High-temperature series for the ±J random-bond Ising model

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High-temperature-series expansions for the magnetic and spin-glass susceptibilities are computed to fifteenth order in $\tanh(J/T)$ for the $\pm J$ random-bond Ising model on simple-cubic and hypercubic lattices, as a function of the fraction of ferromagnetic bonds, p. The ferromagnet–spin-glass multicritical point occurs for $p=0.810\pm0.020$ on the simple-cubic lattice, for $p=0.720\pm0.015$ on the $d=4$ hypercubic lattice, and for $p=0.680\pm0.010$ on the $d=5$ hypercubic lattice.

I. RANDOM-BOND ISING MODEL

High-temperature-series expansions for the magnetic susceptibility χ of various versions of the random-bond Ising model have been computed for almost 20 years, going back to the work of Rapaport[1] on the quenched dilute ferromagnet. A few years later, he also published[2] series for a model in which there is a random mixture of ferromagnetic bonds of strength $+J$ and antiferromagnetic bonds of strength $-J$, which we shall refer to as the $\pm J$ model. The Hamiltonian for this model is

$$H_{\rm rb} = -\sum_{\langle ij\rangle} J_{ij} S_i S_j \ , \quad (1)$$

TABLE I. χ series coefficients for the $\pm J$ random-bond Ising model on a simple-cubic lattice, up to $n=15$. The c_{mn} coefficients are defined by Eq. (3), and those that are not listed are zero.

m	n	c_{mn}	m	n	c_{mn}
1	1	6	2	2	30
3	3	150	4	4	726
3	5	−24	5	5	3534
2	6	−24	4	6	−192
6	6	16 926	1	7	−24
3	7	−192	5	7	−1608
7	7	81 318	2	8	−192
4	8	−1608	6	8	−10 464
8	8	387 438	1	9	24
3	9	−1608	5	9	−10 608
7	9	−67 248	9	9	1 849 126
2	10	−48	4	10	−10 032
6	10	−67 776	8	10	−394 896
10	10	8 779 614	1	11	−264
3	11	312	5	11	−62 304
7	11	−399 408	9	11	−2 295 456
11	11	41 732 406	2	12	−1104
4	12	864	6	12	−352 800
8	12	−2 298 624	10	12	−12 742 464
12	12	197 659 950	1	13	1272
3	13	−8328	5	13	19 824
7	13	−1 958 304	9	13	−12 699 744
11	13	−70 236 960	13	13	936 945 798
2	14	8016	4	14	−28 800
6	14	175 584	8	14	−10 398 528
10	14	−69 113 736	12	14	−376 566 480
14	14	4 429 708 830	1	15	−3672
3	15	75 960	5	15	−87 360
7	15	1 559 904	9	15	−54 324 664
11	15	−366 758 856	13	15	−2 008 556 112
15	15	20 955 627 110			

where each spin S_i is an Ising variable, which takes on the values ± 1, and $\langle ij \rangle$ indicates a sum over nearest neighbors on some lattice. Each bond J_{ij} is an independent random variable, whose value is chosen from the probability distribution

$$P(J_{ij}) = p\delta(J_{ij} - J) + (1-p)\delta(J_{ij} + J) , \tag{2}$$

and then quenched, i.e., fixed for all time. The fraction of ferromagnetic bonds, p, is a parameter of the model, which can be varied between 0 and 1.

The calculations of Ref. 2 were done for a body-centered-cubic (bcc) lattice, and presented χ to 12th order in $w = \tanh(J/T)$. Eighth-order series for d-dimensional hypercubic lattices were given by Rajan and Riseborough,[3] and results to 10th order for the square and simple-cubic lattices were calculated by Reger and Zippelius.[4] The series of these latter two groups were not long enough for accurate quantitative analysis, as the results of this work will demonstrate.

The phase diagram is of the Sherrington-Kirkpatrick[5] type, which is why this model is considered to be one of the standard Ising spin-glass models. For bipartite (two-colorable) lattices, such as the simple-cubic (sc) lattice, the phase diagram is symmetric about $p = \frac{1}{2}$. It contains the usual paramagnetic phase at high temperatures. At low temperatures, there is a ferromagnetic phase near $p = 1$, a spin-glass phase near $p = \frac{1}{2}$, and an antiferromagnetic phase near $p = 0$.

II. HIGH-TEMPERATURE SUSCEPTIBILITY SERIES

For a probability distribution of the form of Eq. (2), computation of the high-temperature χ series for Eq. (1) is only slightly more difficult than the same calculation for the standard Ising ferromagnet. For general probability distributions,[3] the bookkeeping becomes somewhat more elaborate, and the number of coefficients which are necessary to specify the series becomes unwieldy. For the $\pm J$ model, the χ series can be expressed in the form[2]

TABLE II. χ series coefficients for the $\pm J$ random-bond Ising model on a $d = 4$ simple-hypercubic lattice, up to $n = 15$. The c_{mn} coefficients are defined by Eq. (3), and those which are not listed are zero.

m	n	c_{mn}	m	n	c_{mn}
1	1	8	2	2	56
3	3	392	4	4	2696
3	5	−48	5	5	18 584
2	6	−48	4	6	−576
6	6	127 160	1	7	−48
3	7	−576	5	7	−6672
7	7	871 016	2	8	−576
4	8	−6672	6	8	−62 400
8	8	5 943 416	1	9	48
3	9	−6672	5	9	−62 880
7	9	−567 072	9	9	40 578 760
2	10	−288	4	10	−60 960
6	10	−570 048	8	10	−4 775 712
10	10	276 416 120	1	11	−912
3	11	−1680	5	11	−543 552
7	11	−4 807 584	9	11	−39 518 848
11	11	1 883 598 472	2	12	−7584
4	12	−30 720	6	12	−4 488 768
8	12	−39 582 912	10	12	−314 793 728
12	12	12 815 996 216	1	13	4272
3	13	−84 432	5	13	−221 088
7	13	−36 310 080	9	13	−314 355 648
11	13	−2 478 815 232	13	13	87 222 094 600
2	14	33 696	4	14	−636 480
6	14	−1 764 672	8	14	−282 826 176
10	14	−2 458 714 128	12	14	−19 085 547 232
14	14	592 970 187 800	1	15	−23 088
3	15	440 688	5	15	−5 000 640
7	15	−9 126 336	9	15	−2 172 168 496
11	15	−18 820 938 960	13	15	−145 769 126 656
15	15	4 031 986 634 376			

TABLE III. χ series coefficients for the $\pm J$ random-bond Ising model on a $d=5$ simple-hypercubic lattice, up to $n=15$. The c_{mn} coefficients are defined by Eq. (3), and those which are not listed are zero.

m	n	c_{mn}	m	n	c_{mn}
1	1	10	2	2	90
3	3	810	4	4	7210
3	5	−80	5	5	64 250
2	6	−80	4	6	−1280
6	6	570 330	1	7	−80
3	7	−1280	5	7	−18 800
7	7	5 064 970	2	8	−1280
4	8	−18 800	6	8	−228 160
8	8	44 898 170	1	9	80
3	9	−18 800	5	9	−229 280
7	9	−2 658 720	9	9	398 087 370
2	10	−800	4	10	−224 800
6	10	−2 668 800	8	10	−28 995 040
10	10	3 525 940 570	1	11	−2160
3	11	−9840	5	11	−2 587 840
7	11	−29 123 360	9	11	−309 243 840
11	11	31 234 124 010	2	12	−26 720
4	12	−187 200	6	12	−27 876 800
8	12	−309 733 120	10	12	−3 192 141 440
12	12	276 501 673 210	1	13	10 000
3	13	−385 520	5	13	−2 111 200
7	13	−293 318 720	9	13	−3 190 445 120
11	13	−32 495 108 800	13	13	2 447 956 433 930
2	14	92 000	4	14	−4 155 520
6	14	−24 319 040	8	14	−2 987 468 800
10	14	−32 347 480 560	12	14	−324 526 747 360
14	14	21 662 862 474 650	1	15	−82 000
3	15	1 477 520	5	15	−45 326 720
7	15	−237 497 280	9	15	−29 978 024 080
11	15	−321 843 938 800	13	15	−3 210 483 883 680
15	15	191 714 039 709 130			

TABLE IV. χ_Q series coefficients for the $\pm J$ random-bond Ising model on a simple-cubic lattice, up to $n=14$. The d_{mn} coefficients are defined by Eq. (5), and those which are not listed are zero.

m	n	d_{mn}	m	n	d_{mn}
0	2	6	2	2	0
0	4	30	2	4	0
4	4	72	0	6	150
2	6	0	4	6	528
6	6	1320	0	8	582
2	8	0	4	8	3600
6	8	8592	8	8	25 344
0	10	2454	2	10	0
4	10	16 320	6	10	49 584
8	10	169 488	10	10	502 608
0	12	6870	2	12	0
4	12	77 688	6	12	191 472
8	12	906 000	10	12	3 326 928
12	12	10 078 656	0	14	25 782
2	14	0	4	14	224 592
6	14	745 680	8	14	3 330 528
10	14	16 176 072	12	14	65 505 984
14	14	203 914 080			

TABLE V. χ_Q series coefficients for the $\pm J$ random-bond Ising model on a $d=4$ simple-hypercubic lattice, up to $n=14$. The d_{mn} coefficients are defined by Eq. (5), and those which are not listed are zero.

m	n	d_{mn}	m	n	d_{mn}
0	2	8	2	2	0
0	4	56	2	4	0
4	4	144	0	6	392
2	6	0	4	6	1632
6	6	4560	0	8	2408
2	8	0	4	8	16 032
6	8	48 096	8	8	171 840
0	10	15 272	2	10	0
4	10	122 880	6	10	434 016
8	10	1 863 456	10	10	6 517 280
0	12	85 352	2	12	0
4	12	926 448	6	12	3 155 424
8	12	16 408 800	10	12	70 398 368
12	12	254 853 888	0	14	508 808
2	14	0	4	14	5 989 536
6	14	22 231 008	8	14	119 106 240
10	14	601 390 160	12	14	2 734 908 864
14	14	10 168 193 952			

$$T\chi=1+\sum_{n=1}^{\infty}\sum_{m=1}^{n} c_{mn}(2p-1)^m w^n , \quad (3)$$

where $w=\tanh(J/T)$. The series coefficients c_{mn} are integers, and if $m+n$ is odd, then c_{mn} is zero. These coefficients were computed up to $n=15$, using the NFE method of Harris[6] and the d-dimensional hypercubic lattice weak embedding constants of Harris and Meir.[7] The c_{mn} for the sc and the $d=4$ and 5 simple-hypercubic (shc) lattices are given in Tables I–III, respectively. For the sc and $d=4$ shc lattices, it would be possible to add additional terms to the series, by using the star graph method.[8] A series for the simple-cubic lattice of the same length has also been calculated by Singh and Fisher.[9] Singh has informed me that there is complete agreement between their calculation and the series coefficients given in Table I.

High-temperature series were also computed for the Q susceptibility.[10] For the random-bond Ising model, χ_Q can be defined as

TABLE VI. χ_Q series coefficients for the $\pm J$ random-bond Ising model on a $d=5$ simple-hypercubic lattice, up to $n=14$. The d_{mn} coefficients are defined by Eq. (5), and those which are not listed are zero.

m	n	d_{mn}	m	n	d_{mn}
0	2	10	2	2	0
0	4	90	2	4	0
4	4	240	0	6	810
2	6	0	4	6	3680
6	6	10 800	0	8	6730
2	8	0	4	8	47 200
6	8	157 280	8	8	615 040
0	10	56 810	2	10	0
4	10	501 120	6	10	1 914 400
8	10	9 202 400	10	10	36 652 000
0	12	452 970	2	12	0
4	12	5 127 440	6	12	19 762 720
8	12	111 673 440	10	12	549 232 480
12	12	2 275 346 560	0	14	3 697 770
2	14	0	4	14	48 115 040
6	14	195 576 160	8	14	1 167 869 760
10	14	6 576 734 960	12	14	34 026 018 560
14	14	145 501 840 000			

$$T\chi_Q = 1 + \frac{1}{N} \sum_{1 \le i < j \le N}^{N} \langle S_i S_j \rangle^2 , \tag{4}$$

where N is the number of spins. χ_Q may be thought of as the fourth derivative of the free energy with respect to a random external field. It is *not* the same as the fourth derivative of the free energy with respect to a uniform external field $\chi^{(2)}$ except for a $P(J_{ij})$ distribution which is symmetric about 0, e.g., for the special case $p=\frac{1}{2}$.

The $\pm J$ model series coefficients for χ_Q are defined by

$$T\chi_Q = 1 + \sum_{n=1}^{\infty} \sum_{m=0}^{n} d_{mn}(2p-1)^m w^n . \tag{5}$$

All of the d_{mn} are integers, and for bipartite lattices they vanish unless both m and n are even. The d_{mn} coefficients for the sc and the $d=4$ and 5 shc lattices are given in Tables IV–VI, respectively. Note that, in contrast to the c_{mn}, coefficients, d_{0n} is nonzero for even n, but d_{2n} is always zero.

III. ANALYSIS OF THE SERIES

The usual procedure[11] for analyzing a series expansion such as Eq. (3) or (5) is to assume that near the critical point the behavior is dominated by a power-law singularity. Thus we make the ansatz

$$\chi(w,p) \sim C(p)[w_c(p)-w]^{-\gamma(p)} , \tag{6}$$

as w approaches w_c from below (i.e., as $T \to T_c^+$). One then uses ratio or Padé methods to find the "best" values of w_c and γ. The criteria for choosing the best values are somewhat subjective, especially if an attempt is made to allow for corrections to the simple scaling form [Eq. (6)].

When this recipe is followed for the $\pm J$ model, it turns out[1] that the apparent value of γ, which we will refer to as γ_{eff}, depends on p. It is generally believed, however, that γ should be "universal," i.e., that the critical behavior of this model is controlled by a renormalization-group fixed point[12] which does not depend on the details of $P(J_{ij})$. Therefore, we anticipate that the true value of γ is the same for all values of p in the range $p_{\text{mc}} < p < 1$. (The subscript "mc" denotes the ferromagnet–spin-glass multicritical point.) γ might be obtained, in principle, by calculating a large number of terms of the series. In practice, we have not actually calculated enough terms in the series to see the true value of γ. The slow approach of this series to its true asymptotic form is argued to be a consequence of the known fact that the specific-heat exponent α is close to zero.[13] The scenario is thus self-consistent.

Assuming that the Sherrington-Kirkpatrick phase diagram remains qualitatively correct for $d<6$, we can use the standard methods of series analysis to estimate the location of the ferromagnetic–spin-glass multicritical point. This gives the results $p_{\text{mc}} = 0.810 \pm 0.020$ on the simple cubic lattice, $p_{\text{mc}} = 0.720 \pm 0.015$ on the $d=4$ hypercubic lattice, and $p_{\text{mc}} 0.680 \pm 0.010$ on the $d=5$ hypercubic lattice.

It must be said, however, that the existence of simple scaling and a single critical point for H_{rb} has not been proven. It is conceivable, for instance, that the phase transition actually occurs at a higher temperature than one would expect based on the examination of the high-temperature series. This can be tested by comparing the series-analysis predictions with the results of computer simulations. The estimates of T_c from the χ series agree precisely with the Monte Carlo renormalization-group results of Ozeki and Nishimori[14,15] for $p=0.83$ and 0.90. Their estimate of $p_{\text{mc}} = 0.767 \pm 0.004$ is too low to be consistent with the series analysis. However, their phase diagram does not satisfy the condition[16] that the slope of the critical line become infinite at p_{mc}. This should hold if the multicritical point lies on Nishimori's line,[17] as they have assumed in their calculation. Therefore, the p_{mc} estimate of Ozeki and Nishimori appears to be internally inconsistent.[15]

Although the values of p_{mc} given here for the $d=4$ and 5 shc lattices differ somewhat from those of Rajan and Riseborough,[3] these differences can be easily understood as resulting from the substantially longer series used in the current work. The difference between the value of p_{mc} given here for the sc lattice and that of Reger and Zippelius[4] has another origin. Those authors chose p_{mc} based on an esthetic criterion: They required that the Nishimori line[17] be at the midpoint of their estimates of the divergences of the series for χ and $\chi^{(2)}$. This was not a wise choice, since the $\chi^{(2)}$ series is not well behaved. If one looks at their phase diagram and places p_{mc} at the point where the ferromagnetic transition line intersects the Nishimori line, the agreement with the estimate given here is much improved.

IV. GAUGE SYMMETRY

Using rather general gauge-invariance arguments, Le Doussal and Harris[16] have argued that the multicritical point for a random-bond Ising model should lie on Nishimori's line.[17] For the $\pm J$ model, this line obeys the relation

$$w = 2p - 1 . \tag{7}$$

If one is prepared to assume that there is a unique multicritical point, then it necessarily follows from the gauge symmetry that this point must lie on Nishimori's line. The reader can easily check that substituting Eq. (7) into Eqs. (3) and (5) gives series for χ and χ_Q which are identical, term by term, as is required by the gauge invariance. This, of course, means that the nature of the divergences of χ and χ_Q must be identical at the multicritical point. Unfortunately, it also means that if one approaches the multicritical point along Nishimori's line, then all of the odd coefficients of the χ series vanish. The number of nonzero coefficients in the resulting series is therefore rather small, and this makes a convincing series analysis difficult to perform along Nishimori's line.

The difficulty is compounded by the fact that for $d<6$ the slope of the critical line should be infinite[16] at p_{mc}. It turns out, therefore, that the values of T_{mc} given by a series analysis at fixed p for $d<6$ are systematically larger than predicted by Eq. (7), with the discrepancy increasing as d decreases. This is the natural consequence of trying to approximate the infinite slope of the critical line at $p_{\rm mc}$ by a series of finite length. We can, however, use these facts to advantage, by locating $p_{\rm mc}$ at the p for which the apparent value of T_c drops most rapidly as the number of terms which are used in the series analysis is increased.

V. CONCLUSIONS

In this work we have extended the high-temperature susceptibility series for the $\pm J$ random-bond Ising model. We have used these series to obtain more precise estimates of the location of the ferromagnet–spin-glass multicritical point. We have also found that the p-dependent corrections to scaling do not become small as d is increased above 4. It seens reasonable that these corrections to simple scaling can be associated with the crossover from the multicritical point to the ferromagnetic critical point. A quantitative calculation which proceeds along this line would be a useful contribution to our understanding.

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