occur in a model with bounded energy density.

The paper is organized as follows. In Sec. II the scaling argument of Ref. 5 is reviewed, with special emphasis on the points that are clarified in the later sections. In Sec. III a simplified random friction model discussed in Ref. 5 is examined in more detail. This discussion is relatively brief because work that can be applied directly to this model has been done in another context. Section IV consists of a consideration of the CDW equation of motion and a demonstration of the applicability of the scaling argument in various special limits. Section V contains a discussion of other systems involving external forces, interactions, and randomness. Although the arguments in this paper may have relevance to flux lattices in type-II superconductors and to glasses, it is shown that issues that do not arise for the CDW must be taken into account. Finally, concluding remarks are in Sec. VI.

The question of experimental observability of phase slips in weakly pinned CDW's is quite subtle and is discussed in detail in a separate publication.[8] Numerical estimates for the phase-slip probability are made there and it is found that phase slips could well be observed in experimental samples. The arguments in this paper yield lower bounds to the phase-slip probability that are numerically accurate only in the limit of strong pinning and are hence not adequate for direct comparison to experiment.

II. SCALING ARGUMENT

In Ref. 5 the scaling argument indicating the importance of phase slips is presented considering the threshold state of the phase-only model. Here we review this argument and generalize it to the case of arbitrary field; the threshold behavior emerges as a special case.

The argument proceeds by examining the model where amplitude fluctuations are not allowed and showing that the local energy density is unbounded. The scaling arguments employed here do not depend on the details of the nonlinear pinning force, so they may be expected to apply to a class of models with competing randomness and interactions. However, several steps of the argument give rise to subtleties that are discussed in Sec. IV below, so that examination of a simpler but more tractable model described in Sec. III is also useful.

The argument relies on the fact that the surface-to-volume ratio of a region of linear dimension L vanishes as L gets large. Different regions might want to move at different velocities but are prevented from doing so by the spring forces. However, the regions communicate via springs that are only on the boundaries of the regions, while fluctuations in the impurity concentration and hence local threshold field are volume effects. Therefore the force exerted by each spring on the boundaries of the regions can become arbitrarily large as L becomes large.

More specifically, consider a system of infinite size with threshold $F_T(\infty)$ in the presence of a nonzero force F^*. Inside this system, select a region R with linear dimension L (so that its volume scales as L^d and its surface scales as L^{d-1}), which has threshold field $F_T(L)$, where $F_T(L)$ is defined as the threshold field of the region when the springs connecting its boundary to the rest of the system are cut and replaced with, say, a free boundary condition.[9] If $F_T(L)$ is less than F^*, then there must be a force $[F^*-F_T(L)]L^d$ exerted on the region through its boundary by the surrounding regions to keep it from moving. This force can only come from the springs at the boundary. However, since the number of springs at the boundary scales only as L^{d-1}, at least one spring must be stretched by an amount that scales as L. The largest strain at the boundary (denoted here as $\delta x_{\max}$) is at least

$$\delta x_{\max} \sim L[F^*-F_T(L)] \ . \tag{2}$$

Equation (2) makes it obvious that a bounded fluctuation in $F_T(L)$ can lead to large strains when L is large.

The main subtlety in the argument leading to Eq. (2) is in the definition of $F_T(L)$. Intuitively it is reasonable that the maximum pinning force a given set of impurities can exert is well defined up to terms of order $1/L$, but we have succeeded in showing this only in some special limits. One might also worry that the boundary of region R might not scale as L^{d-1}, but since we are free to choose the region R so that it has this property, this concern does not cause serious difficulties. These issues are discussed in Sec. IV below.

Given the distribution of the $F_T(L)$, one can use Eq. (2) to calculate the probability that a region of size L has $\delta x_{\max} > S_{\max}$. The number of regions with large strain along their boundaries can be estimated following the method Randeria, Sethna, and Palmer[10] used to discuss magnetic relaxation in spin glasses. If one assumes that this distribution is Gaussian (as would be the case if the local threshold scales the same way as the impurity density in the region), then $P(F_T(L))$, the probability of observing the value $F_T(L) > 0$, obeys

$$P(F_T(L)) \sim \exp-(\{[F_T(\infty)-F_T(L)]L^d\}^2/2\sigma_F^2 L^d) \; . \quad (3)$$

If one imagines a situation where there is a uniform distribution of pinning centers of similar strength V which is much larger than the spring constant k, then the width of the distribution σ_F is of order $F_T(\infty)$; this result follows because the threshold field is of order V, and in the absence of impurities the local threshold field is zero. The fluctuations in a region of a size L are expected to scale as $L^{d/2}$ (unless they are larger). The simplest case of this is a situation where each site independently has a probability p of having an impurity of strength V on it. In the strong pinning limit, the threshold field of the region is proportional to the number of impurities in the region, leading to Eq. (3). The weak pinning limit $k \ll V$ is more complicated and a non-Gaussian distribution of pinning energies is expected. For this case, Eq. (3) vastly underestimates the variations in the local threshold field.[8] However, Eq. (3) provides a usable lower bound that illustrates the essential points of the argument.

A region of size L in an applied force F^* has strain greater than $S_{\max}$ if $F_T(L) < F^* - S_{\max}/L$. The number of regions n_R which violate this bound is

$$n_R \sim \int_0^{L_{\max}} dL \int_0^{F_1} dF_T(L) P(F_T(L)) \; , \quad (4)$$

where $F_1 = \max(0, F^* - S_{\max}/L)$ and $L_{\max}$ is the size of the system, which acts as an upper cutoff. One might worry that the integration over L might lead to the overcounting of regions with large strains because the same site is included in regions on all different length scales. Whenever $F^* \neq F_T(\infty)$ in any dimension d and for $d > 2$ at any F^* a length scale emerges that dominates the integral over L, so that overcounting error is negligible. Just at threshold in one and two dimensions, the scaling analysis cannot be justified this way, but it yields predictions consistent with the exact results of Sec. III.

The integral over $F_T(L)$ is effected by letting $u = [F_T(\infty) - F_T(L)](L^{d/2}/\sigma_F)$ and noticing that the resulting integral is dominated by the smallest values of u. Thus one has

$$n_R \sim \int_{S_{\max}/F^*}^{L_{\max}} dL \frac{\sigma_F}{L^{d/2}} e^{-[F_T(\infty)-F^*+S_{\max}/L]^2 L^d/(2\sigma_F^2)} \; . \quad (5)$$

The integral is evaluated keeping in mind the constraint $F_T(L) > 0$. For $d > 2$ the integral over L is dominated by its lower limit, corresponding to regions with no impurities at all, of size $L^* = S_{\max}/F^*$. One obtains

$$n_R \sim \left[\frac{\sigma_F}{(S_{\max}/F^*)^d} \right]^{1/2} \times \exp(-\{[F_T(\infty)]^2 (S_{\max}/F^*)^d/2\}) \; .$$

This expression is explicitly nonzero for any $F^* > 0$. These rare regions with very small $F_T(L)$ are similar to those that lead to Griffiths singularities in the context of dilute ferromagnets[11] and spin glasses.[10] However, in contrast to the situation for random magnets, for the CDW system these rare fluctuations result in nonzero $\langle v \rangle$. This difference arises because the external force ensures that every depinned region moves in the same direction, whereas in random magnets the locally ordered regions have different orientations and do not yield a nonzero contribution to the net magnetization.

The Gaussian form is not expected to be valid so far in the tail of the distribution, so it is useful to estimate n_R for $d > 2$ as follows (d is the number of dimensions): Imagine the system is composed of a lattice of sites, each of which has a probability p of having an impurity on it. The estimate above makes it clear that the dominant contribution is by regions with no impurities at all. The chance of a region with L^d sites having no impurities at all is $(1-p)^{L^d}$, and a region of this type will have strain greater than $S_{\max}$ if $L > S_{\max}/F^*$, so the number of these regions goes as $\exp[-(S_{\max}/F^*)^d]$. The threshold field $F_T(\infty)$ does not enter except as an upper bound for F^*.

In one and two dimensions, as one approaches threshold, larger and larger regions have strains that exceed $S_{\max}$. For $d < 2$ the integral over L is dominated by values of L near $L^* = (2-d)S_{\max}/\{d[F_T(\infty) - F^*]\}$, which diverges at $F^* = F_T(\infty)$. Using steepest descents to evaluate the integral, for $f \equiv F_T(\infty) - F^* \neq 0$, in one dimension one finds

$$n_r \sim \left[\frac{\sigma_F}{f} \right] \exp(-2fS_{\max}/\sigma_F) \; . \quad (6)$$

Just at threshold, after making the substitution $u = 1/L$ it is straightforward to show that Eq. (5) leads to $n_R \propto L_{\max}^{1/2}$.

For $d = 2$ and $f = 0$, Eq. (5) straightforwardly leads to $n_R \propto \ln(L_{\max})$. When $f \neq 0$ in two dimensions Eq. (4) is more easily evaluated by first integrating over L, leading to $n_R \sim -\ln[f/F_T(\infty)] e^{-S_{\max}^2/2\sigma_F^2}$.

These results for one and two dimensions follow from a simple scaling argument. When a field F^* is applied, a fraction of order unity of regions of size L have $F_T(L) \le F^*$ when $F_T(\infty) - F^* \sim L^{-d/2}$. From Eq. (2) one expects $n_R \sim L^{(1-d/2)}$, so $n_R \sim L^{1-d/2} \sim f^{-[(2/d)-1]}$.

We now turn to the question of how models with bounded energy density behave. The simplest generalization of Eqs. (1) is to modify the elasticity term so that it remains bounded for all particle separations. If one does this, the arguments presented above are easily generalized to show that when F is nonzero different regions of the system must have different time-averaged velocities. Consider the generalization of Eqs. (1)

$$\dot{x}_i = k \sum_{\delta, nn} f_{\rm spring}(x_{i+\delta} - x_i) - V_i \sin(x_i - \beta_i) + F \ , \tag{7}$$

where the spring forces $f_{\rm spring}$ are now bounded, $|f_{\rm spring}(x_{i+\delta}-x_i)| < kS_{\max}$ for all i. The velocity v_L of a region of size L with no impurities in it must satisfy $v_L \geq F - kS_{\max}/L$. Clearly when L is large enough v_L is strictly greater than zero, even when the average pinning force for the entire system is greater than the applied force F. In the general case where the region has a nonzero density of impurities, as long as the threshold field of the region $F_T(L)$ can be defined and $F - F_T(L) - kS_{\max}/L > 0$, a similar argument applies. Similarly, one can show that anomalously strongly pinned regions remain stationary; regions with $F_T(L)$ such that $F - F_T(L) + kS_{\max}/L < 0$ will have $v_L = 0$ even if F is greater than the threshold field of the model in the absence of phase slips.

The applicability of the argument for inhomogeneous velocity depends crucially on the details of the dynamics of the CDW system. Equation (7) assumes that the pinning potential seen by each degree of freedom depends only on the index labeling the particle and not on the particle's position. This assumption is correct for the CDW because the variable x_j is interpreted as the phase at the jth impurity site. The impurity pinning strength depends only on the charge density and not on how many times the phase has wound at the site. However, if the x_j were to describe physical particles, then when a particle moves its local environment changes and Eq. (7) cannot be used to describe the moving state. The argument also relies on the fact that CDW wavelengths can be created and destroyed locally at phase slips so that the Eqs. (7) can apply in the interiors of the regions (which have no phase slips) without the need to transport particles into and out of the regions.[12] Figure 1 shows schematically how in one dimension CDW wavelengths are created and destroyed at phase slips to allow motion in an isolated portion of the CDW. In higher dimension we expect the phase-slip process to have nontrivial dynamics. For instance, in two dimensions we do not expect the CDW amplitude to vanish simultaneously everywhere along the perimeter of a region; rather, we expect point defects to form and then travel along the boundary of the region. We do not consider the dynamics of these processes here.

We now discuss the nature of the transition when a local region starts to move. Although the velocity versus force relation for model (1) in finite systems with periodic boundary conditions is continuous, periodic boundary conditions are not applicable when a finite region embedded in an infinite system depins. When a region breaks free the force exerted by the springs of the boundary changes markedly. The simplest case to analyze is a "rubber band" model where the spring force is linear in

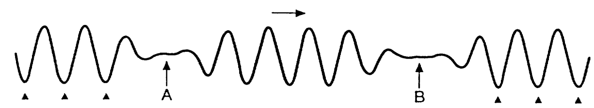

FIG. 1. Schematic drawing of how moving and stationary regions can coexist in a one-dimensional CDW. In this figure strongly pinned regions surround a region with no impurities. CDW wavelengths are created at a phase slip at point A and are destroyed at point B.

the particle separations up to a critical separation $S_{\max}$, after which the spring force is zero. For this case it is clear that the spring force exerted at the boundary of a moving region and stationary region is zero, since the separation between neighboring particles is unbounded at long times. Therefore, when the region breaks free, the spring force changes discontinuously as a function of driving force, leading to a jump in the velocity of the region. This argument makes it clear that there is no symmetry argument requiring the transition to a moving state be continuous; one must consider the dynamics of the depinning process.

In real CDW's the behavior at the phase slips is complicated. The elasticity force oscillates as the separation between neighboring particles increases; one simple form exhibiting this behavior is a spring force of the form $f_{\rm spring} \propto \sin[\alpha(x_{j+\delta} - x_j)]$. When the force is increased to the point where the region just starts to move, we expect the region to have adjusted so that each spring is exerting very close to its maximum force. As the region moves, each spring force oscillates. Although every spring will for some portion of the time exert close to its maximum force, unless the motion along the entire boundary is synchronized so that all the maxima are attained simultaneously, the spring force exerted along the boundary will never again reach the value it achieved at the start of the motion. This argument, which again leads to a discontinuous velocity versus force relation, is supported by some preliminary numerical investigations.[13]

The heuristic arguments are supported by the results of calculations for models which incorporate the possibility of phase slips. A mean field theory for the equation of motion (7) using a sinusoidal spring force with $\alpha = 1$ yields a jump in the velocity as the applied force is increased.[14] Simulations by Inui *et al.*[15] of a model with one phase and one amplitude degree of freedom also yield a discontinuous velocity characteristic because as they discuss, the phase stiffness is proportional to the CDW amplitude, which in turn is suppressed by strain. As the stiffness decreases, the strain increases, causing the stiffness to decrease still more. This feedback mechanism leads to two coexisting solutions, one with and one without a phase slip.

Although isolated regions have different values of the force where they depin and hysteresis of a single region is not observable in the thermodynamic limit, understanding how a region depins is crucial to determining the nature of the behavior for $d > 2$. In one and two dimensions we expect the CDW to break up into larger and larger disconnected regions moving at different velocities, implying that the threshold behavior present in Eqs. (1) is completely destroyed. For $d > 2$ a typical region does not have diverging strain at its boundary, so one expects a connected region to start moving all together at a well-defined value of the force.[16] However, when this region starts to move, a nonzero density of regions with stronger than typical pinning will remain stationary; the argument that shows this is exactly analogous to those presented above for the anomalously weakly pinned regions. When the connected region starts to move, a number of springs that is proportional to the system volume must break.[17] This process should lead to a jump in the velocity as a function of the applied force. Thus allowing for phase slips can fundamentally alter the nature of the transition even in situations where a connected region depins simultaneously.

The expectation that phase slips drive the depinning transition first order does not agree with current-voltage characteristics on many samples.[18] However, the arguments given above that phase slips drive the depinning transition first order in three dimensions ignore the possibility that the nontrivial dynamics of the regions with phase slips lead to fluctuations that act as effective noise sources for the system, causing rounding of the transition. This possibility is supported by the observation of rather ubiquitous broadband noise displayed by sliding CDW's, which is suppressed by mode locking to an external ac drive.[19] It is reasonable to associate phase slips with broadband noise because a model with one phase and one amplitude degree of freedom exhibits chaos in the phase-slipping regime,[15] and because different regions of the CDW which would move at different velocities under the influence of a dc force can all be synchronized to the same ac field, thus eliminating velocity inhomogeneity and the necessity of phase slips.[20] In addition, macroscopically inhomogeneous CDW velocity has been shown to be associated with substantially enhanced broadband noise amplitude.[21] However, detailed understanding of the role of phase slips in rounding the apparent threshold behavior and in generating the broadband noise must involve detailed understanding of the dynamics of the phase slips. This subject is not addressed in this paper.

To summarize this section, we have presented a scaling theory that indicates that the threshold transition present in the phase-only model of CDW's is fundamentally altered by the possibility of phase slips. In one and two dimensions as the force is increased the system breaks up completely into regions moving at different velocities and the threshold is destroyed. In three dimensions the velocity is also inhomogeneous, but since the size of the moving regions does not diverge at threshold, there is a connected region extending across the sample which starts to move at a well-defined value of the field. However, the depinning of this region is expected to be either driven first order or rounded by the generation of phase slips.

Some of the results of the scaling theory can be made rigorous for a simpler model which we now discuss.

III. RANDOM FRICTION MODEL

We examine in this section a simpler analytically tractable model first introduced by Mihály, Crommie, and Grüner.[22] In this model the pinning is modeled by a random static friction instead of a pinning potential that depends nonlinearly on the configuration variables x_j. The critical behavior (i.e., the v versus F relation) differs from that of Eq. (1), but the physics of the competition between randomness and interactions is retained. The equations of motion are

$$f_j = k \sum_{\delta} (x_{j,\delta} - x_j) + F - d_j \ ,$$
$$\dot{x}_j = \begin{cases} f_j & \text{if } f_j > 0 \\ f_j + 2d_j & \text{if } f_j < -2d_j \\ 0 & \text{otherwise} \ . \end{cases} \tag{8}$$

The x_j are the positions of the particles, F is the uniform force, the δ are the vectors to the nearest neighbors, and the d_j describe the static friction, which is assumed to have a random component. The reason this model is more tractable than Eqs. (1) is that the pinning force felt by each particle is independent of the configuration.

First we show that the threshold state for this model occurs at $F_T = \langle d_j \rangle$, where the brackets denote a spatial average, and that the threshold state satisfies $f_j = 0$ for every j. Thus, although Eqs. (8) are nonlinear ($\dot{x}_j$ depends nonlinearly on f_j), at (and above) threshold the equations are linear in the x_j's. This feature enables one to make substantial analytic progress.

The fact that the nonlinearity in the velocity versus force relation for each particle is not important at threshold enables one to make direct contact with work on a system of coupled oscillators with equations of motion of the form

$$\dot{\phi}_i = k \sum_{\delta} \sin(\phi_{i+\delta} - \phi_i) + \omega_i \ , \tag{9}$$

where ϕ_i describes the phase of the ith oscillator and ω_i describes its bare frequency.

For model (8), since each particle is stationary at threshold, by summing over j we see that the threshold field for this model F_T obeys $F_T \le \langle d_j \rangle$. We will show explicitly below that a solution to Eqs. (8) with each $f_j = 0$ exists. Since this configuration saturates the bound, it must be the threshold configuration. Another way to see the result is to consider a stationary configuration which has some $f_j^* = -\eta < 0$. If one uniformly translates every particle except j^* by an amount $[\eta/(kz)](1 - 1/N)$, where z is the number of nearest neighbors and N is the number of particles in the system, the new configuration has $z+1$ particles with $f_j < 0$. One can again uniformly translate the remaining particles and increase the number with $f_j < 0$. Eventually this process can be used to show that a configuration exists with every $f_j = -\eta/N$, so a static configuration with a larger value of F must exist. This construction can be performed for any $F < \langle d_j \rangle$.

Since $F_T=\langle d_j\rangle$, at $F=F_T$, $h_j\equiv(F_T-d_j)/k$ is a random variable with zero mean. Since by hypothesis the d_j's are uncorrelated random variables, it is clear that the distribution of pinning forces obeys the Gaussian statistics assumed in Sec. II.

Since the x_j's obey linear equations at threshold, it is straightforward to obtain analytic information about strains in the system. In one dimension an explicit solution is easily obtained: One defines new variables α_j and Δ_j which satisfy $\alpha_{j+1}-\alpha_j=h_j$, and $\Delta_j\equiv x_{j+1}-x_j+\alpha_{j+1}$. Just at threshold each $f_j=0$. Thus $\Delta_j-\Delta_{j-1}=0$ for all j, so that Δ_j must be independent of j. It is readily verified that $\alpha_j=\sum_{m=1}^{j-1}h_m+\alpha_0$, where α_0 is an arbitrary constant, so that

$$x_{j+1}-x_j=-\left[\sum_{m=1}^{j}h_m\right]+\alpha_0 . \tag{10}$$

Since the h_i are random variables with zero mean, the *separations* obey a random walk. Thus, for a system of size L, the maximum particle separation obeys $\delta x_{\max}\sim L^{1/2}$, which is unbounded. Thus this exactly soluble case yields results consistent with the scaling arguments given above [after Eq. (6)].

In $d\geq 2$ a simple closed form solution is lacking, but it is straightforward to calculate the spatial average $\langle(x_{j+\hat{x}}-x_j)^2\rangle$. The calculation arises in the context of oscillator entrainment described by Eq. (9).[23] One Fourier transforms Eqs. (8) (keeping in mind that each f_j is zero just at threshold). In two dimensions this yields

$$x_j=\sum_{\mathbf{q}\neq 0}e^{i\mathbf{q}\cdot\mathbf{R}_j}\frac{-\tilde{h}_{\mathbf{q}}}{2(2-\cos q_x-\cos q_y)} , \tag{11}$$

where

$$\tilde{h}_{\mathbf{q}}=\sum_j e^{i\mathbf{q}\cdot\mathbf{R}_j}h_j , \tag{12}$$

and $h_j=(F_T-d_j)/k$. Thus

$$\langle(x_{j+\hat{x}}-x_j)^2\rangle=\sum_{q\neq 0}|e^{i\mathbf{q}\cdot\hat{x}}-1|^2\frac{|\tilde{h}_{\mathbf{q}}|^2}{4(2-\cos q_x-\cos q_y)^2} . \tag{13}$$

The sum is dominated by very small values of q, where the cosine and exponential may be expanded. The random nature of the h's implies that $|h_{\mathbf{q}}|\to h_0$ as $q\to 0$, where h_0 is constant, so that one finds $\langle(x_{j+\hat{x}}-x_j)^2\rangle\propto\ln L$. In one dimension the analogous calculation yields $\langle(x_{j+\hat{x}}-x_j)^2\rangle\propto L$, in agreement with the exact solution given above.

In $d>2$ the typical strains remain bounded, but it can be shown that a nonzero density of regions with unbounded strains does occur for the random friction model because rare fluctuations of $F_T(L)$ cause unbounded strains. The argument for entrained oscillators[24] is easily adapted to the random friction model. Below threshold, the dominant contribution comes from regions with no impurities at all. Since the pinning force is zero in impurity-free regions, the argument outlined in Sec. II for impurity-free regions of the original CDW equations applies (and is easily made mathematically rigorous).

IV. CDW MODEL

In this section we discuss the applicability of the arguments given above to the FLR model of CDW's. The presence of Griffiths-like singularities leading to velocity inhomogeneities is demonstrated. The scaling argument describing the strains at threshold is shown to apply to the CDW equations of motion (1) in certain limiting cases.

Once again we assume that Eqs. (1) apply. By summing Eqs. (1) over any region R, one finds that the time average of the velocity of R, v_R, obeys

$$\Omega_R v_R=\sum_{B(R)}k(x_{i+\delta}-x_i)+F\Omega_R-\langle F_{\text{pin}}(R)\rangle\Omega_R , \tag{14}$$

where Ω_R is the volume of the region, $B(R)$ signifies the boundary of R, and $\langle F_{\text{pin}}(R\rangle$ is the mean pinning force $(1/\Omega_R)\sum_{i\in R}V_i\cos(x_i-\beta_i)$. Equation (14) applies to any region R, including regions whose volumes scale like L^d and boundaries scale like L^{d-1}. The cancellation of the spring forces in the interior of the region follows from Newton's third law, so the scaling of the spring force with size of the region is the same for Eqs. (1) as for the random friction model discussed in Sec. III. However, the pinning force $\langle F_{\text{pin}}(R)\rangle$ is more complicated than in the random friction model.

Definition of local pinning force

We would like to understand the distribution of pinning forces $\langle F_{\text{pin}}(R)\rangle$ for different regions. However, first we would like to demonstrate that $\langle F_{\text{pin}}(R)\rangle$ is well defined. This point is not an issue for the extremely rare regions with either no impurities or with very strong pinning, but the scaling arguments in one and two dimensions depend crucially on it. Changing the boundary conditions for a region of size L is expected to lead to a change in $\langle F_{\text{pin}}(R)\rangle$ of order $1/L$ because any small change in the boundary condition changes the configuration over a finite length scale, which we expect to be the Lee-Rice length. Since the choice of boundary condition at the boundary of the region is arbitrary, $\langle F_{\text{pin}}(R)\rangle$ has an uncertainty of order $1/L$. However, in addition to this ambiguity, a possible $O(1)$ effect arises because the pinning force depends on the system's configuration, which can change everywhere inside the region as the strain at the boundary rises. However, there is no obvious reason why the pinning force in the strained configuration should be systematically larger than when the external strain vanishes, and if it is not, then the arguments demonstrating phase slips go through.

The general discussion of the definition of $F_T(L)$ must account for the dependence of the pinning force on the system's configuration. An example of the possible problems arises when the pinning is by a periodic potential $[V_j=V_0=\text{const}, \beta_j=2\pi(\alpha j \bmod 1)$, with α a constant$]$. If one allows for changes in the mean particle spacing, then α can change, which can lead to large changes in the pinning force.[25] For example, if $\alpha=1$, then the local pinning force is V_0, whereas if $\alpha=(\sqrt{5}+1)/2$ then for $V_0<0.972$ the local pinning force is zero.[26] We do not expect the pinning force to depend so strongly on the configuration when the pins are randomly distributed, but we have no proof of this.

A situation where some information about the variation of threshold force with configuration is available is the simple case of commensurate pinning, which corresponds to Eqs. (1) with $V_i=V_0$ and $\beta_i=$const. Since the sine is bounded above by unity, the pinning force per particle is bounded above by V_0. Thus $F_T(L)\le V_0$ for any configuration. One could imagine a system with $V=V_0$ inside the region R and $V_1>V_0$ outside R. If the force F is adjusted to be between V_1 and V_0, the force exerted along the boundary of R to keep it from moving is bounded below by $(F-V_0)L^d$.

Similarly, in the strong pinning limit $V_i\gg k$, one expects the fluctuations in the V_i to set the scale of the fluctuations in the local threshold. Indeed, one can show that up to boundary terms, $F_T(L)\le\Omega^{-1}\sum V_i$, where Ω is the volume of the region. However, since large distortions build up close to threshold even for arbitrarily strong pinning, the system probably does not saturate this bound. Thus, though it is clear that the force needed to keep the region is greater than $(F-\Omega^{-1}\sum V_i)L^d$, this is insufficient to show that the force exerted by the boundary springs is bounded below by $[F-F_T(L)]L^d$.

Showing that the force exerted by the boundary springs is bounded above by the scaling form $[F-F_T(L)]L^d$ is difficult because metastable configurations exist where the boundary springs exert a force scaling as FL^d. We expect that the physically relevant process of slowly increasing the external driving force F leads to the minimum possible value of the boundary force from the springs, but in any case this scaling only makes phase slips more likely, so it does not affect any of the conclusions of this paper.

A final caveat concerns the scaling argument in two dimensions. We expect $F_T(L)$ of a region of size L to be defined only up to order $1/L$ because changes in the boundary conditions at the edge of the region will cause the threshold force to change by an amount of the order the surface-to-volume ratio of the region. In one dimension this uncertainty in the threshold force is much smaller than the $1/\sqrt{L}$ fluctuations caused by fluctuations in the impurity density. However, in two dimensions the ambiguity in $F_T(L)$ is of the same order as the typical threshold fluctuation of a region of size L, so strictly speaking our scaling argument is ill defined. This complication does not occur for the random friction model because the local pinning force is independent of the system configuration. For the CDW model detailed numerical investigation may be necessary to resolve whether the scaling argument applies in two dimensions.

Although the definition of $F_T(L)$ is not clearly unambiguous, in three dimensions the behavior is dominated by rare regions for which these complications do not arise. For instance, above threshold one can focus on regions that are so strongly pinned that their configuration does not change as the force is increased. These regions clearly remain stationary even if most of the system is moving and hence must act as a source of phase slips. Similarly, regions with no impurities at all are easily treated and lead to the coexistence of moving regions when the bulk of the CDW is stationary.

V. OTHER MODELS

In this section we discuss the relevance of our results to other systems which have been described by phase-only equations similar to the FLR model considered here. There are two issues: one is whether the phase-only description breaks down and the other is what it is replaced by. The essential assumptions are that: (a) the driving force acts uniformly through the system while the spring force holding back a more weakly pinned region acts only as its boundary and (b) a phase slip, a local region where the CDW amplitude is driven to zero and where the CDW current is converted to normal current or vice versa, is not forbidden by any conservation law.

Assumption (a) breaks down in cases where the driving force is applied at a boundary and transmitted by elastic forces through the sample. Therefore the arguments given here cannot be applied to the question of whether a glassy material will flow to relieve an applied strain, although they may be applicable to the case of glass in a gravitational field.[27] Assumption (a) can also break down if the elastic interactions are long range.

We now consider the situation when assumption (a) holds but assumption (b) does not. For instance, one could imagine a situation of a CDW with short-range elastic interactions where interconversion between CDW and normal carriers is not allowed. In this case the arguments for the breakdown of the FLR model clearly apply, but a finite density of unpinned sites in an otherwise pinned CDW cannot occur. In the unpinned regions a current (proportional to $\dot{\phi}$) must flow; if there are no normal carriers this current cannot flow in the pinned regions. Thus the simple picture of phase-slip centers is not relevant, and it is not clear how the diverging strains predicted by the FLR model are accommodated. It is clear that the system is not describable in terms of long-wavelength deformations of an elastic medium, but it is entirely possible that the resulting configuration is static. One might expect this situation to apply to the case of a weakly pinned CDW with a negligible density of normal carriers, but for this case one expects the CDW Coulomb interactions to be unscreened and hence long range, and so our arguments, which assume short-range forces, do not apply.

Magnetic flux lines in type-II superconductors are another interesting system which have been described by FLR-like equations of motion and for which assumption (a) but not assumption (b) holds. In the absence of impurities at zero temperature, the flux lines form a triangular lattice which is described by an order parameter of the form $\sum_{\mathbf{Q}}\rho_{\mathbf{Q}}\cos(\mathbf{Q}\cdot\mathbf{x}+\phi_{\mathbf{Q}})$. Impurities lead to distortions of the flux lattice.[28] When a current is passed through the superconductor, a Magnus force proportional to the current acts on each flux line. If one assumes that only long-wavelength distortions of the flux lattice occur, then one can write equations of motion for the distortions that are similar to the FLR equations of motion.[7] The arguments presented above then show that when the current density is nonzero, long-wavelength distortions alone cannot describe the flux configuration.

However, going beyond the FLR equations involves a number of subtleties. First, Eqs. (7) only apply if the flux lattice is ordered in the region under consideration. In general, because flux lines have an existence independent of the lattice, a model of independent entities moving in the sample is more appropriate than one of a locally ordered lattice. Further, a flux line running through a sample involves circulating currents which extend to the boundaries of the sample. The total circulating current is fixed by the magnetic field, and therefore the total number of flux lines is conserved. Thus nucleation of a flux line (the analog of a phase-slip center) can only occur at the boundaries of the sample. In the bulk of the sample one must create or destroy flux loops (in three dimensions) or vortex-antivortex pairs (in two dimensions). There is not yet a theory of the creation of these objects by an applied current at zero temperature.

To see the differences caused by these effects, consider the behavior of a region with no impurities in it. It is possible for the flux lines in the interior of this region to move, leaving behind a void. Therefore, once the vortices move out of the region into a region with stronger pinning, the motion stops. Inside the void the flux lattice has been completely destroyed, so that Eqs. (7) no longer apply. Unless this case can be excluded, the argument that the spatial average of the velocity is nonzero does not go through; the arguments in this paper imply that the system cannot be described as a lattice with small distortions, but they do not rule out the possibility of a well-defined threshold transition.

Despite this complication, the arguments in this paper can be applied straightforwardly to show that stationary regions of the flux lattice remain even when the bulk of the system is moving. Therefore understanding the dynamics of the moving state involves a detailed knowledge of the interactions between stationary and moving regions, and previous work ignoring the possibility of coexisting stationary and moving regions is incomplete.

The case of two-dimensional flux flow is very interesting and deserving of further study. Since as threshold is approached larger and larger regions have strains larger than $S_{\max}$, it is not possible to empty the weakly pinned regions and create voids while maintaining the connectedness of the system. Thus it is plausible that the depinning transition in the absence of grain boundaries is fundamentally altered. However, at this stage our understanding of the behavior is incomplete.

In type-II superconductors, it is very important to understand the effects of thermal fluctuations on the motion of the flux lines, in contrast to the CDW case where thermal effects are very small. The most naive picture of finite temperature is to consider it as a situation where each particle in the system has a nonzero (though very small) mobility. However, this view is clearly oversimplified and investigation of the model at finite temperature and large driving forces is clearly in order.

VI. DISCUSSION

This paper concerns the zero-temperature properties of elastic medium theories in the presence of uncorrelated randomness and an external force. It is shown that such theories are inadequate for a description of physical systems such as charge-density waves and flux lattices in type-II superconductors because they contain regions with strains that diverge as the system is made larger. These results are demonstrated using scaling arguments as well as exact results on a simplified model. For charge-density waves, the arguments are easily generalized to show that the CDW velocity is nonzero for all applied fields and that critical behavior present in the elastic medium model does not survive when phase slips are allowed. Other systems such as flux lattices in type-II superconductors are more complicated, and detailed understanding of how defective regions affect the dynamics remains a challenging issue.

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[1]In this paper we ignore the coupling of the CDW to the underlying lattice except insofar as it affects the CDW viscosity. This assumption is generally accepted to be valid in systems with sliding CDW's such as $NbSe_3$, TaS_3, and $K_{0.3}MoO_3$; see, e.g, L. Sneddon, M. C. Cross, and D. S. Fisher, Phys. Rev. Lett. **49**, 292 (1982).

[2]For review of CDW's see, e.g., G. Grüner, Rev. Mod. Phys. **60**, 1129 (1988).

[3]H. Fukuyama and P. A. Lee, Phys. Rev. B **17**, 535 (1978); P. A. Lee and T. M. Rice, *ibid.* **19**, 3970 (1979).

[4]See, e.g., D. S. Fisher, Phys. Rev. B **31**, 1396 (1985); S. N. Coppersmith and D. S. Fisher, Phys. Rev. A **38**, 6338 (1988); P. Sibani and P. B. Littlewood, Phys. Rev. Lett. **64**, 1305 (1990); P. B. Littlewood and C. M. Varma, Phys. Rev. B **36**, 480 (1987); J. B. Sokoloff, Phys. Rev. B **31**, 2270 (1985); A. A. Middleton and D. S. Fisher, Phys. Rev. Lett. **66**, 92 (1991); C. R. Myers and J. P. Sethna (unpublished).

[5]S. N. Coppersmith, Phys. Rev. Lett. **65**, 1044 (1990).

[6]The relationship between this model and the original FLR model is discussed by L. Pietronero and S. Strässler, Phys. Rev. B **28**, 5863 (1983); P. B. Littlewood in *Charge-Density Waves in Solids,* edited by L. P. Gor'kov and G. Grüner (Elsevier, Amsterdam, 1989); D.S. Fisher, Ref. 4.

[7]A. Schmid and W. Hauger, J. Low Temp. Phys. **11**, 667 (1973).

[8]S. N. Coppersmith, Phys. Rev. B **44**, 2887 (1991).

[9]We note that there is no approximation involved in the selection of a region and its boundary; for the phase-deformation model to be consistent, the spring forces must be bounded for

every choice of region.

[10]M. Randeria, J. Sethna, and R. Palmer, Phys. Rev. Lett. **54**, 1321 (1985).

[11]R. B. Griffiths, Phys. Rev. Lett. **23**, 17 (1969).

[12]Motion of local regions has been shown to have measurable consequences for the current-voltage characteristic; see R. P. Hall, M. F. Hundley, and A. Zettl, Phys. Rev. Lett. **56**, 2399 (1986). This result is reasonable because there is a parallel conduction channel of uncondensed electrons. For a given total current flow, if the CDW carries current even partway across the sample, the total voltage drop decreases compared to having the uncondensed carries only. Additionally, it is important that normal carriers be present so that their motion can prevent buildup of charge at the phase-slip centers, so that a model with short-range forces only is appropriate.

[13]J. Kuo (unpublished).

[14]S. H. Strogatz *et al.*, Phys. Rev. Lett. **61**, 2380 (1988).

[15]M. Inui *et al.*, Phys. Rev. B **38**, 13 047 (1988).

[16]It is clear that an infinite connected region has spatially uniform time-averaged velocity because otherwise the spring forces must become unbounded in the limit of long times.

[17]This result follows because there is a nonzero *density* of regions with phase slips at their boundaries.

[18]S. Bhattacharya, M. J. Higgins, and J. P. Stokes, Phys. Rev. Lett. **63**, 1508 (1989).

[19]M. S. Sherwin and A. Zettl, Phys. Rev. B **32**, 5536 (1985).

[20]S. N. Coppersmith, Phys. Rev. A **36**, 3375 (1987).

[21]M. P. Maher, T. L. Adelman, J. McCarten, D. A. DiCarlo, and R. E. Thorne, Phys. Rev. B **43**, 9968 (1991).

[22]L. Mihály, M. Crommie, and G. Grüner, Europhys. Lett. **4**, 103 (1987). A similar model is also discussed by I. Webman, Philos. Mag. B **56**, 743 (1987).

[23]H. Sakaguchi *et al.*, Prog. Theor. Phys. **77**, 1005 (1987).

[24]S. H. Strogatz and R. E. Mirollo, J. Phys. A **21**, L699 (1988); Physica D **31**, 143 (1988).

[25]See, e.g., S. Aubry, in *Solitons and Condensed Matter,* edited by A. Bishop and T. Schneider, Solid State Sciences, Vol. 8 (Springer, Berlin, 1978), pp. 264–278; S. N. Coppersmith and D. S. Fisher, Phys. Rev. B **28**, 2566 (1983).

[26]See, e.g., M. Peyrard and S. Aubry, J. Phys. C **16**, 1593 (1983).

[27]C. C. Yu and S. N. Coppersmith, J. Non-Cryst. Solids **131-133**, 476 (1991).

[28]A. I. Larkin and Yu. N. Ovchinnikov, J. Low Temp. Phys. **34**, 409 (1979).