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Universal conductivity of two-dimensional films at the superconductor-insulator transition

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The zero-temperature universal conductivity of two-dimensional (2D) films at the superconductor-insulator transition is studied. The existence of a finite conductivity at $T = 0$ and the universality class for this transition is discussed. Neglecting disorder as a first approximation, so the transition is from a commensurate Mott-Hubbard insulator to a superconductor, we calculate analytically the universal conductivity for the 2D pure boson Hubbard model up to the first order in a large-N expansion and numerically by Monte Carlo simulation of the (2+1)-D XY model. From the Monte Carlo results we find the universal conductivity to be $\sigma^* = (0.285 \pm 0.02)\sigma_Q$, where $\sigma_Q^{-1} \equiv R_Q \equiv h/(2e)^2 \approx 6.45$ kΩ. An analysis in one dimension suggests that in the presence of disorder, the universal conductivity in films might be somewhat smaller than this value. The possible existence of universal dissipation in ^{4}He films is also discussed briefly.

I. INTRODUCTION

Recent experimental attention has focused on the destruction of superconductivity by disorder in thin two-dimensional (2D) films.[1–4] By systematically varying the thickness of amorphous films *in situ*, it has been possible to tune through a ($T = 0$) superconductor-insulator transition.[1,2] Films thinner than some critical thickness show insulating behavior at low temperatures, whereas thicker films become superconducting. This transition was probed in earlier work[3] by varying the microscopic disorder in amorphous-composite indium oxide (α-InO$_x$) films, at fixed thickness. A closely related superconductor-to-insulator transition has been studied in artificially constructed Josephson-junction arrays[4] by systematically varying the ratio of charging and Josephson energies. In each of these experiments, the normal-state sheet resistance, R_N (measured at some specified temperature well above the material's bulk transition temperature) can be viewed as a control parameter used to tune through the superconductor-insulator transition, in much the same way as temperature is used to tune through thermal ($T \neq 0$) phase transitions. Over the years a number of groups[5–7] have observed that the critical value of this *normal state* resistance, R_N^*, which corresponds to the border between superconducting and insulating (or nonsuperconducting) behavior at *low* temperatures, is close to $R_Q = h/(2e)^2 \approx 6.45$ kΩ, the quantum unit of resistance. A number of theoretical papers[8–14] have been written trying to explain this observation.

Recently a scaling theory for the superconductor-insulator transition has been advanced, by two of us and Geoffrey Grinstein,[15,16] which predicts that in 2D, right at the transition, the system will be metallic with a finite resistance at $T = 0$. Moreover, the value of the resistivity in this $T = 0$ metallic state, R^*, was predicted to be universal, insensitive to microscopic details and depending only on the universality class of the transition.[15] The notion of a universal resistivity *at* $T = 0$ at the superconductor-insulator transition in 2D should *not* be confused with the earlier and different idea of a "universal" critical[8–12,14] value of the *normal* state resistance, R_N^*. Since the resistance has a temperature dependence in the normal state, the latter quantity, R_N^*, will clearly depend on the temperature at which it is defined, and thus cannot be truly "universal." In contrast, R^* is uniquely defined, being a property of the film *at* zero temperature. We argue below that R^* will be close to, but not precisely equal to the quantum of resistance R_Q.

From the modern scaling theories of the 2D free-electron system,[17] it is understood that even with small disorder all electron states are localized and diffusion is absent. Here we wish to emphasize differences from the disordered 2D fermionic system. We argue below that the relevant degrees of freedom near the superconductor-insulator transition are bosonic. Moreover, the repulsive interactions between the bosons cannot be neglected, since otherwise at $T = 0$ all particles will Bose condense in the lowest energy state which, in the presence of disorder, is localized.[18,19] We assume that these differences define a distinct universality class for the superconductor-insulator transition in disordered 2D films and R^* can be determined from the properties of this universality class.

Since all systems on the critical surface in parameter space flow to the same fixed point, it seems plausible that R^* might be universal. However, not all quantities related to a critical surface are universal. Indeed, at usual finite temperature transitions, for example, in the Ising model, the values of the microscopic coupling constants which place the system on the critical surface are nonuniversal due to the existence of irrelevant operators. We might worry that R^* depends on these nonuniversal microscopic coupling constants. However the analysis we present below provides strong evidence that the critical conductivity, $\sigma^* \equiv R^{*-1}$, is more like a universal amplitude than a coupling. Moreover, according to the "two-scale-factor universality" hypothesis,[20] all *dimensionless* critical point amplitudes are universal.[21] A well-known example is the universal jump of the superfluid density at the Kosterlitz-Thouless transition in the 2D classical XY model.[22] The conductivity is dimensionless for the 2D quantum case and is therefore also universal.[15,23,24]

What universality class is the fixed point in? We hypothesize that the $T = 0$ superconductor-insulator phase transition is correctly described by a model of interacting charge $2e$ bosons moving in a 2D random potential.[15] Loosely speaking we assume that the Cooper pairs have already formed prior to the transition to superconductivity. More specifically, we expect that the Cooper pair size (defined below) remains finite at the transition, even though a pair coherence length, ξ, diverges. In this case, on the scale of ξ the Cooper pairs look like point bosons near the transition. To be more concrete, consider the spatial dependence of the pair correlation function:

$$G_2(\mathbf{x}_1, \mathbf{x}_1'; \mathbf{x}_2, \mathbf{x}_2') = \langle c_\uparrow^\dagger(\mathbf{x}_1) c_\downarrow^\dagger(\mathbf{x}_1') c_\downarrow(\mathbf{x}_2) c_\uparrow(\mathbf{x}_2')\rangle, \tag{1.1}$$

where $c_\sigma(\mathbf{x})$ is an electron annihilation operator with spin σ. In the superconducting phase as $|\mathbf{x}_1 - \mathbf{x}_2| \to \infty$ (with $|\mathbf{x}_1 - \mathbf{x}_1'|$ and $|\mathbf{x}_2 - \mathbf{x}_2'|$ finite) G_2 in Eq. (1.1) factorizes as $G_2 \approx \psi^*(\mathbf{x}_1 - \mathbf{x}_1')\psi(\mathbf{x}_2 - \mathbf{x}_2')$, where $\psi(\mathbf{x})$ is the Cooper pair wave function. One expects that $\psi(\mathbf{x}) \sim \exp(-|\mathbf{x}|/\xi_p)$, which defines the Cooper pair size, ξ_p. In the insulating phase, but close to the transition, as $|\mathbf{x}_1 - \mathbf{x}_2| \to \infty$, G_2 should still factorize, taking the form

$$G_2 \approx e^{-|\mathbf{x}_1 - \mathbf{x}_2|/\xi} \psi^*(\mathbf{x}_1 - \mathbf{x}_1')\psi(\mathbf{x}_2 - \mathbf{x}_2') . \tag{1.2}$$

The Cooper pair size can then be defined from $\psi(\mathbf{x})$, as above, even in the insulating phase. We hypothesize that upon approaching the ($T = 0$) transition to superconductivity, ξ diverges but ξ_p remains finite.

From Eq. (1.2) it is clear that in the insulating phase ξ is essentially a Cooper *pair* localization length. A single electron "localization" length, ξ_1, can also be defined in terms of the imaginary part of the single electron (rather than pair) retarded Green's function $G(\mathbf{x}, \mathbf{x}'; E - \mu)$:

$$\frac{1}{\Omega} \int_{\mathbf{x}'} |\mathrm{Im} G(\mathbf{x} + \mathbf{x}', \mathbf{x}'; 0)|^2 \simeq e^{-|\mathbf{x}|/\xi_1} . \tag{1.3}$$

Here Ω is the system volume. This single electron localization length, while clearly finite in the insulating phase, is also expected to be finite in the superconducting phase since electrons can propagate coherently only as pairs. Indeed, ξ_1 should be roughly comparable to ξ_p, and, like the pair size, should remain finite at the transition. In a Fermi liquid phase ξ_1 would be infinite, but since the transition is directly from insulator to superconductor it seems likely to us that it remains finite (noncritical). This scenario is quite closely analogous to nematic transitions in classical XY models,[25] wherein $\langle e^{2i\varphi}\rangle$ orders but the XY order parameter $\langle e^{i\varphi}\rangle$ remains zero and the associated ferromagnetic correlation length noncritical. The nature of the competition between localization and superconductivity on the insulating side of the transition remains an active area of study and debate.[19,26–29]

Our hypothesis that the superconductor-insulator transition is in the same universality class as bosons moving in a random potential is supported by direct calculation for the 1D case. Giamarchi and Schulz[30] solve a model of fermions with an attractive interaction and show that the universality class for the transition from insulator to s-wave superconductor is the same as the superfluid to insulator transition in a model of disordered, interacting (repulsive) bosons; that is, the fermi degrees of freedom are not relevant. Since (1+1) dimensions is the "lower critical dimension" for this transition,[16] we do not expect any logarithmic corrections to σ^* in (2+1) dimensions, as found for disordered noninteracting electrons[17] (i.e., weak localization). There remains however the possibility that fermionic degrees of freedom cannot be neglected. This would almost certainly mean the existence of marginal variables (as for a Fermi liquid) which would destroy our assumption of a single fixed point. (This point is discussed further in Sec. IX.) We note that the spin susceptibility might prove to be a useful probe of the nature of the insulating state near the transition point.

Another possible effect has recently been noted which is argued to occur in granular systems. For Josephson coupling between large grains mediated by hopping through a small grain, the sign of the Josephson coupling can be reversed.[31,32] This leads to a random frustration,[32,33] in granular systems which, while not necessarily invalidating the boson picture, could change the boson universality class. However, if such effects are important, one would expect to see them also in the classical regime at finite temperatures, perhaps even destroying completely the finite temperature superconducting phase.[32]

In order to discuss the scaling properties of the resistance *near*, but not precisely at, the superconductor-insulator transition, it is useful to define a parameter δ which measures the "distance" to the $T = 0$ transition.

For example, δ can be taken as $\delta = (\ell_c - \ell)/\ell_c$, where ℓ is the film thickness and ℓ_c the critical thickness, or alternatively $\delta = (R_N - R_N^c)/R_N^c$ with R_N the film's normal state resistance. In either case, at $T = 0$ the frequency-dependent resistance near the transition, i.e., at small δ and ω, is expected to satisfy a scaling form[15,16,23]

$$R(\delta, T = 0, i\omega) = (h/4e^2)P(a\delta/\omega^{1/z\nu}), \quad (1.4)$$

where $P(x)$ is a dimensionless scaling function of the dimensionless argument, $x \equiv a\delta/\omega^{1/z\nu}$, and z and ν are critical exponents of the $T = 0$ transition. A particular choice of normalization, say $\partial_x P(x)|_{x=0} = 1$, makes the scaling function $P(x)$ universal. The constant a is nonuniversal. Away from the transition, $\delta \neq 0$, as $\omega \to 0$ the resistance diverges on the insulating side ($\delta > 0$) and vanishes on the superconducting side ($\delta < 0$) of the transition. For small but nonzero frequencies, as δ is taken to zero and the transition is approached, the resistance will have a finite limit, since finite dissipation is always possible at finite frequencies. The limiting value at the transition, $R^* = (h/4e^2)P(0)$, being independent of ω (for small enough ω) will remain finite as $\omega \to 0$ and should be universal. The bulk of this paper is devoted to evaluating R^* from Eq. (1.4).

Experimentally, it is of course impossible to work at zero temperature, and moreover resistance measurements are typically dc. Thus, when comparing with experiments, a more appropriate scaling form is the temperature-dependent dc resistance:[15,16,23]

$$R(\delta, T, \omega = 0) = (h/4e^2)Q(b\delta/T^{1/z\nu}) . \quad (1.5)$$

This form should be valid for small enough δ and T. As in Eq. (1.4), $Q(y)$ is a dimensionless universal scaling function $[\partial_y Q(y)|_{y=0} = 1]$. Again, away from the transition, $\delta \neq 0$, the resistance either vanishes or diverges in the zero temperature limit depending on the sign of δ. For finite temperatures as the transition is approached, say from the insulating side (i.e., $\delta \to 0+$), the resistance is again expected to have a finite (and temperature-independent) limit. This is because at $T \neq 0$ conduction is always possible ($R < \infty$), and being out of the superconducting phase R will be nonzero. The simplest and most natural scenario is that the resistance at the transition in this limit, $(h/4e^2)Q(0)$, will be the *same* as that obtained from Eq. (1.4). Indeed, it seems likely to us that for real disordered films, no matter how the transition ($\delta \to 0$) is approached, be it at finite ω, finite T, or with both T and ω finite, the resistance obtained will have the *same* unique and universal value.

Unfortunately, for the disorder-free boson models that we study in the bulk of this paper, there appear to be further subtleties. Specifically, for the disorder-free boson-Hubbard model at commensurate density which we consider in Secs. III–VI, although the resistance obtained from Eq. (1.4) is finite and universal at the transition, the dc resistance at finite temperatures, Eq. (1.5), is apparently *zero,* at least within the context of a controlled large-N expansion. This can be traced to the fact that for a collection of particles in a translationally invariant system, or for particles on a periodic lattice but in the absence of umklapp scattering, the dc resistance vanishes identically, because the particles are free to accelerate. This "pathology" will clearly not be present in real disordered films which are *not* translationally invariant. (Real disordered films, at finite temperatures away from the superconducting phase, will most certainly have a nonvanishing dc resistance.) Because of this "pathological" feature of the pure boson Hubbard model, we confine our analysis of this model to $T = 0$, and evaluate the universal resistance from Eq. (1.4).

In this paper we discuss various methods to estimate and evaluate the universal $T = 0$ conductivity, σ^*, at the superconductor-insulator transition in 2D film systems. A brief outline is as follows. Duality between vortices and charges will be discussed briefly in Sec. II. In Sec. III, as a first approximation, disorder is neglected and a pure boson Hubbard model is considered. When the boson density is commensurate with the lattice, this model describes a transition from a Mott-Hubbard insulator to a superconductor. Although this transition is expected to be in a different universality class from that of the experimental systems,[16] where the insulating state arises from disorder,[16] it is, nonetheless, useful to study it as a first step toward a theory of the disordered system. Section IV describes a simple mean-field treatment, formulated in a particle-number representation, which allows for an estimate of the universal conductivity in the pure boson Hubbard model. In Sec. V, the first order large-N correction to the Hartree approximation[15,34] is computed exactly for this model and, compared, in Sec. VI, with Monte Carlo calculations for the 3D XY model. Even though zero-disorder systems are not in the appropriate universality class for real disordered films, our results illustrate various approaches to the theoretical calculations and provide an estimate of σ^*. Furthermore these results clearly demonstrate the validity of our central idea: the (2+1)D critical conductivity is a dimensionless universal *amplitude* analogous to (for example) the universal jump in superfluid density in the 2D Kosterlitz-Thouless transition.

Superconductor-insulator transitions in one dimension are considered in Sec. VII, wherein an appropriate universal conductance is defined. This conductance is then evaluated at both the Mott insulator to superconductor transition in the pure case, and the localized insulator to superconductor transition in the presence of disorder. The possible existence of universal dissipation in ^{4}He films is discussed in Sec. VIII and Sec. IX is reserved for a summary and discussion.

II. VORTEX FLOW AND DUALITY PICTURE

Since we are assuming that it is sufficient to use a boson description of the degrees of freedom of the system, we can consider the following *gedankenexperiment* to estimate the universal film resistance. As shown in Fig. 1, let the film be fed by electrodes which are bulk superconductors characterized by a phase $\theta = 0$ for the lower electrode and some time-varying $\theta(t)$ for the upper electrode. The boson degrees of freedom of the film are described by a complex field $\psi(\mathbf{r}, t)$ whose phase connects smoothly to that of the electrodes. The film is a

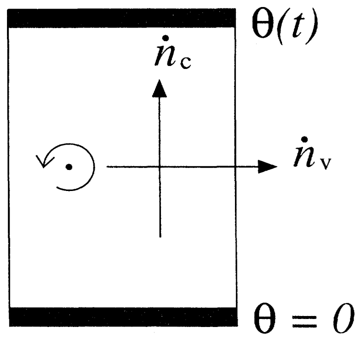

FIG. 1. Schematic diagram for vortex currents and charge current between two electrodes with phases $\theta = 0$ and $\theta = \theta(t)$.

"weak link," that is, a place where phase slips can occur. Resistance in the film is then due to vortex flow. These vortices can be thermally generated which causes the finite-temperature superconducting transition to be of the Kosterlitz-Thouless type. However for films of the critical thickness, T_c is driven to zero. At $T = 0$, vortices are present at the superconductor-insulator transition and in the insulating phase purely due to quantum fluctuations.

The voltage across this weak link is given by the Josephson relation

$$V = \frac{\hbar}{2e}\dot{\theta} = \frac{h}{2e}\dot{n}_v \; , \tag{2.1}$$

where we have used the fact that the rate of phase slip is proportional to the flux $\dot{n}_v$ of vortices flowing across the sample perpendicular to the current. More precisely, $\dot{n}_v$ is the flux of *positive* vortices to the right in Fig. 1, plus the flux of negative vortices to the left. Each vortex gives a phase slip of 2π. Similarly, the current is given by

$$I = 2e\dot{n}_c \; , \tag{2.2}$$

where $\dot{n}_c$ is the flux of Cooper pairs flowing between the electrodes. The resistance is then

$$R = \frac{V}{I} = \frac{h}{(2e)^2}\left(\frac{\dot{n}_v}{\dot{n}_c}\right) \; . \tag{2.3}$$

In the superconducting phase there are no free (i.e., unbound) vortices, so that $\dot{n}_v$ and the (linear) resistance vanishes. In the Mott-Hubbard insulator charge is immobile due to the gap, so that $R = \infty$. In the disordered insulating phase, (Cooper pair) charge is immobile due to localization. It is natural to speculate that right at the superconductor-insulator transition, *both* charge and vortices are mobile. If they are, and, moreover, exactly one vortex crosses the system for each Cooper pair which flows through the system, then the resistance is precisely $h/(2e)^2$, the quantum resistance. Our Monte Carlo results, described in Sec. VI, reveal that at the Mott insulator to superconductor transition, $R^* \approx 3.5R_Q$, a value close to but not equal to R_Q. Why might this be?

It turns out that there exists a duality transformation of the path integral description of the system which interchanges the roles played by charges and vortices.[15,34–38] Roughly speaking, the idea is the following. A vortex is an excitation which gives a Berry's phase of $\pm 2\pi$ if a particle adiabatically encircles it. Thus a stationary vortex has a particle current winding around it. Conversely, we know from the fractional quantum Hall effect[39] that if a vortex is adiabatically dragged around a closed path, the Berry's phase is $\pm 2\pi N$ where N is the expected number of charges inside the path. Hence (as pointed out very early in a seminal paper by Haldane[40]), if we view vortices as particles, they see the original charges as their vortices. Thus a particle can be localized by having a current of vortices flowing around it. This causes the local phase to be completely uncertain and hence allows the particle number to be certain.

We can imagine interchanging the roles played by the charges and vortices. If the action were invariant under this operation, the superconductor-insulator transition would be self-dual. In this case, right at the transition one would expect $\dot{n}_v = \dot{n}_c$, and the critical resistance would be precisely R_Q. There are two reasons why the superconductor-insulator transition in real films is *not* expected to be self-dual. Firstly, in zero magnetic field there is a symmetry between positive and negative vortices. Such a "particle-hole" symmetry is presumably not present for the Cooper pair charges in real 2D films. (The model studied in Secs. III and IV below which describes a transition from a commensurate Mott-Hubbard insulator to superconductor in the *absence* of disorder *does* have a boson "particle-hole" symmetry.) Secondly, vortices interact logarithmically whereas (Cooper-pair) charges interact via a $1/r$ potential. However, it seems plausible that the transition might be "close" to being self-dual.

An applied perpendicular magnetic field breaks the symmetry between positive and negative vortices. Thus the magnetic-field-tuned superconductor-insulator transition,[38] which is believed to be in a different universality class from the zero-field transition, is perhaps "closer" to being self-dual. Indeed, the field-tuned transition for a model of logarithmically interacting (Cooper-pair) charges should be exactly self-dual.[38] In this case, the squares of the longitudinal and Hall resistances should sum to exactly $[h/(2e)^2]^2$. Experimentally at the field-tuned transition,[41] the Hall resistance appears to be very small and the longitudinal resistance is roughly 4.5–5.5 kΩ, close to but not equal to the quantum of resistance $h/(2e)^2 \approx 6.45$ kΩ.

In zero magnetic field a lattice model of logarithmically interacting bosons with a density commensurate with the lattice (an integer number per site) will have a particle-hole symmetry at the transition from a Mott-Hubbard insulator to superconductor. The transition in this model should then be self-dual[37,42] with a critical resistance of exactly $h/(2e)^2$. Clearly the (zero-field) superconductor-insulator transition in *real* disordered 2D films will be more complicated, though.

The duality picture can give us further physical insight into the nature of the insulating and superconducting phases and the reason for an unstable fixed point separating them. The superconducting phase is a condensate of bosons and a vacuum phase for the vortices. A logarithmic confining potential develops between vortices with strength proportional to the superfluid density of the particle condensate.[43] Under rescaling in the superconducting phase, the fixed point is unstable because confinement of the vortices raises the superfluid density, further increasing the confinement. On the other hand, the insulating phase can be viewed as a condensate of vortices and a vacuum (or frozen Bose glass) of charges.[37] In this state one can have a finite current of vortices (phases slips leading to a finite voltage) at no cost in current. At the critical point, both charges and vortices are mobile. This represents an unstable fixed-point separating the two phases. In summary, the fact that there is an approximate duality between charges and vortices at the superconductor-insulator transition, suggests that the universal resistance should be near but not precisely the quantum resistance, R_Q.

III. BOSON HUBBARD MODEL

Rather than study a model of charge $2e$ bosons moving in a random potential, for calculational simplicity we neglect the disorder and consider a pure boson Hubbard model with an integer number of bosons per site. Then the vanishing of the Mott-Hubbard gap yields the insulator-superconductor transition. This transition, however, is not in the same universality class as that in a model with impurities,[16] wherein the transition is from a superconductor into a gapless, localized Bose-glass insulator. Consequently, the universal resistance deduced from the pure boson Hubbard model should not be compared directly to the experiments.

For simplicity we assume the bosons move on a square lattice. Changes in the symmetry of the lattice are not expected to modify the critical phenomena, though. The Hamiltonian of the pure boson Hubbard model is given by

$$\mathcal{H} = U \sum_i (n_i - n_0)^2 - t \sum_{\langle ij \rangle} b_i^\dagger b_j \ , \tag{3.1}$$

where $b_j^\dagger$ and b_j are boson creation and destruction operators at site j, respectively, and n_j are number operators. The parameters t and U represent, respectively, the nearest-neighbor-hopping matrix elements and the intrasite (Coulomb) charging energy. The average density is controlled by the parameter n_0 which we take to be a large integer. The summation $\langle ij \rangle$ ranges over all nearest-neighbor pairs.

The boson Hubbard model in Eq. (3.1) is closely related to (i.e., in the same universality class as) the quantum rotor model,[16,44,45]

$$\mathcal{H}' = U \sum_j \left(\frac{1}{i} \frac{\partial}{\partial \theta_j} - n_0 \right)^2 - J \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j) \ , \tag{3.2}$$

which is expressed in terms of a phase angle at each site θ_j, and conjugate momenta $(1/i)(\partial/\partial\theta_j)$. The Hamiltonian in Eq. (3.2) can be obtained (approximately) from Eq. (3.1) by putting[45] $b_j^\dagger = \sqrt{n_0} e^{i\theta}$ and $n_j \to (1/i)(\partial/\partial\theta_j)$, provided we take $J = n_0 t$. If n_0 is an integer, as we consider below, it can be eliminated from Eq. (3.2) by a shift, $n_j \to n_j + n_0$, reducing the first term to the more familiar form, $-U(\partial/\partial\theta_j)^2$. The ratio of the two parameters J and U determines the phase of this model. In the limit $U/J \gg 1$, the hopping is suppressed and the wave function of the system is roughly $\psi(\theta_1, \ldots, \theta_{N_s}) \approx 1$ where N_s is the number of sites of the system. This is the Mott insulating phase. We can interpret this wave function as saying that every rotor is in its ground state (of kinetic energy). Hence we have a charge vacuum. Equivalently we can interpret $|\psi|^2 \approx 1$ as saying that all possible phase angle configurations are nearly equally probable. The disordered phase angle configuration is due to a condensation of quantum-fluctuation-induced vortices.

In the other limit, $U/J \ll 1$, a variational wave function with a parameter, λ, could be taken as $\psi_\lambda(\theta_1, \cdots, \theta_{N_s}) \approx \exp\left(\lambda \sum_{\langle kl \rangle} \cos(\theta_k - \theta_l)\right)$ representing (for large λ) the superfluid phase. Somewhere between these two limits, a phase transition occurs.[46]

The Mott insulator to superconductor transitions in the Hamiltonians in Eqs. (3.1) and (3.2) at integer filling, n_0, are in the universality class of the (2+1)-D classical XY model.[16,44] This can be understood most easily by representing the partition function of the Hamiltonian $\mathcal{H}'$ as a path integral in a basis diagonal in θ_j. The associated action can be written as $S = \int d\tau \mathcal{L}$ with a Lagrangian:

$$\mathcal{L} = \frac{1}{4U} \sum_j (\partial_\tau \theta_j)^2 - J \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j) \ . \tag{3.3}$$

Note that the first term in Eq. (3.3) is simply a ferromagnetic coupling in the (imaginary) time direction, so that in space-time the action is essentially a (2+1)D XY model. It should be emphasized that the action has this classical (2+1)D XY-like form only when n_0 is an *integer*. For noninteger n_0, the action would have an additional (imaginary) term of the form[35]

$$\delta S = i n_0 \sum_j \int d\tau \, \partial_\tau \theta_j \ . \tag{3.4}$$

IV. BOSON HUBBARD MODEL IN THE HARTREE-FOCK APPROXIMATION

In this section we solve the boson Hubbard model at integer filling in a simple mean-field approximation. This has the advantage of giving a different perspective on the problem and is physically rather clear.

The standard path integral formulation of the Hamiltonian in Eq. (3.2) is in the phase representation, as shown in Eq. (3.3). Here we will work in the conjugate (particle-number) representation. In this description vortices do not explicitly appear. The physical picture is that, in the Mott-Hubbard insulating phase there is a charge excitation gap between the "upper Hubbard band" and the "lower Hubbard band." Thus we can view the system as a semiconductor. We have a simple physical picture of the optical (i.e., finite frequency) conductivity of a semiconductor; photons with energies above the gap can be absorbed by the production of particle-hole pairs. The rate of absorption is given by Fermi's golden rule and is essentially the square of a matrix element times a particle-hole density of states. Thus, even in the absence of disorder, the conductivity is not infinite because there are no carriers present except for the pairs produced by the photoabsorption.

Imagine approaching the $T = 0$ phase transition by first letting the Mott-Hubbard gap go to zero and then taking the frequency to zero. The semiconductor analogy makes clear the source of dissipation at the transition; it is the production of gapless "particle-hole" pairs of bosons. This physical picture of a critical (gapless) state is very different from that used in previous[9–12,47,48] models which invoked some external source of dissipation to explain early experiments on granular films which found that an apparently universal *normal state* resistance correlated with the destruction of superconductivity at low temperatures. In these models, a local phase degree of freedom (a phase "particle" in a washboard potential) corresponding to the boson field was coupled to some external source of dissipation, due to quasiparticle tunneling across the weak links. The dissipation was in turn related to the *normal state* resistance of the film. An essential difficulty was that quasiparticles freeze out at low temperatures so that the normal state dissipation is essentially unrelated to the very low dissipation at low temperatures.

In the present model, the dissipation does not have an extrinsic source, but is due to the gapless excitations in the boson system itself at its critical point. Thus it is *not* the normal state resistance which determines the dissipation at low temperatures, but rather a self-consistent dissipation generated internally by the dynamical fluctuations at the critical point. Indeed, it is the Cooper pairs which are diffusing at the zero-temperature superconductor-insulator transition. We believe that the value of the critical *normal state* resistance which determines the destruction of superconductivity at low temperatures, will in general *not* be universal. Rather, it is the actual resistance of the film *at* zero temperature which is predicted to be universal at the transition.

Taking the Hamiltonian in Eq. (3.1) with n_0 integer, the conductivity is given by[49]

$$\mathrm{Re}\,\sigma(\omega) = -(L^2\hbar\omega)^{-1}\mathrm{Im}\,\Pi^{\mathrm{R}}_{xx}(\omega + i\delta) \ , \tag{4.1}$$

where L is the system size. The retarded correlation function is

$$\Pi^{\mathrm{R}}_{xx}(\omega) \equiv \int_{-\infty}^{\infty} dt' e^{i\omega t'} \{ -i\theta(t') \langle [J_x(t'), J_x(0)] \rangle \} \ , \tag{4.2}$$

and the current operator for charge e^* particles is

$$\mathbf{J} \equiv \frac{ie^* t}{\hbar} \sum_{j,\delta} \boldsymbol{\delta} b^\dagger_{j+\delta} b_j \ , \tag{4.3}$$

where $\boldsymbol{\delta}$ is a near-neighbor vector.

As mentioned above we will begin with the large-U insulating state. When $t = 0$, the ground state has n_0 particles on every site and the cost to add an extra particle is U. There is a large manifold of degenerate states corresponding to the fact that the particle can be added to any one of the sites. This degeneracy is lifted by the kinetic energy ($t \neq 0$) which spreads the states out into the "upper Hubbard band," which has a width (in mean field) of $8t$. Similarly, it costs energy U to create a hole in the "lower Hubbard band," but this band also acquires a finite width. Thus the actual band gap is

$$2\Delta = 2(U - 4t) \ . \tag{4.4}$$

For a square lattice with lattice constant a, the dispersion relation for both bands is

$$\varepsilon(\mathbf{k}) = U - 2t[\cos(k_x a) + \cos(k_y a)] \ , \tag{4.5a}$$

$$\varepsilon(\mathbf{k}) \approx U - 4t + t(ka)^2 \ . \tag{4.5b}$$

Converting to plane-wave variables

$$b^\dagger_j = \frac{1}{\sqrt{N_s}} \sum_k e^{-i\mathbf{k}\cdot\mathbf{r}_j} b^\dagger_{\mathbf{k}} \ , \tag{4.6}$$

the current becomes

$$J_x = \frac{e^* t a}{\hbar} \sum_k 2\sin(k_x a) b^\dagger_{\mathbf{k}} b_{\mathbf{k}} \ . \tag{4.7}$$

The correlation function is then readily evaluated, provided that we view the current operator as creating a "particle" in the upper Hubbard band and a "hole" in the lower Hubbard band, each of which propagate freely and do *not* interact:

$$\mathrm{Im}\,\Pi^{\mathrm{R}}_{xx}(\omega) = -\pi \frac{(e^* t a)^2}{\hbar^2} \sum_k 4\sin^2(k_x a) \delta\left(\omega - \frac{2\varepsilon(\mathbf{k})}{\hbar}\right) \ . \tag{4.8}$$

Since we are interested only in frequencies just above the gap, we can make the long-wavelength (quadratic) approximation to the dispersion relation for $\varepsilon(\mathbf{k})$ to obtain

$$\mathrm{Im}\,\Pi^{\mathrm{R}}_{xx}(\omega) = -\frac{e^{*2}L^2}{8}(\omega - \omega_{\mathrm{T}})\Theta(\omega - \omega_{\mathrm{T}}) \ , \tag{4.9}$$

where $\omega_{\mathrm{T}} \equiv 2\Delta/\hbar$. This expression has a simple physical interpretation. It is the (transition) density of states for particle-hole pairs of energy ω. We see that it vanishes for energies below the gap and rises linearly above threshold. Note however one additional very important feature of Eq. (4.9): The factor of t^2 from the current matrix element has been canceled out because the larger is t, the wider are the bands and hence the smaller the one-body density of states.

With this result the conductivity is readily found to be

$$\mathrm{Re}\,\sigma(\omega) = \frac{e^{*2}}{h}\frac{\pi}{4}\frac{\omega - \omega_{\mathrm{T}}}{\omega}\Theta(\omega - \omega_{\mathrm{T}}) \,. \tag{4.10}$$

Setting the gap to zero at the critical point and then taking the limit of zero frequency, we obtain the result

$$\sigma^* = \frac{e^{*2}}{h}\frac{\pi}{4} \,. \tag{4.11}$$

This result differs by a factor of 2 from the leading large-N limit to be derived in the next section. This is because at the $N = \infty$ critical point, the model describes noninteracting, *linearly* dispersing bosons, whereas we have assumed a quadratic dispersion. Taking instead (near the critical point)

$$\epsilon(\mathbf{k}) = \sqrt{\Delta^2 + (ck)^2} \,, \tag{4.12}$$

the current operator becomes

$$\mathbf{J} = \frac{e^*}{\hbar}\sum_k \frac{c^2 k_x}{\epsilon(\mathbf{k})} b^\dagger_{\mathbf{k}} b_{\mathbf{k}} \,, \tag{4.13}$$

so that the final result is

$$\mathrm{Re}\,\sigma(\omega) = \frac{e^{*2}}{h}\frac{\pi}{8}\frac{\omega^2 - \omega_{\mathrm{T}}^2}{\omega^2}\Theta(\omega - \omega_{\mathrm{T}}) \,, \tag{4.14}$$

in agreement with the leading large-N result of Sec. V.

V. LARGE-N EXPANSION OF THE UNIVERSAL CONDUCTIVITY FOR THE ISOTROPIC 3D CLASSICAL XY MODEL

As discussed in Sec. III, the superfluid to insulator transition in the boson Hubbard model at integer filling is in the same universality class as the isotropic 3D classical XY model. We now consider this model problem. For the XY model, the order parameter dimensionality, N, is 2. In the limit where N goes to infinity, the model is exactly soluble. Here we calculate the universal conductivity as a power series in $1/N$.

Since universal quantities, such as the conductivity, depend only on the universality class of the transition and are insensitive to microscopic details, we are free to choose a convenient form of model action, provided it is in the same universality class. Here we take a Ginzburg-Landau-type action to calculate the universal conductivity. Then, with M complex fields $\phi_1, \ldots, \phi_M (N = 2M)$, the action of the model is given by

$$S = \int d^3x \bigg(\nabla\phi^*_\alpha(\mathbf{x}) \cdot \nabla\phi_\alpha(\mathbf{x}) + r_0\, \phi^*_\alpha(\mathbf{x})\phi_\alpha(\mathbf{x}) + \frac{u_0}{2}[\phi^*_\alpha(\mathbf{x})\phi_\alpha(\mathbf{x})]^2 \bigg) \,, \tag{5.1}$$

where $\alpha = 1, \ldots, M$, and the summation convention for the repeated indices is assumed. The parameter u_0 is a constant representing a short-range repulsive interaction. This 3D classical action is equivalent to the 2D quantum model at zero temperature in a path integral representation, provided one of the three coordinates is interpreted as the imaginary time axis. Let us take the imaginary time axis along the z direction, so that k_z in momentum space now means the frequency. Imposing a vector potential source field and taking the second derivative of the free energy with respect to the source field, as detailed in Appendix A, we arrive at a formula for the conductivity

$$\sigma_{\mathrm{M}}(ik_z) = \frac{e^{*2}}{\hbar}\frac{2}{k_z}\int d^3x [\, \langle \phi^*_\alpha(\mathbf{x})\phi_\alpha(\mathbf{x})\rangle \delta(\mathbf{x}) - 2\langle J_x(\mathbf{x})J_x(\mathbf{0})\rangle]\, e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{5.2}$$

where e^* is the carrier charge, and we have defined $\mathbf{k} = k_z\hat{z}$,

$$\mathbf{J}(\mathbf{x}) = \frac{1}{2i}[\phi^*_\alpha(\mathbf{x})\nabla\phi_\alpha(\mathbf{x}) - \phi_\alpha(\mathbf{x})\nabla\phi^*_\alpha(\mathbf{x})] \,, \tag{5.3}$$

and

$$\langle \mathcal{O} \rangle = \frac{\int \mathcal{D}[\phi]\; \mathcal{O}\; e^{-S}}{\int \mathcal{D}[\phi]\; e^{-S}} \,. \tag{5.4}$$

Using the Fourier transformation

$$\phi_\alpha(\mathbf{x}) = \frac{1}{\sqrt{\Omega}}\sum_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{x}}\; \phi_{\alpha\mathbf{q}} \,, \tag{5.5}$$

where Ω is the volume of the 3D system we have

$$S = \sum_{\mathbf{q}} (q^2 + r_0)\phi^*_{\alpha\mathbf{q}}\,\phi_{\alpha\mathbf{q}} + \frac{1}{2\Omega}\sum_{\mathbf{q}} u_0 \Big(\sum_{\mathbf{p}} \phi^*_{\alpha\mathbf{p}}\,\phi_{\alpha\mathbf{p}+\mathbf{q}}\Big)^* \Big(\sum_{\mathbf{p}'} \phi^*_{\beta\mathbf{p}'}\phi_{\beta\mathbf{p}'+\mathbf{q}}\Big) \tag{5.6}$$

and the conductivity (per flavor) is given by

$$\sigma(ik_z) = \frac{e^{*2}}{\hbar}\frac{1}{k_z}\rho(k_z) \tag{5.7}$$

with

$$\rho(k_z) = 2\int \frac{d^3q}{(2\pi)^3}\langle \phi^*_{\mathbf{q}}\,\phi_{\mathbf{q}}\rangle - 4\Omega \int \frac{d^3q\, d^3p}{(2\pi)^6}\langle \phi^*_{\mathbf{q}-\mathbf{k}/2}\phi^*_{\mathbf{p}+\mathbf{k}/2}\phi_{\mathbf{p}-\mathbf{k}/2}\phi_{\mathbf{q}+\mathbf{k}2}\rangle q_x p_x \,, \tag{5.8}$$

where we have dropped the flavor indices of the ϕ field to represent one among the M fields. Power counting shows that $\rho(k_z) \propto k_z$ and hence $\sigma(k_z)$ is just a combination of some fundamental physical constants. Note that the conductivity and the sheet conductance are equivalent quantities in the (2+1)-D case under consideration.

To calculate the expectation value in Eq. (5.8) we want to treat the ϕ^4 term in Eq. (5.1) perturbatively in the large-N limit. We must choose

$$u_0 = \frac{g_0}{N}, \tag{5.9}$$

where g_0 is a constant, to obtain a sensible limit. The $1/N$ expansion involves selecting diagrams of successive orders in $1/N$. One way to simplify the diagram selection and counting is the introduction of an auxiliary field,[50] ζ, to decouple the ϕ^4 term:

$$S = \int d^3x \Big(\nabla\phi^*_\alpha(\mathbf{x}) \cdot \nabla\phi_\alpha(\mathbf{x}) + r_0\, \phi^*_\alpha(\mathbf{x})\phi_\alpha(\mathbf{x}) + \frac{1}{2}\frac{N}{g_0}\zeta(\mathbf{x})^2 + i\zeta(\mathbf{x})\phi^*_\alpha(\mathbf{x})\phi_\alpha(\mathbf{x}) \Big). \tag{5.10}$$

The functional integral with respect to ζ gives back the ϕ^4 term. In this new action, the interactions are only mediated by the ζ field and the ζ propagator gives all $1/N$ factors. The parameter r_0 represents the square of the bare one-particle excitation energy gap, but many-body effects due to the interactions renormalize r_0. At the critical point, the renormalized excitation energy gap vanishes and the correlation length becomes infinite. In order to get the correlation function at the critical point, we extract the excitation energy gap renormalization term by shifting the ζ field,

$$\zeta - \bar{\zeta} \to \zeta, \tag{5.11}$$

with $\bar{\zeta}$ determined by

$$\left.\frac{\delta S_{\rm eff}}{\delta \zeta}\right|_{\zeta=\bar{\zeta}} = 0, \tag{5.12}$$

where $S_{\rm eff}$ is defined by[50]

$$e^{-S_{\rm eff}[\zeta]} = \int \mathcal{D}[\phi]\, e^{-S[\phi,\zeta]}. \tag{5.13}$$

Then the action becomes

$$S = \int d^3x \Big(\nabla\phi^*_\alpha(\mathbf{x}) \cdot \nabla\phi_\alpha(\mathbf{x}) + r\, \phi^*_\alpha(\mathbf{x})\phi_\alpha(\mathbf{x}) + \frac{1}{2}\frac{N}{g_0}\zeta(\mathbf{x})^2 + i\zeta(\mathbf{x})\phi^*_\alpha(\mathbf{x})\phi_\alpha(\mathbf{x}) + \frac{N}{g_0}\bar{\zeta}\zeta \Big), \tag{5.14}$$

up to some constant, where the physical (renormalized) gap parameter is

$$r = r_0 + i\bar{\zeta}. \tag{5.15}$$

Now the free ϕ propagator is given by

$$G_0(\mathbf{q}) \equiv \langle \phi_{\mathbf{q}}\, \phi^*_{\mathbf{q}} \rangle = \frac{1}{q^2 + r}. \tag{5.16}$$

The two-body interaction coupling with momentum transfer $\mathbf{k}$, which is mediated by the ζ field and renormalized by the polarization insertion, can be calculated up to leading order in $1/N$ in the large-N limit from the diagrams shown in Fig. 2:

FIG. 2. The dressed interaction to $O(1/N)$.

$$u(\mathbf{k}) = \frac{u_0}{1 + M u_0 \Pi(\mathbf{k})}, \tag{5.17}$$

where the factor of M comes from the sum of the contributions of each of the M complex ϕ fields, and $\Pi(\mathbf{k})$ [shown in Fig. 3(a)] represents the polarization of one of the fields with momentum transfer $\mathbf{k}$, and is defined by

$$\Pi(\mathbf{k}) \equiv \int \frac{d^3q}{(2\pi)^3}\, G_0(\mathbf{k}+\mathbf{q})G_0(\mathbf{q}) \tag{5.18}$$

in the 3D case. We also need to define the self-energy [Fig. 3(b)]:

$$\Sigma(\mathbf{k}) \equiv \int \frac{d^3q}{(2\pi)^3}\, u(\mathbf{q})[G_0(\mathbf{k}+\mathbf{q}) - G_0(\mathbf{q})], \tag{5.19}$$

where we have subtracted the zero-momentum part because it has already been absorbed in the renormalization of r.[51]

Now the $1/N$ expansion is implemented by replacing u_0 by $u(\mathbf{q})$ and r_0 by r in Eq. (5.6), and calculating physical quantities in powers of u. The perturbative expansion of $\rho(k_z)$ up to the first two leading-order terms can be represented by the diagrams shown in Fig. 4. At $N = \infty$ we can simply neglect the ϕ^4 term and the action becomes Gaussian so that

$$\rho^{(0)}(k_z) = 2\int \frac{d^3q}{(2\pi)^3}[G_0(\mathbf{q}) - 2q_x^2 G_0(\mathbf{q})G_0(\mathbf{k}+\mathbf{q})]. \tag{5.20}$$

At the critical point r vanishes and the integrals in Eq. (5.20) can be evaluated giving

$$[\rho^{(0)}(k_z)]_{\rm crit} = \tfrac{1}{16}k_z. \tag{5.21}$$

The first $1/N$ correction can be expressed as

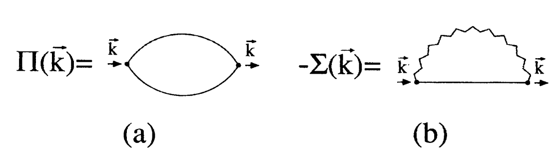

FIG. 3. (a) Polarization bubble; (b) self-energy.

$$\rho^{(1)}(k_z) = -2\left(\int \frac{d^3q}{(2\pi)^3}\Sigma(\mathbf{q})[G_0(\mathbf{q})]^2 - 4\int \frac{d^3q}{(2\pi)^3} q_x^2\Sigma(\mathbf{q})G_0(\mathbf{q}+\mathbf{k})[G_0(\mathbf{q})]^2 \right.$$
$$\left. - 2\int \frac{d^3q\, d^3p}{(2\pi)^6} u(\mathbf{q}-\mathbf{p})G_0(\mathbf{q}+\mathbf{k})G_0(\mathbf{q})G_0(\mathbf{p}+\mathbf{k})G_0(\mathbf{p})q_x p_x \right). \tag{5.22}$$

At the critical point these integrals can be evaluated exactly (see Appendix B) giving

$$[\rho^{(1)}(k_z)]_{\rm crit} = -\tfrac{1}{16}\left(\tfrac{8}{3}\eta\right)k_z \ , \tag{5.23}$$

where the correlation function critical exponent η is[51]

$$\eta = \frac{1}{M}\left(\frac{4}{3\pi^2}\right) . \tag{5.24}$$

Finally, inserting Eqs. (5.21) and (5.23) into Eq. (5.7) and taking $k_z \longrightarrow 0$ yields the universal dc conductivity to $\mathcal{O}(1/N)$:

$$\sigma^* = \sigma_0^*\left(1 - \frac{1}{M}\frac{32}{9\pi^2}\right) \tag{5.25}$$

with

$$\sigma_0^* = \frac{\pi}{8}\left(\frac{(2e)^2}{h}\right) . \tag{5.26}$$

In Eq. (5.26) we have replaced e^* by the Cooper pair charge $2e$. For the XY model $M = 1$. Thus the first order correction to the universal conductivity in the $1/N$ expansion, reduces the value by about 36% from the $N = \infty$ result yielding

$$\sigma^* \approx 0.251\frac{(2e)^2}{h} . \tag{5.27}$$

It would also be interesting to compute the universal conductivity within an ϵ expansion, where $\epsilon = 4 - d$, along the lines of Hohenberg *et al.* and Bervillier.[20]

VI. MONTE CARLO CALCULATION OF σ^*

In this section we discuss the results of numerical quantum Monte Carlo calculations for the universal conductivity at the Mott insulator to superfluid transition. Here we find it convenient to study a "hard-spin" representation of the 3D classical XY model, rather than the "soft-spin" Ginzburg-Landau form studied in Sec. V. Consider then the following partition function:

$$Z = {\rm Tr}_\theta \exp\left(K\sum_{\mathbf{x},\boldsymbol{\delta}}\cos[\theta(\mathbf{x}) - \theta(\mathbf{x}+\boldsymbol{\delta})]\right) , \tag{6.1}$$

where the sum is over near-neighbor bonds of a simple cubic lattice, $\boldsymbol{\delta} = \hat{x}, \hat{y}$, or $\hat{z}$, and periodic boundary conditions are assumed. We simulated this using a variety of techniques, including the Metropolis method with the "checkerboard" algorithm for vectorization on the NCSA Cray-XMP and the Wolff algorithm.[52] We also performed some limited simulations on the dual-transformed model which involves integer-valued currents flowing on the lattice links.[53] Most of the data was taken with the Metropolis code since it produced the most statistically independent samples in the least time.

Our first task is to find the critical point. A series expansion study by Ferer, Moore, and Wortis[54] found

$$K^* = 0.4539 \pm 0.0013 \equiv K_0^* \pm 0.0013 . \tag{6.2}$$

In order to reduce the uncertainty in K^*, we have located the transition by finite-size scaling studies of the stiffness with respect to imposing a twist on the boundary. This is proportional to the superfluid density and, as we shall see, takes a universal value at the critical point. In addition, it is closely related to the universal conductivity.

Specifically, consider the change in free energy when we change the boundary conditions from being periodic in all directions to periodic in $d-1$ dimensions but with a twist, Θ, in the remaining dimension, x say. Throughout this section we will refer to the spatial dimension as $d = 3$ since we are dealing with a classical model in (2+1) dimensions. If $\mathbf{x}_0$ is the coordinate of a site in the first layer perpendicular to the x axis, then we have

$$\theta(\mathbf{x}_0 + L\hat{x}) = \theta(\mathbf{x}_0) + \Theta . \tag{6.3}$$

This imposes an average twist $\nabla\Theta = (\Theta/L)$. Note that we can absorb this twist if we redefine the angles as follows:

$$\theta(\mathbf{x}) \longrightarrow \theta'(\mathbf{x}) = \theta(\mathbf{x}) + \Theta\frac{x}{L} \tag{6.4}$$

so that the θ' satisfy periodic boundary conditions but the Hamiltonian is modified to

$$\beta\mathcal{H} = -K\sum_{\mathbf{x},\boldsymbol{\delta}}\cos[\theta(\mathbf{x}) - \theta(\mathbf{x}+\boldsymbol{\delta}) - A_\delta(\mathbf{x})] , \tag{6.5}$$

where

$$A_x(\mathbf{x}) = \frac{\Theta}{L}, \quad A_\delta(\mathbf{x}) = 0(\delta \neq x). \tag{6.6}$$

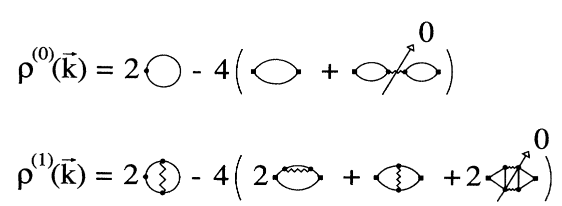

FIG. 4. Diagrams for the first two terms in large-N expansion of $\rho(k_z)$.

Here in the (2+1)D case the "temperature" $T = 1/\beta$, is a parameter which controls quantum fluctuations, not the thermodynamic temperature. Equation (6.5) is just the Hamiltonian used to compute the conductivity in Appendix A. Following the same steps, one finds, to lowest order in $\nabla\Theta$, that the change in free energy per site is given by

$$\beta f_s = \tfrac{1}{2}\rho(0)(\nabla\Theta)^2, \tag{6.7}$$

where, following Appendix A, one can define a wave vector dependent stiffness, $\rho(\mathbf{k})$, such that

$$\rho(\mathbf{k}) = K\langle\cos[\theta(\mathbf{x}) - \theta(\mathbf{x}+\hat{x})]\rangle - K^2\sum_{\mathbf{x}}\langle J_x(\mathbf{x})J_x(0)\rangle e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{6.8}$$

where the current $J_\delta(\mathbf{x})$ is given by

$$J_\delta(\mathbf{x}) = \sin[\theta(\mathbf{x}) - \theta(\mathbf{x}+\delta)]. \tag{6.9}$$

Note that $\rho(0)$ is proportional to the superfluid density, ρ_s, by

$$\rho(0) = \frac{\rho_s}{k_B T} \tag{6.10}$$

and, as shown in Appendix A, the dc conductivity is related to $\rho(k)$ by

$$\sigma = \lim_{k\to 0}\frac{e^{*2}}{\hbar}\frac{\rho(k)}{k}\ . \tag{6.11}$$

Next we discuss the scaling of $\rho(k)$. General arguments require that, since the singular part of the free energy is in fact a density, it must obey

$$\beta f_s = C\xi^{-d}, \tag{6.12}$$

where ξ is the correlation length, d is the dimensionality and, according to "two-scale-factor universality,"[20] C is a *universal* constant. The origin of the critical free energy is phase fluctuations

$$\beta f_s = \tfrac{1}{2}\rho(0)\langle(\boldsymbol{\nabla}\theta)^2\rangle \tag{6.13}$$

and, using $\langle(\boldsymbol{\nabla}\theta)^2\rangle \sim \xi^{-2}$, one finds[55]

$$\rho(0) = C'\xi^{2-d}, \tag{6.14}$$

where C' is also universal according to two-scale-factor universality. Generalizing Eq. (6.14) to finite sizes we assume that the size enters only through the ratio L/ξ. Since $\xi \simeq t^{-\nu}$, where t is the reduced "temperature," we can take the scaling variable associated with L to be $L^{1/\nu}t$. The finite-size scaling ansatz is then

$$\rho(0) = \frac{1}{L^{d-2}}\tilde{\rho}(aL^{1/\nu}t), \tag{6.15}$$

where $\tilde{\rho}$ is a scaling function and a is a nonuniversal metric factor associated with the reduced temperature. The generalization of two-scale-factor universality to finite-size systems[56] is the statement that everything in Eq. (6.15) is universal apart from the metric factor a. In particular, there is no additional metric factor associated with L. Note that for large x, $\tilde{\rho}(x) \simeq x^{(d-2)\nu}$ so one recovers Eq. (6.14) in the bulk limit. Generalizing the stiffness to finite k, the wave vector can only appear in the combination kL according to finite-size scaling, so

$$\rho(k) = \frac{1}{L^{d-2}}\bar{\rho}(aL^{1/\nu}t, kL), \tag{6.16}$$

where $\bar{\rho}$ is another universal scaling function. Note that a universality class for finite systems is specified not only by the bulk universality class but also by the shape of the system and the boundary conditions. Hence $\bar{\rho}$ will depend on shape and boundary conditions except in the limit which describes bulk behavior.

Specifying now to $d = 3$ and $T = T_c$, we have

$$\rho(k) = \frac{1}{L}Q(kL) \quad (T = T_c,\ \ d = 3), \tag{6.17}$$

where $Q(x) = \bar{\rho}(0, x)$. Hence for $k = 0$, $L\rho(0)$ is universal at T_c and plots of $L\rho(0)$ against T for different sizes should intersect at T_c. This is the analog of the universal superfluid density[22] at T_c in 2D.

We computed $L\rho(0)$, and also $L[\partial\rho(0)/\partial K]$ using the technique of Ferrenberg and Swendsen[57] at the nominal critical point $K_0^* \equiv 0.4539$ to produce the curves shown in Fig. 5. As a check, other data was taken by actually varying the coupling K. We see that, as expected, the lines approximately intersect at a value of 0.454, near the nominal value K_0^* and inside of the region defined by the error bars of Ferer, Moore, and Wortis[54] in Eq.

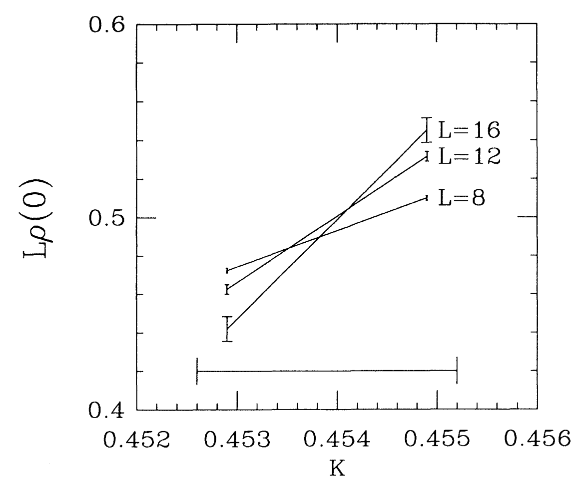

FIG. 5. Stiffness $\rho(0)$ times L as a function of lattice size L and coupling constant K from Monte Carlo simulations. The simulations were performed at the nominal value $K_0^* = 0.4539$ of the critical coupling for lattice sizes 8^3, 12^3, and 16^3. Values of $\rho(0)$ for $K \neq K_0^*$ were determined from simulated values of $\partial\rho(0)/\partial K$. The vertical bars at the end of the lines are estimates of the statistical error in $L\rho(0)$. Finite-size scaling implies that the lines in the figure will cross at the critical point. This gives $K^* = K_0^* \pm 0.0005$, and $L\rho(0) = 0.49 \pm 0.01$ at $K = K^*$. The value of $L\rho(0)$ at $K = K^*$ is expected to be universal, as discussed in the text. Horizontal line indicates error bars from the series expansion of Ref. 54.

(6.2). The value of the critical coupling which we obtain is consistent with that of Li and Teitel[58] who used a similar finite-size scaling method.

From Eq. (6.11) we see that the conductivity is obtained by taking first the limit $L \to \infty$ with k small but finite and then letting $k \to 0$. Hence the argument of the scaling function in Eq. (6.17) tends to infinity. In this limit, presumably, one has $Q(x) \simeq x$ in order to have a well-defined limit as $L \to \infty$ and so

$$\frac{\sigma^*}{\sigma_Q} = 2\pi \lim_{x\to\infty}\left(\frac{Q(x)}{x}\right). \tag{6.18}$$

Note that $k = 2\pi n/L$ where n is an integer, so we can write

$$\frac{\sigma(n)}{\sigma_Q} = \frac{Q(2\pi n)}{n}, \tag{6.19}$$

where to get σ^*, we need to take n in the range $1 \ll n \ll L$. For large x we expect that $Q(x)$ can be expanded in powers of $1/x$, i.e., $Q(x) = ax + b + \mathcal{O}(1/x)$, so, if n is not too large $\sigma(n) = \sigma^* + \text{const}/n$. Unfortunately this form does not fit our data well because, for the sizes studied, we cannot simultaneously satisfy the conditions $n \gg 1$ and $n \ll L$. We must therefore include corrections to scaling which arise because the ratio n/L is finite, so σ is a function of both n and L/n, i.e., $\sigma(n, L/n)$. Since $\sigma(n, L/n)$ refers to the behavior of a finite system and tends to a finite value as both its arguments tend to infinity, it is reasonable to assume that it varies smoothly in this limit, i.e.,

$$\frac{\sigma(n, L/n)}{\sigma_Q} = \frac{\sigma^*}{\sigma_Q} + c\left(\frac{\alpha}{n} - \frac{n}{L}\right) + \cdots \tag{6.20}$$

for some value of the constants α and c.

To find σ^* we have plotted $\sigma(n, L/n)$ for various sizes and values of n against the variable

$$x = \frac{\alpha}{n} - \frac{n}{L} \tag{6.21}$$

and choose the value of α which causes the data to fall onto a single curve. This is shown in Fig. 6 for five different values of K near the nominal critical value K_0^*. The solid curves correspond to lattices of size 8^3, the dashed curves 12^3, and the dotted curves 16^3. At each value of K, a smooth curve fit to the data for a given system size L is compared to the corresponding smooth curve for a different size L'. The discrepancy between the curves is defined to be

$$\delta = \int_{-\xi}^{+\xi} dx[\sigma_L(x) - \sigma_{L'}(x)]^2 , \tag{6.22}$$

where ξ is a cutoff taken to be 0.3. This error is summed over the different pairs of curves and for five values of K near the nominal critical value K^* is shown plotted in Fig. 7 as a function of α. These results suggest the optimal value of α is 0.74 ± 0.08 and are consistent with a transition at $K^* = K_0^*$. We can also obtain an analytic estimate for α by calculating the correction to σ^* due to a lattice cutoff and a finite sample size at $N = \infty$. Taking a spherical rather than a cubic Brillouin zone, whose boundary is determined by the condition that both have the same volume, we calculate the dependence on the lattice cutoff and find that $c = 0.228$. The correction due to a finite sample size mainly comes from the fact that in a finite system, r in Eq. (5.16) does not vanish even at the bulk critical point. Following the method provided by Brézin,[59] we obtain $r = 3.786/L^2$ at the bulk $N = \infty$ critical point. Substitution of Eq. (5.16) with this value of r into Eq. (5.20) yields $c\alpha = 0.155$. The value of α at the $N = \infty$ critical point is then determined from these two corrections to be roughly 0.68, which is in good agreement with the optimal value of $\alpha = 0.74$ deduced above from the Monte Carlo at the XY ($N = 2$) critical point.

There is some interaction between the uncertainty in K^* and the uncertainty in α. We have investigated this by simulations at the nominal critical value K_0^* which compute

$$\left.\frac{\partial\sigma(n, L/n)}{\partial K}\right|_{K=K_0^*} , \tag{6.23}$$

as well as by simulations at nearby values of K. There is a fairly well-defined range of K near the nominal critical

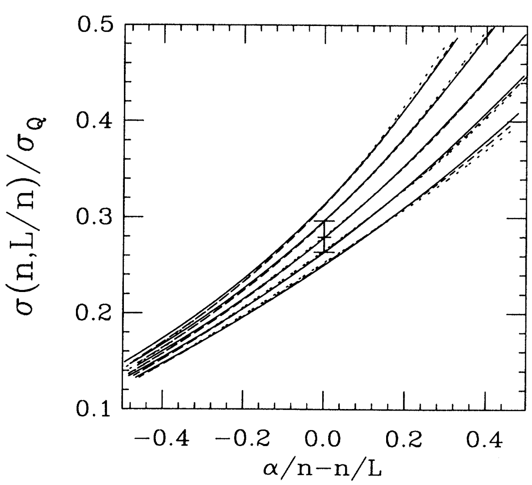

FIG. 6. Monte Carlo result for the conductivity $\sigma(n, L/n)$ divided by σ_Q as a function of the scaling variable $\alpha/n - n/L$ for various lattice sizes and coupling constants. The studied lattice sizes are 8^3 (solid curves), 12^3 (dashed curves), and 16^3 (dotted curves). The curves are obtained by smooth (spline) interpolation between the Monte Carlo data points. The simulations were done at $K = K_0^*$, and the conductivity at other values of K was computed from simulated values of $\partial\sigma/\partial K$. The three uppermost curves have $K = K_0^* + 0.001$, and then, in order from top to bottom, $K = K_0^* + 0.0005$, $K_0^*, K_0^* - 0.0005$, and $K_0^* - 0.001$. For each value of K, the corresponding value of α was chosen to make the curves for different lattice sizes to be as close together as possible; cf. Figs. 7 and 8. The best scaling behavior is found for $K = K_0^*$, and this shows that $\sigma^*/\sigma_Q = 0.285 \pm 0.02$, where the error estimate corresponds to the error in K^* [the statistical error in $\sigma(n, L/n)$ is negligible].

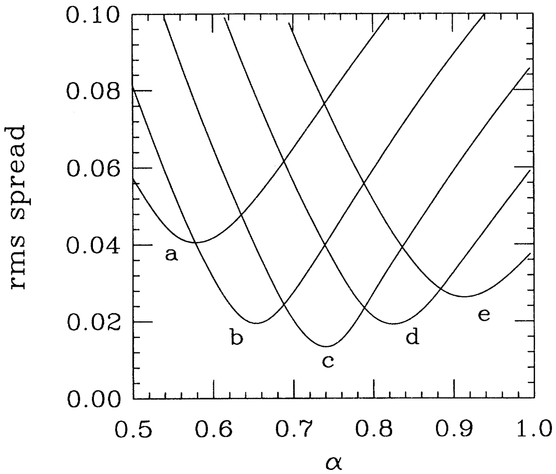

FIG. 7. Integrated discrepancy between the Monte Carlo data for the conductivity at different lattice sizes for $\alpha/n - n/L$ between -0.3 and 0.3. The integration is performed by smooth (spline) interpolation between the Monte Carlo data points. Curve (a) has $K = K_0^* + 0.001$, (b) has $K = K_0^* + 0.0005$, (c) has $K = K_0^*$, (d) has $K = K_0^* - 0.0005$, and (e) has $K = K_0^* - 0.001$. The minimum of each curve determines the value of α where the scaling is best, and these values are used in Fig. 6. The best scaling and least discrepancy is found at $K = K_0^*$, and the corresponding optimal value of α is $\alpha = 0.74 \pm 0.08$, where the error estimate comes from the error in K.

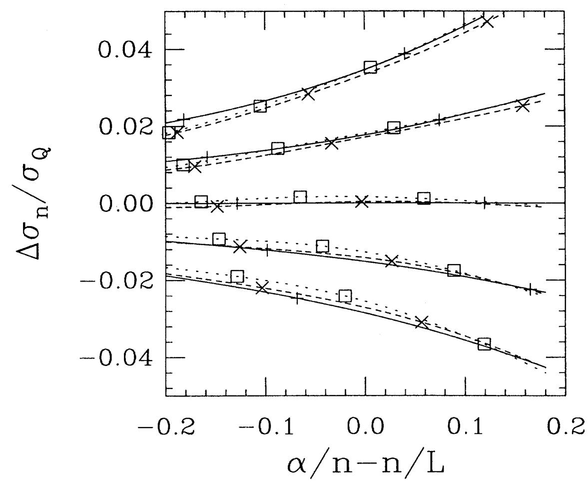

FIG. 8. Monte Carlo results for $\Delta\sigma_n/\sigma_Q \equiv [\sigma(n, L/n) - \sigma_8^*(n)]/\sigma_Q$, plotted as a function of $\alpha/n - n/L$ for different lattice sizes and couplings. The constant $\sigma_8^*(n)$ is the conductivity at size 8^3 and coupling $K = K_0^*$. The lattice sizes are 8^3 (plus signs are MC points and solid curves are interpolated), 12^3 (crosses and dashed curves), and 16^3 (squares and dotted curves). The uppermost group of three curves has $K = K_0^* + 0.001$, and, following in order from top to bottom, $K = K_0^* + 0.0005$, K_0^*, $K_0^* - 0.0005$, and $K_0^* - 0.001$. For each K, the corresponding value of α was determined in Fig. 7 to minimize the discrepancy between the curves. The least discrepancy and hence best scaling is obtained for the middle three curves, which have $K = K_0^*$; cf. Fig. 6.

point K_0^* for which there exists an α which gives good scaling behavior as shown in Fig. 8. Taking all this into account (and using the results from the scaling of the superfluid density) we estimate

$$K^* = K_0^* \pm 0.0005 \tag{6.24a}$$

and

$$\alpha = 0.74 \pm 0.08\ . \tag{6.24b}$$

From the value of the scaling curve for the conductivity at $x = 0$ we obtain for the critical value

$$\frac{\sigma^*}{\sigma_Q} = 0.285 \pm 0.02\ , \tag{6.25}$$

where all errors represent approximately one standard deviation. This numerical result is in good agreement with the result of the large-N expansion to order $1/N$, when extrapolated to $N = 2$; $\sigma^*/\sigma_Q \approx 0.251$. (Recall that at $N = \infty$ the result is $\pi/8 \approx 0.393$.) Assuming geometric convergence of the $1/N$ expansion (for which there is no justification) the discrepancy is consistent with the expected size of the next $(1/N^2)$ term in the series. Most of the uncertainty in the numerical result is in the value of K^* and α rather than direct statistical uncertainty in $\sigma(n, L/n)$. The latter was found to be about $\pm 10^{-4}$ for runs of 4×10^6 sweeps of the 16^3 lattice (with a measurement being made every second sweep). The autocorrelation time for the conductivity was about one sweep for the checkerboard algorithm. The autocorrelation time for the energy was an order of magnitude larger.[60]

The good agreement obtained between the $1/N$ expansion of the ϕ^4 theory and the Monte Carlo results for the XY model are further confirmation of our universality hypothesis since the former is a "soft spin" model while the latter is a "hard spin" model. They differ significantly in their microscopic details but lie in the same universality class.

VII. UNIVERSAL CONDUCTANCE AT 1D SUPERCONDUCTOR-INSULATOR TRANSITION

In Secs. III–VI we have ignored the effects of disorder and focused on the pure boson Hubbard model which, at integer filling, has a transition from a superconductor into a commensurate Mott-Hubbard insulator. In real amorphous thin film systems, though, the insulating phase is presumably due to the localizing effects of disorder, rather than lattice commensurability, so that the experimentally relevant transition is from superconductor into a localized "Bose glass" insulator. This transition is believed to be in a different universality class from the Mott insulator to superconductor transition and will thus have a different value of its universal conductivity (in two dimensions). Can one estimate how close these two universal conductivities will be? As we show below, some progress can be made by studying these two transitions in the one-dimensional case. First, though, it is necessary to generalize the notion of a universal conductivity away from two dimensions.

In two dimensions, conductivity and conductance have the same dimension, and are dimensionless when expressed in units of $e^2/\hbar$. Away from two dimensions only the conductance remains dimensionless. It is thus useful to define a dimensionless conductance $g(\omega, k_x)$ in terms of a frequency and wave-vector dependent conductivity $\sigma_{xx}(\omega, k_x)$:

$$g(\omega, k_x) = (h/e^{*2})\sigma_{xx}(i\omega, k_x)k_x^{2-d} . \tag{7.1}$$

Here the wave vector $\mathbf{k}$ is taken to lie *parallel* to the applied electric field and resultant current, i.e., in the x direction. At the 2D superconductor-insulator transition (Secs. III–VI) we have implicitly put $k_x = 0$, *before* taking the zero frequency limit. At the superconductor insulator transition in $d \neq 2$, though, the ω and $k \to 0$ limits must be specified with care in order to end up with a finite result for the conductance. To see this, first note that from scaling, right *at* the superconductor-insulator transition, $g(\omega, k)$ can be written in the form

$$g(\omega, k) = \tilde{g}(ck/\omega^{1/z}) \tag{7.2}$$

with $\tilde{g}(X)$ a universal scaling function, z the dynamical exponent, and c a nonuniversal constant. At finite ω and in the $k \to 0$ limit, so that the scaling variable $X \equiv ck/\omega^{1/z} \to 0$, the conductance must vary as $\tilde{g}(X) \simeq X^{2-d}$ [since the conductivity $\sigma(\omega \neq 0, k = 0)$ is finite], and thus diverges or vanishes away from $d = 2$. In the large-X limit the behavior of $\tilde{g}(X)$ will also in general depend on dimensionality. In 2D at the superconductor to Mott-Hubbard insulator transition, $\tilde{g}(X)$ can be calculated exactly within the large-N limit of Sec. V. At $N = \infty$ the result is[34]

$$\tilde{g}(X) = \frac{1 + 2X^2}{\sqrt{1 + X^2}} , \tag{7.3}$$

so that $\tilde{g}(X)$ varies as X for large arguments. As we show below, in one spatial dimension $\tilde{g}(X) \simeq 1/X$ for large X, at both the Mott-insulator to superconductor transition in the pure case *and* at the Bose glass to superconductor transition with disorder.

Since in one dimension $\tilde{g}(X)$ vanishes for both small and large arguments, it must have a unique *maximum* at some special value of X, say X^*. Consequently, the conductance at this point

$$g^* \equiv \tilde{g}(X^*) , \tag{7.4}$$

obtained from Eq. (7.2) by taking k and $\omega \to 0$, keeping the scaling combination $X = ck/\omega^{1/z} \equiv X^*$ fixed, is both *universal* and *unique*. Below we evaluate this universal conductance in the one-dimensional case, at both the pure and disordered superconductor-insulator transitions.

To this end, consider a 1D system of repulsively interacting bosons, with short-range interaction, moving in a periodic potential, $u\cos(2\pi x)$, plus a random potential $V(x)$. This model has been studied in detail in Refs. 16 and 30, by employing a representation of 1D bosons due to Haldane,[61] which expresses the relevant low-energy features of the Hamiltonian in terms of an operator $\Pi(x)$ which represents (small) deviations of the Bose density from its mean value. As reviewed in Appendix B of Ref. 16, a functional integration representation can be obtained by working in a basis of states diagonal in an operator $\tilde{\theta}(x)$, defined as $\partial_x\tilde{\theta}(x) \equiv \pi\Pi(x)$. The field $\tilde{\theta}(x)$ should *not* be confused with the phase of the boson field, denoted as θ in other sections. From Eqs. (4.6)–(4.10) in Ref. 16, the partition function at $T = 0$ can be expressed as a path integral over the field $\tilde{\theta}(x, \tau)$ (with τ imaginary time):

$$Z = \int \mathcal{D}\tilde{\theta}(x, \tau) \exp(-S) \tag{7.5}$$

with $S = S_0 + S_{\rm P} + S_{\rm R}$, and

$$S_0 = \frac{K}{2\pi} \int dx\, d\tau [C_2^{-1}(\partial_\tau\tilde{\theta})^2 + C_2(\partial_x\tilde{\theta})^2] , \tag{7.6}$$

$$S_{\rm P} = -u \int_{x,\tau} \cos[2\pi(\rho_0 - 1)x + 2\tilde{\theta}(x, \tau)] , \tag{7.7}$$

$$S_{\rm R} = -\int_{x,\tau} \delta\mu(x)\partial_x\tilde{\theta}(x, \tau) + \int_{x,\tau} (\xi e^{i2\tilde{\theta}(x,\tau)} + \text{c.c.}) . \tag{7.8}$$

Here, S_0 describes the second sound mode of the superfluid phase in the absence of an external potential, with second sound velocity $C_2 = (\rho_s/m\kappa)^{1/2}$ where ρ_s is the superfluid density, m the boson mass, and $\kappa = \partial\rho/\partial\mu$ the compressibility. $S_{\rm P}$ and $S_{\rm R}$ represent the contributions from the periodic and random potentials, respectively. When $S_P = S_R = 0$, superfluid correlations decay as a power law, $\langle b(x)b^\dagger(0)\rangle \simeq |x|^{-K/2}$ with an exponent K. This parameter is related directly to ρ_s and the compressibility

$$(\pi K)^2 \equiv m/\rho_s\kappa . \tag{7.9}$$

The parameter K defined in Eq. (7.9) should *not* be confused with the dimensionless coupling constant for the 3D XY model in Sec. VI, which is also denoted by K.

In arriving at Eq. (7.7) it was assumed that the Bose density ρ_0 was nearly commensurate with the periodic potential, $\rho_0 \simeq 1$. In Eq. (7.8), $-\delta\mu(x)$ and ξ are the contributions of the random potential $V(x)$ with Fourier components near $k = 0$ and $k = \pm 2\pi\rho_0$, respectively. For convenience, these can be taken as satisfying a Gaussian white-noise distribution.

The Mott insulator to superconductor transition occurs at a commensurate density, $\rho_0 = 1$ in Eq. (7.7), and with no randomness, $S_R = 0$. As discussed in Ref. 16, this phase transition is perturbatively accessible in u, the strength of the periodic potential. A renormalization-group analysis shows that the superconducting phase at $u = 0$ is stable with respect to small u, for K smaller than a critical value $K_M = \frac{1}{2}$. For $K > K_M$ the potential u is relevant and the system locks into a Mott insulating phase. The value of K at the transition, K_M, is universal.

A similar analysis[16,30] reveals that in the absence of a periodic potential, the superconducting phase is stable with respect to weak disorder for K less than a critical value $K_G = \frac{2}{3}$, whereas for $K > K_G$ weak disorder is relevant and grows under renormalization. This is taken to signify the formation of a localized, disorder-dominated insulating phase, i.e., a Bose glass. At the Bose glass to superconductor transition, K is universal with a value $K_G = \frac{2}{3}$.

In order to evaluate the universal conductance, g^* in Eq. (7.4), at these two transitions, it is necessary to couple the bosons to a (1D) "vector" potential $A(x)$. Minimal coupling dictates that $\partial_x \varphi$ is replaced by $\partial_x \varphi - ie^* A/\hbar$, where φ is the phase of the boson field (denoted θ in other sections), i.e., $b(x) = \sqrt{\rho} e^{i\varphi}$. As discussed in Appendix B of Ref. 16, $\partial_x \varphi = \pi \Pi_{\tilde{\theta}}$, where $\Pi_{\tilde{\theta}}$ is a momentum conjugate to the field $\tilde{\theta}$ which appears in the action Eqs. (7.6)–(7.8). In the Hamiltonian corresponding to the Lagrangian in Eq. (7.6), one must thus put $\Pi_{\tilde{\theta}} \to \Pi_{\tilde{\theta}} - (ie^* A/\pi\hbar)$. This change corresponds to an additive contribution to the Lagrangian, and hence to the action in Eq. (7.5): $S \to S + S_A$ with

$$S_A = (ie^*/\pi\hbar) \int dx\, d\tau A(x,\tau) \partial_\tau \tilde{\theta}(x,\tau) \ . \tag{7.10}$$

An expression for the conductivity can then be obtained directly by differentiating the free energy twice with respect to the "vector" potential, as in Eq. (A3) in Appendix A. Upon inserting the conductivity into Eq. (7.1), this gives for the dimensionless conductance,

$$g(\omega,k) = \left(\frac{2\omega k}{\pi}\right) \langle |\tilde{\theta}(\omega,k)|^2 \rangle \ , \tag{7.11}$$

where the average is to be taken with respect to the action in Eq. (7.5).

The conductance can now be readily evaluated at the two transitions discussed above, since the action S at the fixed points is Gaussian ($S_R = S_P = 0$). Performing the average in Eq. (7.11) using the action S_0 in Eq. (7.6) gives

$$g(\omega,k) = \tilde{g}(C_2 k/\omega) \ , \tag{7.12a}$$

with

$$\tilde{g}(X) = \frac{2}{K} \left(\frac{X}{1+X^2}\right) . \tag{7.12b}$$

The universal scaling function $\tilde{g}(X)$ in Eq. (7.12b) has a maximum at $X^* = 1$. The universal conductance at the transition, $g^* \equiv \tilde{g}(X^*)$, is then given by

$$g^* = 1/K^* \ , \tag{7.13}$$

where K^* is the universal value of K.

As noted above, at the Mott insulator to superconductor transition K is universal with a value of $\frac{1}{2}$, so that the dimensionless conductance $g^* = 2$. At the Bose glass to superconductor transition, $K^* = \frac{2}{3}$, so that the universal conductance is somewhat smaller, $g^* = \frac{3}{2}$. We thus conclude that in the one-dimensional case, an appropriately defined universal conductance is reduced from its value at the pure Mott insulator to superconductor transition by a factor of $\frac{3}{4}$ when disorder is present and the transition is into a localized (Bose glass) insulating phase. The possible implications of this 1D result for the 2D case are briefly discussed in Sec. IX.

VIII. DISSIPATION IN HELIUM-4 FILMS AT THE SUPERFLUID-INSULATOR TRANSITION

The purpose of this section is to explore the possibility that there exists a mechanical analog of the universal electrical resistance, namely, universal dissipation might be found in torsion oscillator experiments done on ^{4}He films. The experiments must be done in the limit that the critical temperature approaches zero, that is, *at* the $T = 0$ superfluid-insulator transition.[16] We are assuming, here, that destruction of superfluidity at $T = 0$ for low density of absorbed ^{4}He is due to the localizing effects of the disordered substrate (rather than due to formation of commensurate solid phase).

The dissipation in a two-dimensional electrical conductor is

$$P = \mathbf{J} \cdot \mathbf{E} A, \tag{8.1}$$

where P is the power, J is the (2D) current density, E is the in-plane electric field, and A is the area. This can be rewritten in terms of the conductivity

$$P = \sigma E^2 A = \frac{\sigma}{e^{*2}} F^2 A, \tag{8.2}$$

where e^* is the carrier charge and $F = e^* E$ is the force acting on each carrier. Now at the superconductor-insulator transition we have

$$\sigma = \frac{e^{*2}}{h} g^*, \tag{8.3}$$

where h is Planck's constant and g^* is a universal dimensionless conductance of order unity. Hence

$$P = F^2 A g^*/h. \tag{8.4}$$

At the superfluid-insulator transition for ^{4}He absorbed on disordered 2D substrates, we also expect a universal dimensionless conductance g^*. Since ^{4}He is neutral (lacks $1/r$ Coulomb interactions), though, the value of g^* is expected to differ from the value at the superconductor-insulator transition.

We now relate the power dissipation in Eq. (8.4) to the damping (Q factor) in a mechanical oscillator experiment. For simplicity, consider a *linear* rather than a torsional oscillator. Let the linear oscillator have an in-plane coordinate X, which in the absence of damping obeys

$$X = \Delta \sin(\omega t). \tag{8.5}$$

The acceleration of the oscillator and hence of the substrate holding the ^{4}He atoms is

$$a = -\Delta\omega^2 \sin(\omega t). \tag{8.6}$$

If we go to the rest frame of the substrate (which is oscillating in the laboratory frame), then there will be a pseudoforce (inertial force) acting on the ^{4}He atoms of mass m

$$F = ma \tag{8.7}$$

when they are stuck on the substrate. Using this force in Eq. (8.4) yields

$$P = \frac{m^2}{h} g^* A a^2. \tag{8.8}$$

Averaging this over one cycle of the oscillator using the acceleration from Eq. (8.6) yields

$$\langle P \rangle = \frac{1}{2}\Delta^2\omega^4 A g^* \frac{m^2}{h}. \tag{8.9}$$

In the limit of very weak damping, the energy lost per cycle to dissipation in the ^{4}He film will be given by

$$\delta E/\text{cycle} = P\frac{2\pi}{\omega} = \pi\Delta^2\omega^3 A g^* \frac{m^2}{h}. \tag{8.10}$$

On the other hand, the energy *stored* in the oscillator is

$$E = \tfrac{1}{2} M_0 \omega^2 \Delta^2, \tag{8.11}$$

where M_0 is the mass of the oscillator (dominated by the substrate). Taking the ratio of these gives us the Q of the oscillator in the presence of the damping:

$$Q^{-1} = \frac{\delta E/\text{cycle}}{2\pi E} = \frac{m^2}{h} g^* \left(\frac{\omega A}{M_0}\right). \tag{8.12}$$

Note that, very fortunately, the amplitude of oscillation Δ has dropped out of the final expression leaving us with a universal number times a simple geometric property of the oscillator.

The expression for the torsion oscillator is essentially identical, provided that it is the type used by McQueeney[62] in which a Mylar film is rolled up on a cylinder whose axis is that of the oscillator. The driving force varies slightly with radius, but the only effect is to replace one power of m by the moment of inertia of the ^{4}He film and to replace the M_0 by the moment of inertia of the oscillator. To about 20% accuracy, the mean square radius is the same for the two and so the ratio of the moments of inertia is essentially the ratio of the masses.

The most convenient form to evaluate the answer is

$$Q^{-1} = \frac{m}{h} g^* A\omega \left(\frac{m}{M_0}\right), \tag{8.13}$$

since the quantum of circulation is approximately

$$\frac{m}{h} = 10^3 \text{ cm}^{-2}\,\text{s}. \tag{8.14}$$

The frequency of McQueeney's oscillator is $\omega = 8 \times 10^3$ s^{-1}. M_0 is about 4 gm. Hence m/M_0 is about 1.6×10^{-24}. A is about 2×10^4 cm^2. The final result is

$$Q^{-1} = 2.5 \times 10^{-13}, \tag{8.15}$$

which appears to be hopelessly too small to be observable. The only way around this is to have the helium mass a more significant fraction of the oscillator mass. It may be possible to do this with grafoil or some other system. Increasing the area of the Mylar film will not help. Making the Mylar vastly thinner, might help.

IX. SUMMARY AND DISCUSSION

We have studied various approaches to estimating and calculating the value of the universal conductivity of 2D films at the superconductor-insulator transition. We have argued that the relevant model is charge $2e$ bosons moving in a random potential. Our central notion is that the conductivity of a 2D quantum boson system is a universal amplitude in the same sense that the jump in superfluid density of the 2D *classical* XY model is a universal amplitude. A duality mapping between vortices and charges is useful to understand the occurrence of the superconductor-insulator transition.

Neglecting disorder as a first approximation, we have calculated the universal conductivity at the Mott-insulator to superfluid transition in the pure boson Hubbard model at integer filling, which belongs to the universality class of the isotropic 3D classical XY model. We find good agreement between an analytic $1/N$ calculation for a "soft spin" model and numerical Monte Carlo simulations of a "hard spin" model. The result of the Monte Carlo simulation is $\sigma^* = (0.285 \pm 0.02)\sigma_Q$.

The notion of a universal conductivity is generalized to the one-dimensional case, and an appropriately defined universal *conductance* is evaluated at both the Mott insulator to superconductor transition in the pure case and at the Bose glass to superconductor transition in the presence of disorder. The possible existence of universal dissipation in ^{4}He films, which is a mechanical analog of the universal conductivity, was discussed briefly.

Even though our results provide an estimate for the universal conductivity of 2D films, disorder must be included to consider the appropriate universality class and to compare calculation with measurement. The analysis presented in Sec. VII shows that in the one-dimensional case, an appropriately defined universal conductance is reduced from its value at the pure Mott insulator to superconductor transition by a factor of $\frac{3}{4}$ when disorder is present and the transition is into a localized (Bose glass) phase. This suggests that in the case of *disordered* 2D films, the value of the universal conductivity might be somewhat smaller than our estimate for the pure 2D case of $\sigma^* \approx 0.285\sigma_Q$. In addition, though, the long-ranged $1/r$ Coulomb interaction should really be taken into account. In one dimension a $1/r$ interaction actually destroys the superconducting phase[45] (and the transition) entirely. Although this is *not* expected to be the case in two dimensions,[45] the Coulomb interaction is a relevant perturbation at the transition, modifying exponents[15,38,41] and other universal quantities. At this point we can only guess how σ^* will be modified upon inclusion of the Coulomb interaction.

Unfortunately there is no convenient mean-field theory for the disordered case about which to do a $1/N$

expansion, and although formal ϵ expansions exist in principle,[15,16] they are probably not useful for practical calculations. It is for this reason that we have focused on the zero-disorder case here. We have numerical Monte Carlo calculations presently underway to address the questions of disorder and Coulomb interactions.

Real experiments are of course limited to finite temperatures, so it is natural to ask how low a temperature is actually necessary to measure the universal ($T = 0$) conductivity. Firstly, it is necessary that temperatures are low enough that a boson description is valid. In practice, this can be determined as follows. It is first necessary to estimate a "mean-field" transition temperature, T_c^0, for those films on the superconducting side of the superconductor-insulator transition.[63] T_c^0 could be defined as the temperature at which the resistance drops to, say, half of its *normal* state value. Alternatively, a T_c^0 could be extracted by fitting the fluctuation conductivity to Aslamazov-Larkin theory.[63] In either case, due to fluctuations, T_c^0 for each given film will be larger than the true Kosterlitz-Thouless transition temperature, T_c, at which the resistance actually vanishes. This latter temperature can either be determined from the nonlinear $I - V$ characteristics[63] ($V \simeq I^3$) or from the magnetoresistance[63] ($R \simeq B$). Both T_c and T_c^0 will typically decrease when the film is made dirtier or thinner. Extrapolating to the superconductor-insulator transition, T_c vanishes (by definition) but T_c^0 will in general still be finite, with some value, T_c^{0*}. Since boson physics[64] sets in below T_c^0, the temperature at which the Cooper pairs are "formed," to measure the universal conductivity it is necessary, but perhaps not sufficient, to cool *below* T_c^{0*}. The magnitude of T_c^{0*} will clearly be system dependent—in granular films it tends to be close to the bulk transition temperature, but can be much smaller for amorphous films. For example, Valles, Dynes, and Garno[65] have presented tunneling and transport data indicating that both the gap and T_c^0 clearly drop rapidly with increasing disorder suggesting that T_c^{0*} might be inaccessibly low.

In addition to being below T_c^{0*}, it is necessary that temperatures are low enough to be within the critical regime of the $T = 0$ transition. This can be deduced empirically by checking to see if the resistance versus temperature curves scale appropriately (see below). To this end, it is useful to define (as discussed in Sec. I) a parameter δ which measures the "distance" to the $T = 0$ transition and can be taken, for example, as $\delta = (\ell_c - \ell)/\ell_c$ where ℓ is the film thickness or $\delta = (R_N - R_N^c)/R_N^c$ with R_N the film's *normal* state resistance. In the critical regime (small δ and T) the resistance should satisfy a scaling form as in Eq. (1.5):

$$R(T, \delta) = (h/4e^2)Q(b\delta/T^{1/z\nu}), \tag{9.1}$$

where $Q(x)$ is a universal scaling function, with $\partial_x Q(x)|_{x=0} = 1$, b is a nonuniversal constant, and z and ν are critical exponents of the $T = 0$ transition. The scaling function $Q(x)$ should approach a constant as $x \equiv b\delta/T^{1/z\nu} \to 0$ and diverge or vanish exponentially as x tends to positive or negative infinity, respectively.[38] Theory predicts[15] $z = 1$, which is consistent with recent experiments,[41] and from theoretical arguments[16] ν should be bounded below by 1. If a given set of resistance data is at low enough temperatures to be in the critical regime, it should collapse onto a universal function $Q(x)$ when plotted versus the scaling combination $\delta/T^{1/z\nu}$, with $z\nu(\geq 1)$ taken as an adjustable parameter. If it does, then $(4e^2/h)[Q(x = 0)]^{-1}$ could be taken as the measured value of the universal conductivity.

Recent experiments by Hebard and Paalanen[41] on the magnetic-field tuned superconductor-insulator transition in thin 2D films, show precisely such expected scaling, with δ taken as the "reduced" magnetic field, $\delta = B - B_c$. A quick inspection of the recently published data by Haviland, Liu, and Goldman[1] and Lee and Ketterson[2] on the zero magnetic-field superconductor-insulator transition in *amorphous* films, suggests that temperatures have not been taken low enough to enter the critical regime and measure the "universal" conductivity. As noted above, granular films are expected to have a more accessible critical regime (up to higher temperatures) and might be better systems in which to test theoretical predictions of a universal conductivity.

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APPENDIX A

Here we discuss the precise form of the Kubo formula[49] for the conductivity for the models studied in the text.

Consider first the Ginzburg-Landau action in Eq. (5.1). In the presence of a vector potential $\mathbf{A}(\mathbf{x})$, this action becomes

$$S = \int d^dx \left\{ \left[\left(\boldsymbol{\nabla} + \frac{ie^*}{\hbar}\mathbf{A}(\mathbf{x}) \right) \phi_\alpha^*(\mathbf{x}) \right] \left[\left(\boldsymbol{\nabla} - \frac{ie^*}{\hbar}\mathbf{A}(\mathbf{x}) \right) \phi_\alpha(\mathbf{x}) \right] + r_0\, \phi_\alpha^*(\mathbf{x})\phi_\alpha(\mathbf{x}) + \frac{u_0}{2}[\phi_\alpha^*(\mathbf{x})\phi_\alpha(\mathbf{x})]^2 \right\}, \tag{A1}$$

where e^* is the boson charge and, for generality, we have written down the model in d dimensions. We take the $d-1$ "space" dimensions to be of length L and the "time" dimension, denoted by z, to be of length L_z, which is equal to $\beta\hbar$ in order to represent the original $(d-1)$-dimensional quantum problem at inverse temperature β. We are interested in the uniform dc conductivity, which is obtained by setting the space components of the wave vector $\mathbf{k}$ to be zero and only letting the time component, k_z, vanish at the end of the calculation. Noting that $\langle \mathbf{J}_t\rangle = -\hbar(\delta F/\delta \mathbf{A})$, $E=-\partial A/\partial t$, and $\langle \mathbf{J}_t\rangle = \sigma E$, we see that the uniform, frequency dependent conductivity is given by[49]

$$\sigma_{\rm M}(ik_z) = \frac{\hbar}{k_z}\int d^dx \frac{\delta^2 F}{\delta A_x(\mathbf{x})\delta A_x(0)}\, e^{i\mathbf{k}\cdot\mathbf{x}} \ , \tag{A2}$$

where the functional derivatives are with respect to a component of $\mathbf{A}$ in one of the space directions, x say, $\mathbf{k}=(0,0,\ldots,k_z)$, and F, the free energy, is given by

$$F = -\ln Z \tag{A3}$$

with

$$Z = \int \mathcal{D}[\phi] e^{-S} \ . \tag{A4}$$

Performing the derivatives is straightforward and gives

$$\sigma_{\rm M}(ik_z) = \left(\frac{e^{*2}}{\hbar}\right)\frac{2}{k_z}\Bigg(\langle \phi^*_\alpha(\mathbf{x})\phi_\alpha(\mathbf{x})\rangle - 2\int d^dx \langle J_x(\mathbf{x})J_x(0)\rangle e^{i\mathbf{k}\cdot\mathbf{x}}\Bigg), \tag{A5}$$

where the current $\mathbf{J}(\mathbf{x})$ is defined by

$$\mathbf{J}(\mathbf{x}) = \frac{1}{2i}[\phi^*_\alpha(\mathbf{x})\nabla\phi_\alpha(\mathbf{x}) - \phi_\alpha(\mathbf{x})\nabla\phi^*_\alpha(\mathbf{x})]. \tag{A6}$$

The conductivity (per flavor) is given by

$$\sigma(ik_z) = \frac{e^{*2}}{\hbar}\frac{1}{k_z}\rho(k_z), \tag{A7}$$

where

$$\rho(\mathbf{k}) = 2\langle\phi^*(\mathbf{x})\phi(\mathbf{x})\rangle - \frac{4}{M}\int d^dx \langle J_x(\mathbf{x})J_x(\mathbf{0})\rangle e^{i\mathbf{k}\cdot\mathbf{x}} \tag{A8}$$

which gives Eqs. (5.7) and (5.8) of the text.

Let us now discuss the classical XY model, Eq. (6.1), in $d-1$ space dimension plus one time dimension. Adding a vector potential, it can be written, on a simple cubic lattice, as

$$\beta\mathcal{H} = -K\sum_{\mathbf{x},\boldsymbol{\delta}}\cos\Big(\theta(\mathbf{x}) - \theta(\mathbf{x}+\boldsymbol{\delta}) - \frac{e^*}{\hbar}A_\delta(\mathbf{x})\Big), \tag{A9}$$

where $\boldsymbol{\delta} = \hat{x},\hat{y},\ldots,\hat{z}$, runs over the neighbors with a positive displacement along the coordinate axes, and $A_\delta(\mathbf{x}) = \int_{\mathbf{x}}^{\mathbf{x}+\boldsymbol{\delta}}\mathbf{A}\cdot d\boldsymbol{\ell}$. The conductivity is then given by

$$\sigma(ik_z) = \frac{\hbar}{k_z}\sum_{\mathbf{x}}\frac{\partial^2 F}{\partial A_x(\mathbf{x})\partial A_x(0)}e^{i\mathbf{k}\cdot\mathbf{x}} \ . \tag{A10}$$

Proceeding as before one finds that $\sigma(ik_z)$ is again given by Eq. (A8) but with

$$\rho(\mathbf{k}) = K\langle\cos\left[\theta(\mathbf{x}) - \theta(\mathbf{x}+\hat{x})\right]\rangle - K^2\sum_{\mathbf{x}}\langle J_x(\mathbf{x})J_x(0)\rangle e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{A11}$$

where

$$J_\delta(\mathbf{x}) = \sin\left[\theta(\mathbf{x}) - \theta(\mathbf{x}+\boldsymbol{\delta})\right] \ . \tag{A12}$$

This gives Eq. (6.8) of the text.

APPENDIX B

Here we evaluate, to the first order in a systematic $1/N$ expansion, the universal conductivity at the critical point between the Mott insulator and superconductor phase. Specifically, employing the action in Eq. (5.1) we calculate the conductivity using expressions Eqs. (5.7) and (5.8). Let us first calculate it at $N=\infty$. We begin by proving

$$\rho^{(0)}(0) = 0 \ , \tag{B1}$$

where the superscript denotes leading (zeroth) order in $1/N$ [see Eq. (5.20)]. Using a simple relation

$$\frac{\partial}{\partial q_x}[G_0(\mathbf{q})]^\nu = -2\nu q_x[G_0(\mathbf{q})]^{\nu+1} \ , \tag{B2}$$

where $G_0(\mathbf{q})$ is the $N=\infty$ propagator defined in Eq. (5.16) we have

$$\rho^{(0)}(0) = 2\int\frac{d^3q}{(2\pi)^3}\frac{\partial}{\partial q_x}[q_x G_0(\mathbf{q})] = 0. \tag{B3}$$

This means that the first term of Eq. (5.20) can be replaced by the value of the second term evaluated at $\mathbf{k}=\mathbf{0}$. [Actually the boundary terms in Eq. (B3) are not well defined when q_x goes to infinity. But in a system with periodic boundary conditions, the full expression for q_x is $\sin q_x$ and the boundary terms in Eq. (B3) vanish by periodicity.] Using this relation, at the critical point, we have after some algebra

$$[\rho^{(0)}(k_z)]_{\rm crit} = 2\int\frac{d^3q}{(2\pi)^3}\frac{2q_x^2}{q^2}\left(\frac{1}{q^2} - \frac{1}{(\mathbf{q}+\mathbf{k})^2}\right) = \frac{1}{16}k_z \ . \tag{B4}$$

Now let us consider the $1/N$ correction, $\rho^{(1)}(k_z)$ in Eq. (5.22). Again, we want to first prove

$$\rho^{(1)}(0) = 0 \ . \tag{B5}$$

Using Eq. (B2) and integrating by parts with respect to p_x, we have from Eq. (5.22)

$$\rho^{(1)}(0) = -2\left[\int \frac{d^3q}{(2\pi)^3}\Sigma(\mathbf{q})[G_0(\mathbf{q})]^2 + \int \frac{d^3q}{(2\pi)^3}\Sigma(\mathbf{q})\, q_x\left(\frac{\partial}{\partial q_x}[G_0(\mathbf{q})]^2\right) - \int \frac{d^3q\, d^3p}{(2\pi)^6} u(\mathbf{q}-\mathbf{p})G_0(\mathbf{p})\frac{\partial}{\partial q_x}\{q_x[G_0(\mathbf{q})]^2\}\right] = 0. \tag{B6}$$

In the last step we used the definition of $\Sigma(\mathbf{q})$ in Eq. (5.19). Then, using this relation, we have at the critical point

$$[\rho^{(1)}(k_z)]_{\rm crit} = -8\int \frac{d^3q}{(2\pi)^3}\frac{q_x^2}{q^4}\Sigma_c(\mathbf{q})\left(\frac{1}{q^2} - \frac{1}{(\mathbf{q}+\mathbf{k})^2}\right) - 4\int \frac{d^3q\, d^3p}{(2\pi)^6}u(\mathbf{q}-\mathbf{p})\frac{q_x p_x}{q^2p^2}\left(\frac{1}{q^2p^2} - \frac{1}{(\mathbf{q}+\mathbf{k})^2(\mathbf{p}+\mathbf{k})^2}\right)$$
$$= -8\int \frac{d^3q}{(2\pi)^3}\frac{q_x}{q^2}\left(\frac{1}{q^2} - \frac{1}{(\mathbf{q}+\mathbf{k})^2}\right)\left(\frac{q_x}{q^2}\Sigma_c(\mathbf{q}) + \frac{d^3p}{(2\pi)^3}\frac{p_x}{p^4}u(\mathbf{q}-\mathbf{p})\right) + 4\Lambda(k_z)\ , \tag{B7}$$

where

$$\Lambda(k_z) = \int \frac{d^3q\, d^3p}{(2\pi)^6}u(\mathbf{q}-\mathbf{p})\frac{q_x p_x}{q^2p^2}\left(\frac{1}{q^2} - \frac{1}{(\mathbf{q}+\mathbf{k})^2}\right)\left(\frac{1}{p^2} - \frac{1}{(\mathbf{p}+\mathbf{k})^2}\right)\ , \tag{B8}$$

and Σ_c means the self-energy at the critical point. But

$$\int \frac{d^3p}{(2\pi)^3}\frac{p_x}{p^4}u(\mathbf{q}-\mathbf{p}) = -\frac{1}{2}\frac{\partial}{\partial q_x}\int \frac{d^3p}{(2\pi)^3}\frac{1}{p^2}u(\mathbf{q}-\mathbf{p}) = -\frac{1}{2}\frac{\partial}{\partial q_x}\Sigma_c(\mathbf{q})\ , \tag{B9}$$

so that Eq. (B7) becomes

$$[\rho^{(1)}(k_z)]_{\rm crit} = -8\int \frac{d^3q}{(2\pi)^3}\frac{q_x}{q^2}\left(\frac{1}{q^2} - \frac{1}{(\mathbf{q}+\mathbf{k})^2}\right)\left(\frac{q_x}{q^2}\Sigma_c(\mathbf{q}) - \frac{1}{2}\frac{\partial}{\partial q_x}\Sigma_c(\mathbf{q})\right) + 4\Lambda(k_z)\ . \tag{B10}$$

Ultimately we will take the limit that k_z goes to 0, so only very small q and p in the above integrands contribute. In this region we can take[51]

$$\Sigma_c(\mathbf{q}) = -\eta q^2 \ln q + \text{const} \times q^2 \tag{B11}$$

and

$$u(\mathbf{q}-\mathbf{p}) = \frac{1}{M}\frac{1}{\Pi(\mathbf{q}-\mathbf{p})} = 6\pi^2\eta|\mathbf{q}-\mathbf{p}|\ , \tag{B12}$$

where the exponent η to order $1/N$ is given in Eq. (5.24). Plugging Eq. (B11) into Eq. (B10) yields

$$[\rho^{(1)}(k_z)]_{\rm crit} = -4\eta\int \frac{d^3q}{(2\pi)^3}\frac{q_x^2}{q^2}\left(\frac{1}{q^2} - \frac{1}{(\mathbf{q}+\mathbf{k})^2}\right) + 4\Lambda(k_z)$$
$$= -\eta[\rho^{(0)}(k_z)]_{\rm crit} + 4\Lambda(k_z)\ . \tag{B13}$$

But, using the Fourier transform and Green's theorem, we have

$$|\mathbf{q}-\mathbf{p}| = \frac{1}{2\pi^2}\int \frac{d^3y}{y^2}\nabla_y^2(1 - e^{i\mathbf{y}\cdot(\mathbf{q}-\mathbf{p})})$$
$$= \frac{1}{\pi^2}\int \frac{d^3y}{y^4}(1 - e^{i\mathbf{y}\cdot(\mathbf{q}-\mathbf{p})}). \tag{B14}$$

Then

$$\Lambda(k_z) = -6\eta\int \frac{d^3y}{y^4}g_{\mathbf{k}}(\mathbf{y})g_{\mathbf{k}}^*(\mathbf{y})\ , \tag{B15}$$

where

$$g_{\mathbf{k}}(\mathbf{y}) = \int \frac{d^3p}{(2\pi)^3}\frac{p_x}{p^2}\left(\frac{1}{p^2} - \frac{1}{(\mathbf{p}+\mathbf{k})^2}\right)e^{i\mathbf{y}\cdot\mathbf{p}}\ . \tag{B16}$$

The Feynman trick[66] is useful for the calculation of the integral in $g_{\mathbf{k}}(\mathbf{y})$, and we have upon integration by parts

$$g_{\mathbf{k}}(\mathbf{y}) = \frac{i}{8\pi}\frac{y_x}{y}\int_0^1 d\alpha\left(1 - e^{-[\sqrt{\alpha(1-\alpha)}ky + i\alpha\mathbf{y}\cdot\mathbf{k}]}\right). \tag{B17}$$

Now Eq. (B15) becomes

$$\Lambda(k_z) = -\frac{3\eta}{32\pi}k_z\int_0^1 d\alpha\int_0^1 d\beta\int_{-1}^1 du(1-u^2)\int_0^\infty \frac{dy}{y^2}\left(1 - e^{-[\sqrt{\alpha(1-\alpha)}y + i\alpha yu]}\right)\left(1 - e^{-[\sqrt{\beta(1-\beta)}y - i\beta yu]}\right). \tag{B18}$$

The integration with respect to y gives

$$\Lambda(k_z) = -\frac{3\eta}{32\pi}k_z\int_0^1 d\alpha\int_0^1 d\beta\int_{-1}^1 du(1-u^2)[(A+B)\ln(A+B) - A\ln A - B\ln B]\ , \tag{B19}$$

where

$$A = \sqrt{\alpha(1-\alpha)} + i\alpha u, \ B = \sqrt{\beta(1-\beta)} - i\beta u \ . \tag{B20}$$

The imaginary part of $\Lambda(k_z)$ vanishes because the imaginary part of the integrand is an odd function of u. After some algebra we obtain

$$\Lambda(k_z) = -\frac{3\eta}{32\pi}k_z\left(-\frac{\pi}{12} + 2\frac{13\pi}{72}\right) = -[\rho^{(0)}(k_z)]_{\rm crit}\left(\tfrac{5}{12}\eta\right) \ . \tag{B21}$$

Therefore, from Eqs. (B13) and (B21), the $1/N$ correction is

$$[\rho^{(1)}(k_z)]_{\rm crit} = -\tfrac{1}{16}\left(\tfrac{8}{3}\eta\right)k_z \ , \tag{B22}$$

which is Eq. (5.23).

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