Critical behavior of ultrasonic attenuation near interaction-driven metal-insulator transitions

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We consider the critical behavior of the ultrasonic attenuation (UA) near interaction-driven metal-insulator transitions. To first order in the disorder no localization corrections are found. Using a field-theoretical nonlinear σ-model representation, and performing a renormalization-group analysis, we show that the absence of first-order corrections follows from the presence of two scaling parts for the UA. The critical exponents for the UA are shown to be sensitive to both the symmetry class and the interaction range, in contrast to the behavior of the conductivity. We have examined the cases of strong magnetic fields and of magnetic impurities where the critical exponents prove to be universal for Coulomb interactions, but are found to be nonuniversal for short-ranged interactions. The UA was further examined near the pseudomagnetic transition that occurs in the absence of magnetic perturbations. In that case, the UA critical exponents were found to vanish, indicating that the UA stays uncritical at this transition.

I. INTRODUCTION

It is widely believed that a sufficient amount of disorder will cause a system to undergo a metal-insulator transition (MIT) at zero temperature. For a long time the precise nature of this transition and its underlying mechanism were unclear. A major advance in the understanding of MIT emerged by describing it as a continuous second-order phase transition.[1] Accordingly, theoretical approaches analogous to the ones developed for conventional critical phenomena were applied to the analysis of MIT's. Sufficiently close to the transition, much of the detailed information becomes irrelevant. A system can typically be described by a much simpler model that still preserves the important symmetries determining the critical behavior.

In the case of disordered electrons, the critical behavior is dominated by the presence of diffusion modes in both the particle-particle channel ("cooperons") and the particle-hole channel ("diffusions").[2] The interactions of these modes give rise to singular, logarithmic corrections in $d=2$, indicating the breakdown of the conventional diffusive behavior, and lead to a MIT in $2+\epsilon$ dimensions. For noninteracting electrons, Wegner was the first to show how an effective theory, the nonlinear σ model, describing the critical behavior can be derived.[1] This theory systematically includes all the interactions of diffusion modes, but eliminates other degrees of freedom which are believed to be irrelevant to criticality.

When Coulomb interactions are taken into account similar corrections are found.[2] A mathematical description in terms of interacting diffusion modes is still possible, as first demonstrated by Finkelshtein.[3] He constructed the appropriate σ-model description and performed a renormalization-group (RG) analysis for several different universality classes. The external perturbations that couple to the spin degrees of freedom of the electronic system change the overall symmetry of the problem and thus each situation corresponds to a different universality class.[4] In particular, when strong magnetic fields,[3,5] magnetic impurities,[3,5] or spin-orbit scatterers[6] are present, MIT's in $2+\epsilon$ dimensions were found. In the case when external perturbations are absent ("generic case") a more complicated behavior occurs,[3] as the interaction seems to scale to infinity and the conductivity to a constant under renormalization. Very recently, some further progress in the generic case was achieved by identifying a pseudomagnetic phase transition[7] in the disordered Fermi liquid preceding the MIT which is expected to occur at higher disorder. At this transition the magnetic susceptibility diverges and the spin-diffusion constant vanishes, but the charge diffusion remains noncritical. In all the instances where interaction driven MIT's were identified (all except the generic case) the critical exponent s, describing the vanishing of the conductivity σ, was found to be identical to the one obtained in the noninteracting case to leading order in an ϵ expansion. In fact, the exponent was found to be universal, insensitive not only to the symmetry class in question, but also to the range of the interactions.[3,5] In contrast, the same conclusion could not be drawn for the single-particle density of states (DOS) or the quantities that depend on the frequency renormalization coefficient, which proved to depend on both the symmetry class and the interaction range.[5]

In this paper we examine the question of whether the behavior of other *transport coefficients* will be equally insensitive to changing symmetry classes and the interaction range as is the case for the conductivity, or whether a richer behavior might occur. This question is important not only from the purely theoretical perspective, but also from the experimental one since it might facilitate

the identification of different universality classes. Specifically, we focus on the critical behavior of the ultrasonic attenuation near interaction driven MIT's, as an example of a transport coefficient expected to be strongly influenced by localization. In the case of noninteracting electrons, corrections to the ultrasonic attenuation (UA) due to cooperons[8] were found. To lowest order in the disorder, the corrections were similar to the ones for the conductivity. Near the MIT though, the UA proved to have an even richer behavior corresponding to the presence of two scaling parts[9,10] so that the critical exponent cannot be extracted just from perturbation theory.

On a perturbative level, several authors have examined interaction corrections to the UA.[8,11] The corresponding calculations up to date have been limited to the lowest order in disorder, and no interaction corrections were found. If the UA coefficient scales near the MIT as $\alpha \simeq b^x$, with b the length rescaling factor and the critical exponent $x \sim \epsilon$ (at $d=2+\epsilon$), then to lowest order $\alpha \sim 1+x \ln b$. Since the interaction-driven MIT's for $\epsilon \ll 1$ occur at disorder $t^* \sim \epsilon$ (here t is the dimensionless resistance), it appears that there should be a correction even to first order in disorder ($\sim t$). Naively, this might suggest that the UA should not be critical near interaction-driven MIT's. Since the decay of phonons in electron-hole pairs is the main mechanism for UA, it seems unlikely that it will be insensitive to the drastic reduction of electronic mobility occurring as the MIT is approached.

The natural solution of this dilemma is found in the possibility of having more than one scaling part for the UA, the contributions of which exactly cancel to first order in disorder. Of course, to be able to demonstrate this we will have to use a more powerful approach than the conventional perturbation theory. Indeed, by constructing an appropriate σ-model representation for UA and performing a RG analysis, we will be able to confirm the above conjecture and obtain detailed information on the critical behavior. The resulting critical exponents are found to be sensitive to both the symmetry class and the interaction range. Critical behavior of the UA is obtained in the presence of a strong magnetic field or of magnetic impurities. We have also examined the behavior of the UA near the pseudomagnetic phase transition of the generic model, where we find no critical corrections, essentially because the phonons couple to charge but not to spin fluctuations.

The organization of this paper is as follows. In Sec. II we first discuss the nature of the electron-phonon coupling in disordered systems. The microscopic model, which also contains electron-electron interactions, is then presented. Using linear-response theory, the UA coefficient is expressed through a certain electronic correlation function for which we then derive a σ-model representation. Section III discusses the formal structure of the resulting field theory and its parametrization. For completeness, we briefly review the vertex structure of the unperturbed σ model describing interacting disordered electrons, as well as the source terms describing the UA. The renormalization of the UA is presented in Sec. IV. The noninteracting limit is first examined where we can determine the scaling operators by symmetry arguments. Although the corresponding symmetries cease to be valid in the interaction case, this approach will allow us to generate the minimum set of operators that have to be included in the theory. The RG equations for interacing electrons are then calculated to one-loop order, by allowing different parts of the source term to scale independently. The analysis of the resulting RG equations near fixed points representing interaction-driven MIT's is carried out in Sec. V, giving scaling operators and the corresponding critical exponents. Our results are summarized in Sec. VI, where we also discuss the limitations of our work.

II. THE MICROSCOPIC MODEL AND ITS σ-MODEL REPRESENTATION

The ultrasonic attenuation in metals predominantly takes place by phonons decaying into electron-hole pairs and so is essentially determined by the electron-phonon interaction.[12] When disorder is added, the situation becomes more complicated since the impurities now move with the lattice distortion (phonons). This motion creates a local charge imbalance which in turn "drags" the electrons along with the impurities via screening. The problem assumes a considerably simpler form if a canonical transformation to a reference frame moving with the lattice is performed. In this representation the phonon field strain is coupled to the electronic stress tensor and the resulting electron-phonon interaction Hamiltonian takes the form[12] (we use units where $\hbar = c = k_B = 1$)

$$H_{e\text{-ph}} \sum_{a,b} \int d\mathbf{r}\, \mathcal{T}_{ab}(\mathbf{r}) \nabla_a \mathcal{U}_b(\mathbf{r}) , \tag{2.1}$$

where the electronic stress tensor (operator) is

$$\mathcal{T}_{ab}(\mathbf{r}) = \frac{1}{4m_e} \sum_i (\nabla - \nabla')_a (\nabla - \nabla')_b \varphi_i^\dagger(\mathbf{r}) \varphi_i(\mathbf{r}')|_{r=r'} . \tag{2.2}$$

Here $\mathcal{U}_b(\mathbf{r})$ is the phonon displacement field, $\varphi_i^\dagger(\mathbf{r})$ and $\varphi_i(\mathbf{r})$ are the electronic field operators with spin i, and m_e is the electronic mass.

The sound attenuation coefficient is defined by

$$\alpha = \mathcal{Q} / \tfrac{1}{2} \rho_{\text{ion}} v_0^2 c_s , \tag{2.3}$$

with $\mathcal{Q}$ being the dissipated power per unit volume, ρ_{ion} the ionic mass density, v_0 the phonon velocity amplitude, and c_s the speed of sound. Considering the phonons as a weak external field, it is easy to obtain an expression for the attenuation coefficient. In the following we will limit our attention to transverse sound attenuation. According to linear-response theory,[13] we can relate the dissipated power to a spectral function corresponding to the operator conjugate to the external field, giving the expression

$$\alpha(\omega) = \frac{\omega}{\rho_{\text{ion}} c_s^3} \chi''(k=0, \omega) , \tag{2.4}$$

where the stress-stress spectral function is

$$\chi''(k=0,\omega) = \frac{1}{V}\int d\mathbf{r}\,d\mathbf{r}'\int_{-\infty}^{+\infty} dt\, e^{i\omega t}\tfrac{1}{2}\langle[\mathcal{T}_{12}(\mathbf{r},t),\mathcal{T}_{12}(\mathbf{r}',0)]\rangle\ . \tag{2.5}$$

The attenuation coefficient is thus proportional to an electronic correlation function. In the case of interacting electrons, the calculations are most conveniently carried out in the finite temperature, imaginary-time (Matsubara) formalism where the real time (frequency) response functions are obtained by an appropriate analytical continuation. More specifically, the spectral function is just the imaginary part of the corresponding retarded response function[13]

$$\chi''(k,\omega)=\mathrm{Im}\chi_R(k,\omega)=\frac{1}{2i}[\chi_R(k,\omega)-\chi_R(k,-\omega)]\ . \tag{2.6}$$

The retarded function itself is obtained by direct analytical continuation of the temperature-ordered function[14]

$$\chi_R(k,\omega)=\chi(k,i\omega_M\to\omega+i\eta)\ , \tag{2.7}$$

where, in our case,

$$\chi(k=0,i\omega_M) = \frac{1}{V}\int d\mathbf{r}\,d\mathbf{r}'\int_0^{\beta} d\tau\, e^{i\omega_M\tau}\langle T_\tau[\mathcal{T}_{12}(\mathbf{r},\tau)\mathcal{T}_{12}(\mathbf{r}',0)]\rangle\ . \tag{2.8}$$

Here T_τ is the temperature-ordering operator, τ the imaginary time, $\omega_M=2\pi TM$ the corresponding (bosonic) Matsubara frequency, and β the inverse temperature.

Temperature correlation functions such as the required stress-stress correlation functions are most easily obtained using the partition function as a generating functional by adding appropriate source terms. The calculations are performed by formulating quantum averages in terms of functional integrals over Grassman (anticommuting) fields,[15] reflecting the Fermi statistics of the electrons. In this language, using the expression (2.2) for the electronic stress tensor, the temperature correlation function can be written as

$$\chi(k=0,i\omega_M)=\frac{1}{m_e^2}\frac{1}{V}\int d\mathbf{x}\,d\mathbf{x}'d\mathbf{y}\,d\mathbf{y}'\sum_{i,j}\delta(\mathbf{x}-\mathbf{x}')\delta(\mathbf{y}-\mathbf{y}')D_xD_y\frac{1}{\beta}\sum_{\omega_1,\omega_2}\langle\bar{\psi}^{1,i}_{\omega_1}(\mathbf{x})\psi^{1,i}_{\omega_1-\omega_M}(\mathbf{x}')\bar{\psi}^{1,j}_{\omega_2}(\mathbf{y})\psi^{1,j}_{\omega_2+\omega_M}(\mathbf{y}')\rangle\ , \tag{2.9}$$

where $\bar{\psi}$ and ψ are the electronic Grassman fields and $D_x\equiv\partial_{x_1}\partial_{x_2}$. The averages indicated in this expression are defined by

$$\langle A[\bar{\psi},\psi]\rangle\equiv\int D\bar{\psi}D\psi\, A[\bar{\psi},\psi]\exp\left[-\int d\mathbf{r}\,\mathcal{L}[\bar{\psi},\psi]\right]\ , \tag{2.10}$$

where

$$\mathcal{L}[\bar{\psi},\psi]=-\sum_{\alpha,i,\omega}\bar{\psi}^{\alpha,i}_{\omega}(\mathbf{r})\left[i\omega+\frac{1}{2m_e}\nabla^2+\mu-V(\mathbf{r})\right]\times\psi^{\alpha,i}_{\omega}(\mathbf{r})+\mathcal{L}_{\mathrm{int}} \tag{2.11}$$

is the Lagrangian of interacting electrons in a disordered system.[3] Here, as in Eq. (2.9), ω represents the fermionic Matsubara frequency $[\omega=(2n+1)\pi T,\ n=0,\pm1,\pm2,\ldots]$, μ is the chemical potential, i is the spin index, and $\alpha=1,\ldots,N(N\to0)$ is the replica index; $V(\mathbf{r})$ is the random potential coming from impurities and $\mathcal{L}_{\mathrm{int}}$ represents the interaction between electrons. Note that the functional integral in Eq. (2.10) has no normalization prefactor since the replicated partition function $Z_N=\int D\bar{\psi}D\psi\exp(-\int d\mathbf{r}\,\mathcal{L})\to1$ as $N\to0$. In fact, replicas are introduced precisely to facilitate the averaging over impurities by removing this normalization prefactor.

The correlation function of Eq. (2.9) can be generated by introducing a source term of the form

$$\delta\mathcal{L}[\bar{\psi},\psi,h]=-\sum_{\omega_M}h(\omega_M)\sum_{i,\omega}\bar{\psi}^{1,i}_{\omega}(\mathbf{r})D_r\psi^{1,i}_{\omega+\omega_M}(\mathbf{r})\ . \tag{2.12}$$

If we further define a generating functional

$$Z[h]\equiv\int D\bar{\psi}D\psi\exp\left[-\int d\mathbf{r}\,\mathcal{L}[\bar{\psi},\psi]+\delta\mathcal{L}[\bar{\psi},\psi,h]\right]\ , \tag{2.13}$$

the desired stress-stress temperature function can be written as

$$\chi(k=0,i\omega_M)=\frac{1}{m_e^2\beta V}\frac{\partial^2Z[h]}{\partial h(\omega_M)\partial h(-\omega_M)}\bigg|_{h=0}\ . \tag{2.14}$$

The generating functional $Z[h]$ is in fact simply the partition function for disordered interacting electrons in presence of an external field defined by $\delta\mathcal{L}[h]$. We can thus use the same procedure as Finkelshtein[3] to derive a σ-model representation for our problem. After performing the average over disorder, composite fields $Q^{\alpha\beta,ij}_{\omega_1\omega_2}(\mathbf{r})\simeq\psi^{\alpha,i}_{\omega_1}(\mathbf{r})\bar{\psi}^{\beta,j}_{\omega_2}(\mathbf{r})$ are introduced by Hubbard-Stratonovich (Gaussian) transformations. For the universality classes we will consider we can ignore the cooperons (particle-particle channel) so that the resulting Q matrices have a unitary symmetry with entries being complex numbers rather than quaternions. At this stage, the action is quadratic in electronic $\bar{\psi},\psi$ fields which can therefore be integrated out exactly. The generating functional then takes the form of a functional integral over Q fields only

$$Z[h]=\int DQ\exp\left[-\int d\mathbf{r}\,\mathcal{L}[Q,h]\right]\ , \tag{2.15}$$

with

$$\mathcal{L}[Q,h]=-\mathrm{Tr}\ln\left[i\hat{\epsilon}+\left[\frac{1}{2m_e}\nabla^2+\mu\right]\hat{I}+\frac{i}{2\tau}\hat{Q}(\mathbf{r})+\sum_{\omega_M}h_m\hat{L}^m D_\tau\right]+\frac{\pi\nu}{4\tau}\mathrm{Tr}\hat{Q}^2(\mathbf{r})+\mathcal{L}_{\mathrm{int}}\ , \tag{2.16}$$

where $h_m\equiv h(\omega_M)$; τ is the inelastic scattering time; ν is the self-consistent Born approximation (SCBA) single-particle DOS (per spin); and $\hat{\epsilon}$, $\hat{I}$, and $\hat{L}^m$ are second-rank tensors (matrices) with respect to energy, replica, and spin indices,

$$\begin{aligned}\epsilon^{\alpha\beta,ij}_{\omega_1\omega_2}&=\omega_1\delta_{\omega_1\omega_2}\delta_{\alpha\beta}\delta_{ij}\ ,\\ I^{\alpha\beta,ij}_{\omega_1\omega_2}&=\delta_{\omega_1\omega_2}\delta_{\alpha\beta}\delta_{ij}\ ,\\ (L^m)^{\alpha\beta,ij}_{\omega_1\omega_2}&=\delta(\omega_1-\omega_2+\omega_m)\delta_{\alpha 1}\delta_{\beta 1}\delta_{ij}\ .\end{aligned} \tag{2.17}$$

The presence of interactions will not modify the bare form of the UA source term (up to irrelevant corrections; cf. below), so for the purposes of deriving this term, we can limit our attention to the noninteracting model. The interactions *will* be important in the renormalization of the UA, so $\mathcal{L}_{\mathrm{int}}$ will be included in the following sections where the corresponding calculations are performed.

We proceed by performing an expansion around the noninteracting saddle point of the unperturbed ($h=0$) action. One such solution is $\hat{Q}=\hat{\Lambda}$ with

$$\Lambda^{\alpha\beta,ij}_{\omega_1\omega_2}=\mathrm{sgn}(\omega_1)\delta_{\omega_1\omega_2}\delta_{\alpha\beta}\delta_{ij}\ . \tag{2.18}$$

Replacing the Q fields by their saddle-point value is equivalent to using the SCBA which ignores localization. However, at zero frequency, the action has the same value on the *entire manifold* of saddle points defined by *global* transformations

$$\hat{Q}=\hat{U}\hat{\Lambda}\hat{U}^{-1}\ , \tag{2.19}$$

with $\hat{U}$ being an arbitrary unitary matrix.

Near the critical point, the physics is dominated by fluctuations which at long wavelengths (and low frequencies) cost very little free energy.[3] These "massless" (Goldstone) modes, physically corresponding to diffusion, are singled out by restricting the domain of Q fields to range over all matrices of the form

$$\hat{Q}(\mathbf{r})=\hat{U}(\mathbf{r})\hat{\Lambda}\hat{U}^{-1}(\mathbf{r})=\hat{U}(\mathbf{r})\hat{\Lambda}\hat{U}^{\dagger}(\mathbf{r})\ . \tag{2.20}$$

Using this expression for Q and dropping coordinate labels, we can now rewrite $\mathcal{L}[Q,h]$ as

$$\begin{aligned}\mathcal{L}[Q,h]&=-\mathrm{Tr}\ln(\hat{G}_0^{-1}+\hat{B}_0+\hat{B}_\epsilon+\hat{B}_h)+\mathrm{const}\\ &=-\mathrm{Tr}\ln(1+\hat{B}_0\hat{G}_0+\hat{B}_\epsilon\hat{G}_0+\hat{B}_h\hat{G}_0)+\mathrm{const}\ ,\end{aligned} \tag{2.21}$$

where

$$\hat{G}_0=\left[\left[\frac{1}{2m_e}\nabla^2+\mu\right]\hat{I}+\frac{i}{2\tau}\hat{\Lambda}\right]^{-1} \tag{2.22}$$

is the SCBA Green's-function matrix and

$$\begin{aligned}\hat{B}_0&\equiv\hat{U}^{\dagger}\left[\frac{1}{2m_e}\nabla^2,\hat{U}\right]\ ,\\ \hat{B}_e&\equiv i\hat{U}^{\dagger}\hat{\epsilon}\hat{U}\ ,\\ \hat{B}_h&\equiv\sum_{\omega_m}h_m\hat{U}^{\dagger}\hat{L}^m D_\tau\hat{U}\ .\end{aligned} \tag{2.23}$$

We now expand[10,16] $\mathrm{Tr}\ln(\ )$ in powers of $\hat{B}_0\hat{G}_0$, $\hat{B}_\epsilon\hat{G}_0$, and $\hat{B}_h\hat{G}_0$. The terms coming from $\hat{B}_0\hat{G}_0$ and $\hat{B}_\epsilon\hat{G}_0$ alone contribute to the unperturbed σ-model action. Since $\chi(\omega_M)$ is the second derivative with respect to h, it is sufficient for our purposes to go to second order in $\hat{B}_h\hat{G}_0$. Due to symmetry of D_τ, the linear term vanishes, and the desired source term (up to irrelevant contributions) takes the form

$$\delta\mathcal{L}=\tfrac{1}{2}\sum_{\omega_M,\omega_{M'}}h_M h_{M'}\mathrm{Tr}[\hat{U}^{\dagger}\hat{L}^m\hat{U}(D_\tau\hat{G}_0)\hat{U}^{\dagger}\hat{L}^{m'}\hat{U}(D_\tau\hat{G}_0)]\ . \tag{2.24}$$

The obtained expression is nonlocal in coordinate space since such is $D_\tau\hat{G}_0$. However, since the SCBA Green's function is of short range (of the order of the mean free path), this nonlocality is presumably irrelevant in the critical region where only long-wavelength fluctuations contribute.[17] This allows us to ignore the nonlocality in $\delta\mathcal{L}$ and evaluate all the unitary matrices U at the same point in space. Also, note that according to Eq. (2.14) only the terms with $\omega_{M'}=-\omega_M$ contribute to $\chi(\omega)$. The expression in Eq. (2.24) can be further simplified by writing the Green's-function matrix in terms of $\hat{\Lambda}$ as

$$\hat{G}_0(\mathbf{x},\mathbf{y})=\tfrac{1}{2}(1+\hat{\Lambda})G_0^R(\mathbf{x},\mathbf{y})+\tfrac{1}{2}(1-\hat{\Lambda})G_0^A(\mathbf{x},\mathbf{y})\ , \tag{2.25}$$

where

$$G_0^{R/A}(\mathbf{x},\mathbf{y})=\left\langle\mathbf{x}\left|\left[\frac{1}{2m_e}\nabla^2+\mu\pm\frac{i}{2\tau}\right]^{-1}\right|\mathbf{y}\right\rangle \tag{2.26}$$

is the retarded or advanced SCBA Green's function (not matrix). By using this formula with the definition of Q matrices, Eq. (2.20), and also using the fact that

$$\hat{L}^m\hat{L}^{-m}=\hat{I}\ , \tag{2.27}$$

the source term can be rearranged as

$$\delta\mathcal{L}=\tilde{\alpha}\sum_{\omega_M}h_M h_{-M}\mathrm{Tr}(\hat{L}^m\hat{Q}\hat{L}^{-m}\hat{Q})\ , \tag{2.28}$$

where

$$\begin{aligned}\tilde{\alpha}\equiv\tfrac{1}{4}\int d\mathbf{y}[D_y(G_0^R(\mathbf{y})-G_0^A(\mathbf{y}))]\\ \times[D_y(G_0^R(-\mathbf{y})-G_0^A(-\mathbf{y}))]\ .\end{aligned} \tag{2.29}$$

Finally, by introducing a source field $j_M\sim h_m h_{-m}$, the UA coefficient can be written as

$$\alpha(\omega)=\alpha_0(\omega)\frac{1}{\omega}\mathrm{Im}\left[\frac{\partial Z[j]}{\partial j_M}\bigg|_{\substack{i\omega_M\to\omega+i\eta\\ j=0}}\right], \tag{2.30}$$

with

$$\begin{aligned}\delta\mathcal{L}[Q,j]&=-\sum_{\omega_M} j_M\frac{1}{4V}(2\pi T)\mathrm{Tr}[\hat{L}^m\hat{Q}(\mathbf{r})\hat{L}^{-m}\hat{Q}(\mathbf{r})]\\&=-\sum_{\omega_M} j_M\frac{1}{4V}(2\pi T)\sum_{i,j}\sum_{\omega_1,\omega_2} Q^{11,ij}_{\omega_1+\omega_M,\omega_2}\\&\quad\times(\mathbf{r})Q^{11,ji}_{\omega_2-\omega_M,\omega_1}(\mathbf{r})\,.\end{aligned} \tag{2.31}$$

Here, $\alpha_0(\omega)=-2\omega^2\tilde{\alpha}/\pi m_e^2\rho_{\mathrm{ion}}c_c^3$ is the SCBA ("bare") attenuation coefficient. In deriving this result we have ignored contributions independent of ω_M since such terms do not contribute when the analytical continuation from imaginary to real frequencies $i\omega_M\to\omega+i\eta$ is performed. This is a consequence of the fact that according to Eq. (2.6), $\chi''(\omega)$ is antisymmetric in frequency, so there are no $\omega=0$ contributions. As mentioned above, we have also ignored the interactions in deriving the σ-model representation for the UA source term. The interaction corrections to this source term could be included, but they would bring extra powers of temperature which makes them irrelevant to the critical behavior.

III. PARAMETRIZATION AND LOOP EXPANSION

The procedure described in Sec. II eliminates irrelevant degrees of freedom, and the effective Lagrangian for Q fields reduces to the expression

$$\mathcal{L}[Q,j]=\mathcal{L}[Q]+\delta\mathcal{L}[Q,j]\,, \tag{3.1}$$

where the UA source term $\delta\mathcal{L}[Q,j]$ is given by Eq. (2.31), and

$$\begin{aligned}\mathcal{L}[Q]=&\frac{1}{2G}\mathrm{Tr}(\nabla\hat{Q})^2-2H\,\mathrm{Tr}(\hat{\epsilon}\hat{Q})\\&-\frac{\pi T}{2}(K_s-K_t)(\hat{Q}\cdot\hat{Q})_1+\pi TK_t(\hat{Q}\cdot\hat{Q})_2\end{aligned} \tag{3.2}$$

is the unperturbed σ-model Lagrangian describing disordered interacting electrons.[3] Here, the coupling constant $G=4/\pi\sigma$, where σ is the bare (SCBA) conductivity. The parameter H with bare value $H_0=\pi\nu/2$ describes the frequency renormalization, and K_s and K_t are the singlet and triplet interaction coupling constants, respectively. The last two terms in Eq. (3.1) representing electronic interactions involve the two "products" which differ in their spin structure

$$\begin{aligned}[\hat{Q}\cdot\hat{Q}]_1&\equiv\sum_{\alpha}\sum_{i,j}\sum_{\omega_1,\ldots,\omega_4}\delta(\omega_1-\omega_2,\omega_4-\omega_3)Q^{\alpha\alpha,ii}_{\omega_1\omega_2}Q^{\alpha\alpha,jj}_{\omega_3\omega_4}\,,\\[\hat{Q}\cdot\hat{Q}]_2&\equiv\sum_{\alpha}\sum_{i,j}\sum_{\omega_1,\ldots,\omega_4}\delta(\omega_1-\omega_2,\omega_4-\omega_3)Q^{\alpha\alpha,ij}_{\omega_1\omega_2}Q^{\alpha\alpha,ji}_{\omega_3\omega_4}\,.\end{aligned} \tag{3.3}$$

If all the components of the matrix field were independent, calculating the generating functional $Z[j]$ would be trivial since $\mathcal{L}[Q,j]$ is quadratic in Q. However, according to Eq. (2.20), the Q matrices are actually subject to the constraints

$$\hat{Q}=\hat{Q}^\dagger,\quad \hat{Q}^2=\hat{I},\quad \mathrm{Tr}\hat{Q}=0\,. \tag{3.4}$$

When external perturbations, such as magnetic impurities or strong magnetic fields (causing Zeeman splitting) are added, some components of the Q matrices become "massive," i.e., their fluctuations get suppressed.[4] For strong magnetic fields only diagonal spin components should be retained $Q^{ij}\to\delta_{ij}Q^i$, while magnetic impurities completely suppress all the spin structure so that $Q^{ij}\to\delta_{ij}Q$. The explicit calculations for all these different situations are extremely similar, differing only in their respective spin structures. In the rest of this section, and in Sec. IV, in order to display the structure of the theory we shall limit our attention to the case of strong magnetic fields. By doing appropriate modifications, we have repeated the analogous calculations for magnetic impurities and for the generic model, and only the resulting final RG equations will be presented in Sec. V.

Because the electron-electron interactions are spin independent, the total spin of any two electrons is conserved in a collision, making it useful to rewrite our model in terms of the singlet and triplet interaction channels. In the case of strong magnetic fields, the interaction part of $\mathcal{L}[Q]$ can accordingly be written as

$$\mathcal{L}_{\mathrm{int}}[Q]=\sum_{\alpha}\sum_{ij}\sum_{\omega_1,\ldots,\omega_4}\delta(\omega_1-\omega_2,\omega_4-\omega_3)Q^{\alpha\alpha,i}_{\omega_1\omega_2}S_{ij}Q^{\alpha\alpha,j}_{\omega_3\omega_4}\,. \tag{3.5}$$

Here, the spin structure matrix is defined by

$$S_{ij}\equiv\tfrac{1}{2}(2\pi T)(K_sP^s_{ij}+K_tP^t_{ij})\,, \tag{3.6}$$

where the singlet-triplet spin projectors are

$$\hat{P}_s=\tfrac{1}{2}\begin{bmatrix}1&1\\1&1\end{bmatrix},\quad \hat{P}_t=\tfrac{1}{2}\begin{bmatrix}1&-1\\-1&1\end{bmatrix}. \tag{3.7}$$

The idempotency and orthogonality of these projectors,

$$\hat{P}^2_{s/t}=\hat{P}_{s/t},\quad \hat{P}_s\hat{P}_t=\hat{P}_t\hat{P}_s=0\,, \tag{3.8}$$

implement the spin conservation laws and considerably simplify the spin sum evaluation when performing the loop expansion.

In practical calculations, it is convenient to parametrize the Q matrices in terms of a subset of independent components in order to eliminate the constraints. Many parametrizations are possible, but the one that we shall use offers a particularly simple loop expansion,[7] and is defined by

$$\hat{Q}=\begin{bmatrix}\sqrt{1-qq^\dagger} & q\\ q^\dagger & -\sqrt{1-q^\dagger q}\end{bmatrix}, \tag{3.9}$$

with matrix elements $q^{\alpha\beta,i}_{\omega_1\omega_2}$; $\omega_1\geq 0$, $\omega_2<0$, being independent complex numbers.

The loop expansion, i.e., the expansion in powers of the

coupling constant G, can be formulated by expanding the action in a Taylor series in independent fields q. The Gaussian ("free") theory obtained by truncating this expansion at quadratic order describes noninteracting diffusion modes. Higher-order terms such as cubic, quartic, and higher-order vertices account for the interaction of these modes, and can be systematically included in perturbation theory to any desired order in G. Near two dimensions, the MIT is located at $G \simeq d-2$, and the one-loop calculation is sufficient to first order in $\epsilon = d-2$. In the above notation, the free (Gaussian) propagator can be written as[7]

$$\langle q^{*\alpha_1\beta_1,i_1}_{\omega_1\omega_2}(\mathbf{p}_1) q^{\alpha_2\beta_2,i_2}_{\omega_3\omega_4}(\mathbf{p}_2)\rangle_0 = (2\pi)^d \delta(\mathbf{p}_1+\mathbf{p}_2)\delta_{\alpha_1\alpha_2}\delta_{\beta_1\beta_2} G \Big[\delta_{\omega_1\omega_3}\delta_{\omega_2\omega_4}\delta_{i_1 i_2} D_{\omega_1\omega_2}(\mathbf{p}_1) + \delta_{\alpha_1\beta_1}\delta(\omega_1-\omega_2,\omega_3-\omega_4)\frac{2\pi T}{\omega_1-\omega_2} \times [P^s_{i_1 i_2}\Delta D^s_{\omega_1-\omega_2}(\mathbf{p}_1) + P^t_{i_1 i_2}\Delta D^t_{\omega_1-\omega_2}(\mathbf{p}_1)] \Big] , \tag{3.10}$$

where

$$D_\omega(\mathbf{p}) = [\mathbf{p}^2 + HG\omega]^{-1} ,$$
$$D^{s/t}_\omega(\mathbf{p}) = [\mathbf{p}^2 + (H+K_{s/t})G\omega]^{-1} , \tag{3.11}$$
$$\Delta D^{s/t}_\omega(\mathbf{p}) = D^{s/t}_\omega(\mathbf{p}) - D_\omega(\mathbf{p}) .$$

Further details of the vertex structure and the renormalization for the unperturbed ($j_M = 0$) theory are well known[3,7] and will not be elaborated here.

In order to perform the renormalization of the UA, a thorough analysis of the vertex structure of the corresponding source term is required, especially since different parts of $\delta\mathcal{L}$ could behave differently under renormalization. The analysis of the UA in the noninteracting limit indeed showed such a complicated behavior.[9,10] Quite generally, the source term should be expanded in terms of scaling operators that rescale independently (i.e., do not mix) under renormalization. In the absence of interactions, symmetry (group theory) arguments can be invoked to construct the scaling operators.[18] Unfortunately, when electron-electron interactions are present, the mentioned symmetry arguments are no longer valid and a different procedure is needed to determine the scaling operators. To this end, we will allow different parts of $\delta\mathcal{L}$ to rescale independently, obtaining RG equations that will in principle mix different terms. The scaling operators can then be obtained by performing an eigenvalue analysis of the RG equations which also gives the corresponding critical exponents.

For strong magnetic fields, Q's are diagonal in spin indices, so that the source term can be written as

$$\delta\mathcal{L}[j,Q] = -\sum_{\omega_M} j_M \frac{1}{V} I[Q] , \tag{3.12}$$

where

$$I[Q] = \tfrac{1}{4}(2\pi T)\sum_{\omega_1,\omega_2}\sum_i Q^{11,i}_{\omega_1+\omega_M,\omega_2} Q^{11,i}_{\omega_2-\omega_M,\omega_1} . \tag{3.13}$$

To examine the vertex structure of the theory, we expand $\delta\mathcal{L}$, i.e., I in powers of q. Since the quadratic form I is diagonal in both replica and spin, we shall temporarily drop these indices and focus on the frequency structure. We first expand the Q's in term of q's:

$$Q = Q^{(0)} + Q^{(1)} + Q^{(2)} + \cdots , \tag{3.14}$$

where the index corresponds to the power in q

$$Q^{(0)}_{\omega_1\omega_2} = \delta_{\omega_1\omega_2}\mathrm{sgn}(\omega_1) ,$$
$$Q^{(1)}_{\omega_1\omega_2} = \Theta(\omega_1)[1-\Theta(\omega_2)]q_{\omega_1\omega_2} + [1-\Theta(\omega_1)]\Theta(\omega_2) q^\dagger_{\omega_1\omega_2} , \tag{3.15}$$
$$Q^{(2)}_{\omega_1\omega_2} = -\tfrac{1}{2}\Theta(\omega_1)\Theta(\omega_2)(qq^\dagger)_{\omega_1\omega_2} + \tfrac{1}{2}[1-\Theta(\omega_1)][1-\Theta(\omega_2)](q^\dagger q)_{\omega_1\omega_2} ,$$

etc., where $\Theta(x)=1$ for $x>0$ and is zero otherwise. In terms of $Q^{(a)}$, $a=0,1,2,\ldots$, the quadratic form $I[Q]$ can be written as

$$I[Q] = \sum_{a,b} I(a,b) ,$$

where

$$I(a,b) = \tfrac{1}{4}(2\pi T)\sum_{\omega_1,\omega_2} Q^{(a)}_{\omega_1+\omega_M,\omega_2} Q^{(b)}_{\omega_2-\omega_M,\omega_1} . \tag{3.16}$$

To quadratic order in q, only the terms up to $a,b=0,1,2$ are needed. The contributions will have a different structure depending on whether they come from diagonal or off-diagonal parts of Q. To lowest order, only diagonal terms contribute

$$I(0,0) = -\omega_M , \tag{3.17}$$

giving, upon analytical continuation, the SCBA result $\alpha=\alpha_0$. The linear terms $I(0,1)=I(1,0)=0$ vanish due to particle number conservation.

The terms quadratic in q's generate "masses," i.e., provide an infrared regularization for the theory. The diagonal contributions are

$$I(0,2) = -\tfrac{1}{4}(2\pi T)\sum_\alpha\sum_i\sum_{\omega_1=0}^{+\infty}\sum_{\omega_2=-\infty}^{0} q^{*\alpha 1,i}_{\omega_1,\omega_2-\omega_M} q^{\alpha 1,i}_{\omega_1,\omega_2-\omega_M} , \tag{3.18}$$

$$I(2,0) = -\tfrac{1}{4}(2\pi T)\sum_\beta\sum_i\sum_{\omega_1=0}^{+\infty}\sum_{\omega_2=-\infty}^{0} q^{*1\beta,i}_{\omega_1+\omega_M,\omega_2} q^{1\beta,i}_{\omega_1+\omega_M,\omega_2} ,$$

and the off-diagonal contributions are

$$I(1,1)=\tfrac{1}{4}(2\pi T)\sum_i \sum_{\omega_1=0}^{+\infty}\sum_{\omega_2=-\infty}^{0} q^{*11,i}_{\omega_1+\omega_M,\omega_2} q^{11,i}_{\omega_1,\omega_2-\omega_M} + \tfrac{1}{4}(2\pi T)\sum_i \sum_{\omega_1=0}^{+\infty}\sum_{\omega_2=-\infty}^{0} q^{*11,i}_{\omega_1,\omega_2-\omega_M} q^{11,i}_{\omega_1+\omega_M,\omega_2} \, . \tag{3.19}$$

Here, we have dropped contributions independent of the external frequency ω_M which do not contribute to the acoustic attenuation, as explained in Sec. II. We note that the off-diagonal terms have no internal replica index and also that their frequency structure is different than in the diagonal terms. Because of these differences in frequency and replica structure, we should allow for the possibility of diagonal and off-diagonal terms to scale differently. The obtained "mass insertions" could be in principle included in the Gaussian theory, which would lead to appropriate modifications of the propagator. However, according to Eq. (2.30), only the terms *linear* in the source field j_M contribute to UA, so we can restrict our calculation to this order and treat the above mass insertions as a part of the perturbation (as vertices).

The mixed terms such as $I(1,2)$ and $I(2,1)$ give no contribution to quadratic order, but will generate four different cubic vertices. In the zero-temperature formalism,[9,10] which describes noninteracting electrons, only vertices with even number of "legs" (q's) are present in both the unperturbed σ model and the UA source term. In the present formulation cubic source vertices are present, but do not contribute in the noninteracting limit since the unperturbed σ-model vertices with odd number of legs vanish in that case. When the interactions are turned on, the unperturbed σ model acquires cubic (interaction) vertices[7] giving nonzero contributions even from cubic source vertices.

For the one-loop renormalization of mass insertions, it is sufficient to include only cubic and quartic vertices. The off-diagonal terms are linear in q's so the only further contributions come from the diagonal terms $I(0,4)$, $I(4,0)$, and $I(2,2)$, giving five different quartic vertices. Obtaining the explicit form of these vertices using Eqs. (3.15)–(3.16) is straightforward and will not be elaborated here.

As an example of the presented formalism, and to make a contact with previous work, we next calculate the one-loop perturbation theory correction to the UA coefficient. To this order, it is sufficient to keep the expansion to second order in q only, giving

$$\alpha=\alpha_0\{1+(1/\omega)\mathrm{Im}(\langle I^{(2)}[Q]\rangle_0)|_{i\omega_M\to\omega+i\eta}+O(G^2)\} \, , \tag{3.20}$$

where

$$I^{(2)}[Q]=I(0,2)+I(2,0)+I(1,1) \tag{3.21}$$

and the expectation values $\langle\ \rangle_0$ are evaluated with respect to the free (Gaussian) action. Since the $I^{(2)}$ includes only the mass insertions (quadratic in q), the corresponding expectation values are just the appropriate integrated propagators. If we denote the noninteracting and interaction parts of the free propagator as

$$\langle\ \rangle_0=\langle\ \rangle_{\mathrm{NI}}+\langle\ \rangle_{\mathrm{int}} \, , \tag{3.22}$$

the one-loop corrections reduce to the sum of noninteracting and interaction corrections as follows. All noninteracting contributions vanish identically. In fact, $\langle I(0,2)\rangle_{\mathrm{NI}}$ and $\langle I(2,0)\rangle_{\mathrm{NI}}$ vanish due to an internal replica sum $\sim N\to 0$, but $\langle I(1,1)\rangle_{\mathrm{NI}}$ vanishes due to energy conservation (elastic scattering only).

The *interaction* contributions are diagonal in replica so that there is no internal replica summation; furthermore, the interaction allows for energy exchange which was forbidden above. Consequently, the interaction corrections do not vanish and we get

$$\langle I(0,2)\rangle_{\mathrm{int}}=-\tfrac{1}{4}(2\pi T)^2\sum_{\omega_1=0}^{+\infty}\sum_{\omega_2=-\infty}^{0}\int\frac{d\mathbf{p}}{(2\pi)^d}\sum_{s,t}\sum_i P^{s,t}_{ii}\Delta D^{s,t}_{\omega_1-\omega_2+\omega_M}(\mathbf{p}) \, . \tag{3.23}$$

At low temperatures, we can replace the frequency sums by integrals. To first order in disorder (in G) we keep only the corrections linear in ω_M giving

$$\langle I(0,2)\rangle_{\mathrm{int}}=(-\omega_M)\tfrac{1}{4}G\sum_{s,t}\gamma_{s,t}\int_0^{+\infty}dx\int\frac{d\mathbf{p}}{(2\pi)^d}(\mathbf{p}^2+x)^{-1}[\mathbf{p}^2+(1+\gamma_{s,t})x]^{-1}+O(G^2) \, . \tag{3.24}$$

Here we have introduced the rescaled frequency $x=HG\omega$ and the singlet-triplet interaction parameter $\gamma_{s,t}\equiv K_{s,t}/H$. In two dimensions, this integral diverges logarithmically in the infrared, as expected at the lower critical dimension. Such logarithmic corrections can be summed up by a renormalization-group approach as will be done in Sec. IV.

From other mass insertions, we find similar contributions $I(2,0)=I(0,2)$ and $I(1,1)=-2I(0,2)$. Although individual mass insertions give nonzero contributions, when all the terms are added together, they all cancel out so the total UA correction vanishes to give

$$\alpha=\alpha_0[1+O(G^2)] \, . \tag{3.25}$$

The same perturbation theory result has previously been obtained by conventional diagrammatic methods, but only to first order in interaction.[8,11] Our results show that there are no UA corrections to first order in disorder, even if interaction is included to *all* orders as we have presently done.

In a perturbative approach such as presented here, one neglects two effects that can be very important near the critical point.

(i) Different parts of the source term can scale differently even if the bare value of the appropriate prefactors is the same. For example, if the diagonal and off-diagonal terms scaled differently, the above one-loop contributions would not cancel out.

(ii) New terms can be generated under renormalization even if they were absent in the bare theory. If this actually happens, such terms have to be included in the theory from the start.

In a consistent scaling analysis, one has to allow for the possibility of having both of these effects present and we will do so in the following section, where the renormalization of UA is performed.

IV. RENORMALIZATION OF ULTRASONIC ATTENUATION

When a renormalization-group analysis is carried out, one performs a loop expansion in the *renormalized* theory which often differs from the bare or unrenormalized one. Under renormalization some operators become irrelevant and can be ignored, while others can be generated even if originally absent from the theory. For the nonlinear σ model the determination of these operators is quite subtle, since the loop expansion proceeds, as shown in Sec. III, by eliminating the constraints and expanding in independent q fields. The terms generated under RG are then obtained only to a given order in the q expansion, rather than in closed form in terms of Q fields. Since the symmetries of the theory have to be preserved under renormalization,[19] one has to be able to write the renormalized Lagrangian in the closed form.

For noninteracting electrons the determination of the invariant form of the renormalized theory, i.e., the form of the scaling operators, can be determined using symmetry considerations. The unperturbed part of the action is then invariant under arbitrary global unitary transformations and by using group-theoretic arguments it has been shown[18] that the scaling operators correspond to the symmetric and antisymmetric irreducible representations of the permutation group. In the interaction case, the unitary symmetry is broken,[19] so the above group theory arguments cannot be used to construct the scaling operators. Still, the symmetry arguments can be used to determine at least the closed-form expressions for the operators generated under RG in the noninteracting limit. These operators are the minimum set of new operators that have to be added to the bare source terms. By performing the loop expansion we will then check that no further "new" terms are generated when interactions are present.

Although we have used the finite-temperature formalism which has internal frequency summations, in the absence of interactions the unperturbed action still has the unitary symmetry and the same strategy can be used to obtain the "new" operators. These new operators are obtained from the old ones by switching the second set of indices of the Q matrices.[18]

In our case, by using this approach we obtain the new part of the source term operator in the form [compare Eqs. (3.12) and (3.13)]

$$\delta\tilde{\mathcal{L}}[j,Q]\simeq\frac{1}{V}\tilde{I}[Q]\ , \tag{4.1}$$

where

$$\tilde{I}[Q]=-\tfrac{1}{4}(2\pi T)\sum_{\omega_1,\omega_2}\sum_{i}Q^{11,i}_{\omega_1+\omega_M,\omega_1}Q^{11,i}_{\omega_2-\omega_M,\omega_2}\ . \tag{4.2}$$

When renormalization is performed, we shall add this part to the source term from the start, although its bare value is zero. Just as for the original part, we expand this "new" term in powers of the independent q fields to obtain the corresponding vertex structure. Using the notation similar to the one in Eqs. (3.15)–(3.16), we immediately conclude that $\tilde{I}(0,a)=\tilde{I}(a,0)=0$ for all $a=0,1,2,\ldots$. In particular, since $\tilde{I}(0,0)=0$, there is no constant term (i.e., SCBA contribution). The linear terms vanish as before, but this time we also find no diagonal contributions quadratic in q since $\tilde{I}(0,2)=\tilde{I}(2,0)=0$. The only new mass insertion comes from the off-diagonal terms

$$\tilde{I}(1,1)=-\tfrac{1}{4}(2\pi T)\sum_{\omega_1=-\omega_M}^{0}\sum_{\omega_2=-\omega_M}^{0}q^{*11,i}_{\omega_1+\omega_M,\omega_1}q^{11,i}_{\omega_2+\omega_M,\omega_2}\ . \tag{4.3}$$

Furthermore, we also find four cubic vertices coming from $\tilde{I}(1,2)$ and $\tilde{I}(2,1)$, and four quartic vertices coming from $\tilde{I}(2,2)$. Again, the explicit form of these vertices is easily obtained by the procedures of Sec. III, and will not be presented here. We will just mention that in these vertices, just as in the above mass insertion, a number of frequency integrations is restricted to the interval $\omega\in(-\omega_M,0)$. To first order in G, this restriction makes irrelevant most of the diagrams containing the "new" vertices, which considerably simplifies their calculation.

Before starting the actual renormalization calculations we have to specify the general scaling form for our (renormalized) source term. This form has to be expressed in close form, i.e., in terms of the Q fields. Since the source term is diagonal in spin, and has fixed (external) replica indices, different contributions will be characterized by their respective frequency structures. In order to allow for maximum generality, we will allow for all the parts of our source term which have different structure to renormalize independently and write

$$\delta\mathcal{L}=-\frac{1}{V}\sum_{l=1}^{6}a_l A_l\ . \tag{4.4}$$

Here, the bare value of $a_1,\ldots,a_3$ is j_M, of $a_4,\ldots,a_6$ it is zero, and we have included the "new" terms under the symbol $\delta\mathcal{L}$. The operators $A_1,\ldots,A_6$ correspond to different ways in combining positive and negative frequency pieces, i.e., to different ways of combining the diagonal and off-diagonal parts of the Q matrices. In terms of the q expansion these operators are

$$A_1 = I(0,0)+I(0,2)+I(2,0)+I(2,2) + I(0,4)+I(4,0)+\cdots , \tag{4.5}$$

$$A_2 = I(1,1) , \tag{4.6}$$

$$A_3 = I(1,2)+I(2,1)+\cdots , \tag{4.7}$$

$$A_4 = \tilde{I}(2,2)+\cdots , \tag{4.8}$$

$$A_5 = \tilde{I}(1,1) , \tag{4.9}$$

$$A_6 = \tilde{I}(1,2)+\tilde{I}(2,1)+\cdots . \tag{4.10}$$

After expanding in independent q fields, these operators take a form of infinite series containing vertices of all orders in q. Since the theory has to take the same form after each RG step, each term in the expansion of an operator has to renormalize *identically*.[19] We can thus obtain the recursion relations for a coefficient a_l by considering any of the terms in the q expansion of the corresponding operator A_l. This observation is important since it allows us to obtain the same RG flow equation by various routes, which will restrict the number of independent scaling fields. In particular, the RG flow equation for a_1 can be obtained by either calculating the renormalization of the "constant" term $I(0,0)$, or the renormalization of the mass insertions $I(0,2)$ and $I(2,0)$. To one loop order, the cubic and quartic vertices do not contribute to the constant renormalization, but they do contribute to the mass renormalization. By comparing the RG flow equation for a_1 obtained in the two fashions we have derived the relations

$$a_3 = a_1 , \tag{4.11}$$

$$a_5 = a_4 . \tag{4.12}$$

We have also found that the cubic vertices of A_6 do not give any contribution to one-loop order. Furthermore, since $\langle A_6 \rangle_0 = 0$, this operator can be ignored at least to the considered order. Since we now have only three independent parts of the source term, and also have three different mass insertions, the desired RG flow equations can be obtained by calculation of mass renormalization only. We assume that the theory is renormalizable,[19] i.e., that the higher-order vertices renormalize identically. In principle, further checks of the renormalizability could be performed by explicitly renormalizing the higher-order vertices, as it has been done in Ref. 19 for the model without the extra source term. However, in our case the calculations of the higher vertex renormalization would be extremely cumbersome due to the large number of vertices in the theory.

The mass insertion found using the symmetry arguments will be generated under RG even in the noninteracting limit. The above form appropriate to the noninteracting case is diagonal in spin indices. When interactions are turned on, we have found that the singlet and triplet parts of this operator renormalize differently. Accordingly, we have to introduce two different scaling fields corresponding these two spin structures. The operator A_5 breaks in two parts that scale independently and which take the form

$$A_5^{s/t} = -\tfrac{1}{4}(2\pi T) \sum_{i,j} \sum_{\omega_1=-\omega_M}^{0} \sum_{\omega_2=-\omega_M}^{0} q^{*11,i}_{\omega_1+\omega_M,\omega_1} \times P^{s/t}_{ij} q^{11,j}_{\omega_2+\omega_M,\omega_2} . \tag{4.13}$$

Note that in the noninteracting limit $A_5^s + A_5^t = A_5$ since then the renormalization is spin independent and the spin projectors form a complete set $P^s + P^t = I$. The vertices coming from A_4 do contribute to the renormalization of the original mass insertions, but only noninteracting contributions from these vertices are found.

In view of the above considerations, we can rewrite the source operator in the scaling form

$$\delta\mathcal{L} = -\frac{1}{V}(b_1 B_1 + b_2 B_2 + b_3^s B_3^s + b_3^t B_3^t) , \tag{4.14}$$

where

$$B_1 = A_1 + A_3 , \tag{4.15}$$

$$B_2 = A_2 , \tag{4.16}$$

$$B_3^{s/t} = A_4^{s/t} + A_5^{s/t} . \tag{4.17}$$

To each of these operators, there corresponds a mass insertion with different symmetry, and we will obtain the RG equations for the b's by performing the one-loop mass renormalization calculation.

In order to perform the mass renormalization, we have calculated all the one-loop diagrams with two external legs. Different topological classes of such diagrams are schematically represented in Fig. 1. Because we have a total of three different mass insertions, eight different cubic source vertices and nine different quartic source vertices in addition to the unperturbed σ-model vertices, the calculations are of considerable complexity, although straightforward in principle. Fortunately, a large number of diagrams can be eliminated by counting the number of independent frequency integrations in each diagram. Namely, when a frequency is rescaled as in Eq. (3.24), each frequency integration will absorb one power of the coupling constant G. The diagrams with "insufficient" numbers of frequency integrations will be of $O(G^2)$ and can be ignored to first order in disorder. The same diagrams also prove to have extra powers of temperature and thus will be irrelevant at $T=0$ even to higher order in disorder.

V. RESULTS

We first present our results for the case of strong magnetic fields. After the contributions from different diagrams are combined, considerable cancellations occur, and the resulting one-loop corrections can be represented in the following simple form:

$$\delta b_1 = \tfrac{1}{2} G (b_1 - b_2) \sum_{s,t} \mathcal{I}_1^{s/t} + \tfrac{1}{4} G (b_3^s + b_3^t) \mathcal{I}_2 . \tag{5.1}$$

$$\delta b_2 = -\tfrac{1}{4} G (b_1 - b_2) \sum_{s,t} \mathcal{I}_1^{s/t} + \tfrac{1}{4} G (b_3^s + b_3^t) \mathcal{I}_2 , \tag{5.2}$$

$$\delta b_3^{s/t} = \tfrac{1}{2} G b_1 \mathcal{J}_2 + G b_2 (K_{s/t} \mathcal{J}_3 + 4 K_{s/t}^2 \mathcal{J}_4) + \tfrac{1}{2} G b_3^{s/t} \sum_{s,t} \mathcal{J}_1^{s/t} , \tag{5.3}$$

with

$$\mathcal{J}_1^{s/t} = \int \frac{d\mathbf{p}}{(2\pi)^d} \int_0^{+\infty} d\omega \frac{1}{\omega} \Delta D_\omega^{s/t}(\mathbf{p}) , \tag{5.4}$$

$$\mathcal{J}_2 = \int \frac{d\mathbf{p}}{(2\pi)^d} D_{\omega=0}(\mathbf{p}) , \tag{5.5}$$

$$\mathcal{J}_3 = G \int \frac{d\mathbf{p}}{(2\pi)^d} \int_0^{+\infty} d\omega \, D_\omega^2(\mathbf{p}) , \tag{5.6}$$

$$\mathcal{J}_4 = G^2 \int \frac{d\mathbf{p}}{(2\pi)^d} \int_0^{+\infty} d\omega \, \omega \, D_\omega^3(\mathbf{p}) . \tag{5.7}$$

For a one-loop renormalization-group calculation, it is the easiest and physically most transparent to use the momentum-shell method[20] in which we integrate at each step over the momenta in the interval $\Lambda/b < |\mathbf{p}| < \Lambda$, where Λ is the momentum cutoff, and over all the frequencies.

All the integrals are proportional to $\delta l = \ln b$, the thickness of the shell, and by letting δl be infinitesimal, we obtain the desired differential RG flow equations. It is convenient to express these RG flow equations in matrix form by defining the column vector $\underline{b}$ with entries b_1, b_2, b_3^s, and b_3^t and we get

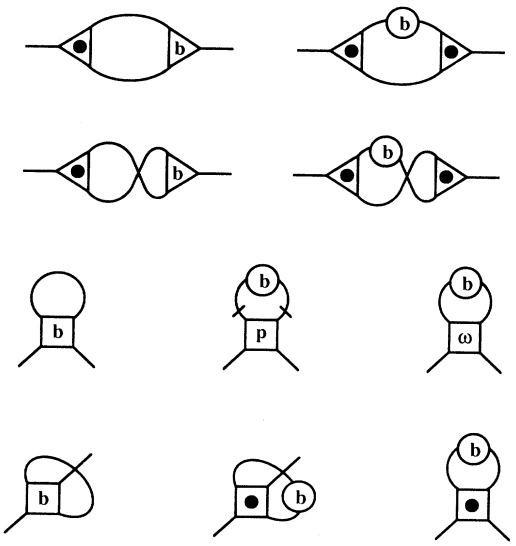

FIG. 1. Different topological classes of diagrams for the one-loop renormalization of the ultrasonic attenuation. Here, the vertices with a symbol "b" represent the UA source term vertices. The solid dot represents the interaction verices, "p" the momentum vertex, and "ω" the frequency vertex of the unperturbed σ model. All the diagrams for UA contain a single source field vertex, since higher powers of the source field b do not contribute to UA. Note that the diagrams with "direct" and "crossed" legs are *not* equivalent since $q_{\omega_1 \omega_2}^{\alpha\beta, ij}$ is a matrix (tensor) field.

$$\frac{d}{dl} \underline{b} = \frac{g}{8\pi} \hat{Y} \underline{b} , \tag{5.8}$$

where the matrix $\hat{Y}$ is

$$\hat{Y} = \begin{bmatrix} 2L & -2L & 1 & 1 \\ -L & L & 1 & 1 \\ 2 & 4\gamma_s(1+2\gamma_s) & 2L & 0 \\ 2 & 4\gamma_t(1+2\gamma_t) & 0 & 2L \end{bmatrix} . \tag{5.9}$$

Here, we have introduced the renormalized coupling constant $g(b) = b^{-(d-2)} G(b)$, and the quantity L is a function of the interaction parameters $\gamma_{s/t} \equiv K_{s/t}/H$, to be specified below.

The critical exponents and the corresponding scaling operators for UA can now be obtained by performing an eigenvalue analysis[20] of our RG flow equations in the vicinity of the MIT fixed point. At first glance, it appears that we will get four independent scaling parts since we diagonalize a 4×4 matrix. A closer inspection reveals that the situation is in fact simpler. We first note that the structure of the matrix $\hat{Y}$ is such that if initially $b_1 = b_2$, as it is indeed the case in the bare theory, this will remain true under renormalization. We can thus restrict our attention to the three-dimensional subspace corresponding to $b_1 = b_2$. Second, we find an eigenvector of the form $\mathbf{b}^{(0)} \simeq (0,0,1,-1)$ with eigenvalue $2L$. This scaling operator is proportional to $P^s - P^t$, which is fully off diagonal in spin, and so does not contribute to the UA either. In this way we conclude that the UA will have only two independent scaling parts. The corresponding eigenvalues of the matrix $\hat{Y}$ can be calculated analytically with the result

$$\lambda_\pm = L \pm (L^2 + 4 + 16\gamma^2)^{1/2} , \tag{5.10}$$

where we have used the fact[5] that at the MIT fixed point $\gamma_s^* = -\gamma_t^* \equiv \gamma$. The results of Eqs. (5.9) and (5.10) are valid for disordered interacting electrons in a strong magnetic field, for both short- and long-ranged interactions. The fixed point values of g and $\gamma_{s/t}$ depend on the interaction range, as does the form of L, so we now separately discuss these two situations.

For short-ranged interactions and near the MIT fixed point $L = \ln(1-\gamma^2)$. Furthermore, the interaction amplitudes $\gamma_{s/t}$ as well as the coupling constant g assume *nonuniversal*[5] values since γ_s remains unrenormalized. Consequently, the critical exponents $x_\pm = (g^*/8\pi)\lambda_\pm$ are also nonuniversal

$$x_\pm^{sr}(\gamma) = \frac{\lambda_\pm(\gamma)}{f(\gamma)} \epsilon , \tag{5.11}$$

where

$$f(\gamma) \equiv 4 - 2\left[1 + \frac{1}{\gamma}\right] \ln(1+\gamma) - 2\left[1 - \frac{1}{\gamma}\right] \ln(1-\gamma) , \tag{5.12}$$

and the parameter γ can assume any value in the interval $(0,1)$.[5] Note that x_+ is positive (relevant) and x_- is negative (irrelevant) so that the UA diverges as $\alpha(b) \simeq b^{x_+}$ when $b \to \infty$.

When Coulomb (long-ranged) interactions are present, the situation changes since in that case the particle conservation condition requires $\gamma_s=-1$, and the coefficients of the matrix $\hat{Y}$, as well as the fixed point value of g, become universal numbers. Furthermore, we note that the condition $\gamma_s=-1$ makes the interaction part of the propagator $\Delta D^s_\omega(\mathbf{p})$ more singular, which can affect the degree of divergence of certain integrals. In our case only $L\simeq\mathcal{I}^s_1$ is affected since this is the same integral entering the single-particle density-of-states renormalization. A careful analysis[3] shows that the integral in question leads to a contribution $\simeq 1/\epsilon$ in $2+\epsilon$ dimensions, giving $L=-z/\epsilon$. To one loop, the dynamical exponent is $z=d$ in the case of strong magnetic fields. In this way, for Coulomb interactions we obtain *universal* exponents which to lowest nontrivial order in ϵ take the values

$$x_+=\frac{5}{4(1-\ln 2)}\epsilon^2 , \tag{5.13}$$

$$x_-=\frac{-1}{(1-\ln 2)} . \tag{5.14}$$

It is interesting to note that the expansion of the relevant exponent x_+ starts with the quadratic order in ϵ. This result can be directly traced back to the fact that the bare part of the source term is by itself unrenormalized, which is also the origin of the absence of first-order corrections to the UA.

Next, we turn our attention to the case when *magnetic impurities* are present, which is experimentally the most common situation. As mentioned in Sec. III, magnetic impurities will completely suppress spin fluctuations,[4] so that $Q^{ij}\simeq\delta_{ij}Q$. The frequency and replica structure is unaffected, and the calculations proceed very similarly to the case of strong magnetic fields. The contributions from the triplet spin channel are now suppressed, and only three scaling fields b_1, b_2, and b_3 are needed. Just as in the previous case, we can show that $b_1=b_2$ remains conserved under renormalization so that again only two scaling parts for UA are found. The scaling field vector $\mathbf{b}$ now reduces to a two-component vector (b_1,b_3), and the RG equations take the same form as in Eq. (5.8), with the matrix $\hat{Y}$ presently taking the form

$$\hat{Y}=\begin{bmatrix} 0 & 1 \\ 1+2\gamma_s(1+2\gamma_s) & 2L \end{bmatrix} . \tag{5.15}$$

In the short-ranged case, the interactions are irrelevant in presence of magnetic impurities, and the problem is reduced to the noninteracting limit. In agreement with Castellani and Kotliar,[10] we find exponents $x^u_\pm=\pm\sqrt{2\epsilon}$ at the (noninteracting) unitary fixed point. When Coulomb interactions are present, the interactions *are* relevant[3,5] since $\gamma_s=-1$ and there is an interaction-driven MIT at $g^*/8\pi=\epsilon/2$. Similarly as for strong magnetic fields, the long-ranged interactions lead to $L=-z/\epsilon$, but in this case $z=d-\epsilon/2$. The critical exponents in the present case of magnetic impurities and long-ranged interactions take the (again universal) values

$$x_+=\tfrac{3}{8}\epsilon^2 , \tag{5.16}$$

$$x_+=-1 . \tag{5.17}$$

Finally, we have also examined the generic case where no external perturbations are present. The electronic system can then undergo a pseudomagnetic phase transition[7] before the MIT is reached. This phase transition is located at $\gamma_t^*=\infty$ and $g^*=0$ such that $y^*\equiv(\gamma_t g)^*$ is finite. The calculations for the UA are similar as before, except that now all the spin components of Q^{ij} have to be considered. The UA corrections have been calculated to one-loop order with results of the similar form as for strong magnetic fields. The most important difference is in the structure of the appropriate spin sums and this time no terms proportional to γ_t are found. Near the magnetic fixed point $g\to 0$ and all the corrections vanish. The ultrasonic attenuation therefore remains *uncritical*, similar to the charge diffusion constant. This is not unexpected since the phonons do not couple directly to spin fluctuations.

VI. CONCLUSIONS

In this paper we have considered the behavior of the ultrasonic attenuation near interaction-driven metal-insulator transitions. Since the decay of phonons occurs in metals by decay to electron-hole pairs, the UA is strongly affected by the reduction of the electronic mobility as the MIT is approached. The critical behavior of the electronic conductivity is known to be insensitive to changing the symmetry classes corresponding to the presence of different magnetic perturbations in the systems. This study has addressed the question whether other transport properties, as exemplified by the ultrasonic attenuation, will behave similarly, or whether a more complicated behavior can occur.

In a perturbative approach to first order in disorder, but including the electron-electron interactions to all orders, we have found no corrections to the UA in contrast to the case of the conductivity, where such corrections are present.[3,4] Using a renormalization-group treatment, we have shown that the absence of first-order corrections follows from the fact that the UA consists of two independent scaling parts, i.e.,

$$\alpha(b)=\alpha_0[b^{x_+}f_+(b^z\omega,b^{1/\nu}\tau)+b^{x_-}f_-(b^z\omega,b^{1/\nu}\tau)] , \tag{6.1}$$

where $\tau\sim|g-g_*|$ determines the distance from the critical point, and $\nu=1/\epsilon$ is the correlation length exponent to one-loop order. The dynamical exponent $z=d$ in all the cases considered, except for magnetic impurities and long-range interactions where $z=d-\epsilon/2$. Equation (6.1) is valid at $T=0$, but the temperature scaling can be obtained by simply replacing ω by T in the above expression. The critical exponents $x_\pm$ (given in Sec. V) describing the two scaling parts of the UA are found to depend both on the symmetry class and the interaction range. A nontrivial critical behavior was found in the case of magnetic impurities and strong magnetic fields (causing Zee-

man splitting). In presence of long-range (Coulomb) electron-electron interactions, the critical exponents are found to be universal, while short-range interactions lead to nonuniversal exponents. In the absence of the perturbations (generic case), near the pseudomagnetic phase transition, the critical exponents vanish, and the UA is therefore noncritical there.

In the present theoretical approach, we have used several simplifications which we will discuss as follows. First, in constructing the σ-model description, we have ignored the particle-particle channel ("cooperons"). The corresponding corrections have been shown to be irrelevant[4] in the presence of perturbations which break the time-reversal invariance, as magnetic fields or magnetic impurities. The cooperons are believed to be relatively unimportant *above* two dimensions even in the generic case,[7] but their precise role continues to be a topic of active research and still represents an open question. Second, the treatment of the Coulomb interactions is based on the screening approximation,[3,5] which is expected to be valid in the metallic phase. The resulting differences from the short-range interaction case are limited to enforcing the compressibility sum rule requiring that $\gamma_s = -1$ be valid at any disorder. Finally, as mentioned earlier, the renormalizability of the considered σ model has been assumed, although it has been proven only within a loop expansion. In particular, we have found that contributions to the UA source term are generated under renormalization. A closed-form expression for these terms was obtained by using a symmetry argument valid in the absence of interactions. By performing a one-loop calculation we have then confirmed that the terms retain their form even in the presence of interactions and that no further terms are generated. All of these simplifications have been used at the same level as in the corresponding previous investigations of the conductivity,[3,5,7] and their further validity is not addressed in this paper.

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