Fractional quantum Hall effect and Chern-Simons gauge theories

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We present a theory of the fractional quantum Hall effect (FQHE) based on a second-quantized fermion path-integral approach. We show that the problem of interacting electrons moving on a plane in the presence of an external magnetic field is equivalent to a family of systems of fermions bound to an even number of fluxes described by a Chern-Simons gauge field. The semiclassical approximation of this system has solutions that describe incompressible-liquid states, Wigner crystals, and solitonlike defects. The liquid states belong to the Laughlin sequence and to the first level of the hierarchy. We give a brief description of the FQHE for bosons and anyons in this picture. The semiclassical spectrum of collective modes of the FQHE states has a gap to all excitations. We derive an effective action for the Gaussian fluctuations and study the hydrodynamic regime. The dispersion curve for the magnetoplasmon is calculated in the low-momentum limit. We find a nonzero gap at ω_c. The fractionally quantized Hall conductance is calculated and argued to be exact in this approximation. We also give an explicit derivation of the polarization tensor in the integer Hall regime and show that it is transverse.

I. INTRODUCTION

The fractional quantum Hall effect (FQHE) is a fascinating condensed state of matter whose study has unveiled some previously unsuspected properties of quantum-mechanical systems in two dimensions. A system of fermions with only repulsive interactions has been found to display properties more characteristic of Bose condensed systems such as long-range correlations in the ground state and excitations with both fractional charge and fractional statistics. In addition, the Hall conductance is an exact fraction of the "quantum" e^2/h. Most of what is now understood about this fundamental phenomenon has been found by one (or more) of the following approaches: (i) the Laughlin wave function,[1] (ii) Jain's generalization,[2] (iii) exact diagonalization of small systems,[1,3] and (iv) the "Bose"–"Ginzburg-Landau"[4–6] picture.

Superficially, the approaches (i) and (ii) resemble to be what in many-body physics is normally called a mean-field approach: a wave function for the ground state is derived, usually by means of a variational calculation, in this case by educated guess. However, the many-body mean-field theories usually involve a starting point in which the particles (say, fermions, in this case) move in the presence of an average field due to the presence of the other particles. Thus, the mean-field wave function is an appropriately generalized Slater determinant. However, the Laughlin wave function and its generalizations cannot be written in a determinant form. For a system of N interacting electrons moving in a plane, in the presence of an external uniform magnetic field of strength B, the Laughlin wave function for N polarized electrons is

$$\Psi(z_1,\ldots,z_N)=\prod_{i<j}(z_i-z_j)^m \exp\left[-\sum_{i=1}^{N}\frac{|z_i|^2}{4l^2}\right], \tag{1.1}$$

where the set $\{z_i\}$ $(i=1,..,N)$ are the coordinates of the N electrons in complex notation $(z=x+iy)$ and l is the cyclotron radius. The *odd* integer m is equal to the inverse of the filling fraction $\nu=N/N_\phi$ of the lowest Landau level, where N_ϕ is the total number of flux quanta going through a sample of linear size L, $N_\phi=(1/2\pi)[BL^2/(\hbar c/e)]$. Quite surprisingly, the Laughlin wave function has turned out to yield an excellent ground-state energy[1,3] and, for the case of a particular potential, it has been shown to be the *exact* wave function[7] for the ground state. It has also yielded a good description of the spectrum of collective modes.[8] The "Bose" picture,[4–6] on the other hand, gives information in a language that makes the FQHE look a lot more like superfluidity than what the "Fermi" picture may suggest. Thus, it is possible to formulate a Ginzburg-Landau–type theory[5] which can account for many qualitative properties of the FQHE.

The Laughlin state was originally constructed in close analogy with the wave functions commonly used in theories of superfluid Helium. Shortly afterwards, Halperin[9] realized that the quasiparticles supported by this state exhibit not only fractional charge but that they are *anyons,* particles with fractional statistics.[10] This observation gave rise to the picture of the FQHE as a ground state of "electrons bound to fluxes." In this picture, the main role of the long-range correlations is to make it possible for the electrons to "nucleate" flux. Jain[11] realized that in the Laughlin state the electrons nucleate enough flux so that the bound states exactly fill up an integer number of the Landau levels associated with the unscreened part of the field. In this formulation, the FQHE is an integer quantum hall effect (IQHE) of the bound states.[2,11] Jain proposed to write the Laughlin wave function Eq. (1.1) in the suggestive factorized form

$$\Psi(z_1,\ldots,z_N)=\prod_{i<j}(z_i-z_j)^{m-1}\chi_1(z_1,\ldots,z_N), \tag{1.2}$$

where χ_1 is the wave function for a completely filled lowest Landau level

$$\chi_1(z_1,\ldots,z_N)=\prod_{i<j}(z_i-z_j)\exp\left[-\sum_{i=1}^{N}\frac{|z_i|^2}{4l^2}\right]. \tag{1.3}$$

The phases associated with the first factor in Eq. (1.2) represent an even number $(m-1)$ of fluxes that are attached to each coordinate z_i where an electron is present. It is a crucial feature of this picture that the electrons bind to an *even* number of flux quanta and, in this way, they retain their fermion character. Jain's approach has allowed for a simple description of the so-called hierarchy states[12,13] in terms of wave functions which have a factorized structure. This picture also suggests a possible way to construct a mean-field theory for the FQHE: the fermions "nucleate" enough flux to partially screen the external field. The effective field in which they move is due to both the external field and to the correlations among the electrons. The wave function χ_1 is just the state described by this mean-field theory. We will show in this paper that it is possible to use the methods of second quantization to derive a mean-field theory which precisely has this physics. However, the physics of the Laughlin state cannot be described by a wave function such as χ_1 which represents a set of completely filled Landau levels. As we will show below, the FQHE and the physics of the Laughlin wave function turn out to follow from the effects of fluctuations about this mean field.

Although the progress in the understanding of the FQHE has been very impressive, many fundamental questions remain without a reasonable answer. Most of all, it remains unclear how a system of electrons in an external magnetic field manages to turn the correlations, which result from the electron-electron interactions, into "nucleated fluxes." Similarly, although a number of very good arguments have been given which explain why the Laughlin wave function is the exact ground state for a class of local potentials,[3,7] it is not clear what sort of corrections this state should have (if any) for arbitrary potentials, local or not. In this paper we develop a formalism which, in principle, can be used to answer these questions and can also serve as a practical computational tool.

In this work we show that the key to the answer of these issues is the Chern-Simons gauge theory. In 1982 Wilczek's observed[10] that a particle current coupled to a Chern-Simons (CS) gauge field produced states with fractional statistics through the binding of particles to fluxes. Thus, if we are to get the Laughlin wave function Eq. (1.1) by attaching $m-1$ fluxes to each electron, as suggested by Eq. (1.2), it is natural to guess that the "right" theory must contain fermions (electrons) coupled to Chern-Simons gauge fields with an appropriate value of the Chern-Simons coupling constant θ. What is less clear is the origin of such a Chern-Simons gauge field for the problem of interacting electrons in a magnetic field. In quantum field theory, where the CS gauge theory was first introduced,[14] the Chern-Simons term in the action originates from the parity anomaly of relativistic fermions in $2+1$ dimensions. Clearly, the electrons which live in quasi-two-dimensional electron gases, such as the metal-oxide-semiconductor field-effect transistors (MOSFET's), are not relativistic. Also the real three-dimensional system one is dealing with, neither breaks Parity by itself nor as a result of the presence of the device. The Chern-Simons gauge field must then be a result of the dynamics of interacting two-dimensional electrons in the presence of an external magnetic field. For our purposes, the most important feature of the Chern-Simons term is not its relativistic invariance, but the fact that it is the only local gauge-invariant theory which yields bound states of particles and fluxes. Chern-Simons theories have also become the essential ingredient of some of the more recent theories of strongly correlated electron systems, such as the chiral spin liquid[15,16] and anyon superconductivity.[17-24]

In this paper we derive a field theory for the FQHE in the fermion language in which the Chern-Simons gauge field appears explicitly in the problem of interacting *electrons* in a magnetic field. We do so by first considering a theory in which the electrons, in addition to their mutual interaction, are coupled to both an external electromagnetic field and a Chern-Simons gauge field. We show that, if the coefficient of the Chern-Simons term is chosen in such a way that an *even* number of flux quanta get attached to each electron, all the physical amplitudes calculated in this theory are identical to the amplitudes calculated in the standard theory, in which the CS field is absent. We further show that the Laughlin state is the *semiclassical* approximation of this theory. In the "classical" (i.e., mean-field) approximation we get a picture identical to the one proposed by Jain. We show that, at this level, the only states that can be described are the Laughlin states and the first level of the hierarchy. This theory, not only explains where the fluxes come from, but also allows for the systematic calculation of corrections around this state in a manner similar to the Wentzel-Kramers-Brillouin (WKB) approximation in quantum mechanics. In particular, we calculate the Hall conductance directly from the field theory, we discuss the excitation spectrum in this approximation, and show that this state has a gap to all excitations and thus that it is incompressible. This formalism also yields a picture of the FQHE for *anyons.*

The approach that we follow here is based on a second-quantized *fermion* path-integral approach. Other path-integral approaches have been used for the study of the FQHE. Kivelson, Kallin, Arovas, and Schrieffer[25] (KKAS) introduced a *first-quantized* path-integral approach based on coherent states labeled by the guiding center coordinates of the single-particle states of the lowest Landau level. The approach of KKAS, refined later on by several authors,[26,27] yields a natural description of the Wigner crystal states as well as a qualitative picture of the phase transition (at zero temperature) into the liquid phase. Zhang, Hansson, and Kivelson[5] used a version of the fractional statistics transformation to map the FQHE problem into a path integral over Bose degrees of freedom coupled to a Chern-Simons gauge field. The "superfluid" state of the equivalent boson problem was identified by these authors with the fractional quantum Hall state. Although this theory yields a correct qualitative description of physics of the low-energy degrees of freedom, in the form of a Landau-Ginzburg theory, it has difficulties with the short-distance behavior of the system. One of the outstanding issues is the problem of the identification of the collective modes, their energy gaps, and in particular the possible existence of magnetorotons. Some of these difficulties have been resolved recently by Lee and Zhang,[28] who have stressed the important role that vortices play in this bosonic description. In our fermion path-integral approach, the semiclassical limit yields a picture very close to that of Jain's: the fermions nucleate enough *even* number of flux quanta so as to, on average, become equivalent to a problem in which an *integer* number of Landau levels of the *effective* flux are completely filled. There is a nonzero gap and a semiclassical expansion around this state yields a spectrum whose quantum numbers coincide with those obtained with the Laughlin wave function. The picture that emerges from our study has a close connection with the physics of anyon superconductors. As expected, the anyon superconductors are the *compressible* partners of the *incompressible* FQHE states.

The paper is organized as follows. In Sec. II we derive the Chern-Simons theory of the FQHE. In Sec. III we consider the semiclassical approximation and show that the behavior of the system in this limit agrees with the results of Laughlin and Jain. We also discuss the case of bosons and, more generally, of anyons. In Sec. IV we discuss the spectrum of low-lying excitations, including a calculation of the magnetoplasmon dispersion curve. We also calculate the Hall conductance. Section V is devoted to the conclusions. In Appendix A we give the details of the proof of the statement that "an even number of flux quanta is the same as nothing" and in Appendix B we present the calculation of the polarization tensor and the proof of its transversality.

II. A CHERN-SIMONS THEORY FOR THE FRACTIONAL QUANTUM HALL EFFECT

Consider a system of N electrons moving on a plane in the presence of an external uniform magnetic field B perpendicular to the plane. The electrons will be assumed to have an interparticle interaction governed by a pair potential $V(|\mathbf{r}|)$, for two electrons separated a distance $|\mathbf{r}|$ on the plane. The magnetic field will be assumed to be so large that the system is completely polarized and that we can ignore the spin degrees of freedom. The eigenstates $\Psi(\mathbf{x}_1,\ldots,\mathbf{x}_N)$ are eigenfunctions of the (first-quantized) Hamiltonian $\hat{H}$

$$\hat{H}=\sum_{j=1}^{N}\left[\frac{1}{2M}\left[\mathbf{p}_j-\frac{e}{c}\mathbf{A}(\mathbf{x}_j)\right]^2+eA_0(\mathbf{x}_j)\right]+\sum_{i<j}V(|\mathbf{x}_i-\mathbf{x}_j|)\ , \tag{2.1}$$

where we have included the coupling to both the electromagnetic vector potential $\mathbf{A}$ and the scalar potential A_0. Hence, we are dealing with N spinless fermions of charge $-e$ and mass M.

Our goal is to show that this system is *equivalent* to the same system but coupled to an additional *statistical* vector potential $\mathcal{A}_\mu$ ($\mu=0,1,2$) whose dynamics is determined by the Chern-Simons *action* $S_{\rm CS}$

$$S_{\rm CS}=\int d^3x\frac{\theta}{4}\epsilon_{\mu\nu\lambda}\mathcal{A}^{\mu}\mathcal{F}^{\nu\lambda}\ , \tag{2.2}$$

for a suitably chosen value of θ. In Eq. (2.2) x_0, x_1, and x_2 represent the time and the space coordinates of the electrons, respectively, and $\mathcal{F}^{\nu\lambda}$ is the field tensor for the statistical gauge field

$$\mathcal{F}^{\nu\lambda}=\partial^{\nu}\mathcal{A}^{\lambda}-\partial^{\lambda}\mathcal{A}^{\nu}\ . \tag{2.3}$$

The equivalent theory has a Hamiltonian $\hat{H}'$ which is identical to $\hat{H}$, given in Eq. (2.1), except for the fact that the electrons are also coupled to the statistical vector potential $\mathcal{A}_\mu$. This is accomplished simply by setting

$$\frac{e}{c}\mathbf{A}\to\frac{e}{c}\mathbf{A}+\mathcal{A},\quad eA_0\to eA_0+\mathcal{A}_0\ . \tag{2.4}$$

As one of us noted in Ref. 21, the Chern-Simons action implies a constraint for the particle density $j_0(\mathbf{x})$ and the statistical flux $\mathcal{B}$

$$j_0(\mathbf{x})=\theta\mathcal{B}(\mathbf{x})\ , \tag{2.5}$$

which can be regarded as a kind of Gauss law, and an equal-time commutation relation among the *spatial* components of $\mathcal{A}_\mu$

$$[\mathcal{A}_1(\mathbf{x}),\mathcal{A}_2(\mathbf{x}')]=\frac{i}{\theta}\delta^{(2)}(\mathbf{x}-\mathbf{x}')\ . \tag{2.6}$$

For *arbitrary* values of θ, it was shown in Ref. 21 that the system is a set of *anyons* with statistical angle $\delta=1/2\theta$, measured with respect to Fermi statistics. Thus, if $\theta=(1/2\pi)(1/2n)$, where n is an arbitrary integer, then $\delta=2\pi n$ and the system still represents fermions. The constraint Eq. (2.5) tells us that θ represents a statistical flux per particle of $1/\theta$. Hence, for $\theta=(1/2\pi)(1/2n)$, *each* fermion picks up a statistical flux equal to $1/\theta=2\pi(2n)$ i.e., an *even* number of flux quanta ($2n$) is attached to each particle. This argument suggests that for both theories to be equivalent we must make the choice

$$\theta=\frac{1}{2\pi}\frac{1}{m-1}\ , \tag{2.7}$$

where m is the odd integer appearing in the Laughlin wave function Eq. (1.1).

In Appendix A we present a detailed proof of the physical equivalence of two theories of particles coupled to a Chern-Simons gauge field with coupling constants θ and θ' such that

$$\frac{1}{\theta'}=\frac{1}{\theta}+2\pi\times 2n\ , \tag{2.8}$$

where n is an arbitrary integer. We show that both theories have the same amplitudes for all of their physical states. In particular, a theory of interacting *fermions* is always equivalent to a family of theories of interacting *fermions* coupled to a Chern-Simons gauge field with coupling constant θ such that $1/\theta=2\pi\times 2n$. This result is the starting point of our analysis of the FQHE.

Let us remark that Eq. (2.8) makes apparent the *periodicity* in the statistical angle $\delta\to\delta+2\pi\times$even integer. While this periodicity is obvious in the "anyon" language, it is far from evident in the Chern-Simons description. In Chern-Simons language, periodicity means that theories with values of the CS coupling constant θ and θ' are equivalent if $1/\theta-1/\theta'=2\pi\times$ even integer. This is important since in the mean-field approximation to both the FQHE and the anyon gas, *one particular period has to be chosen.* The perturbative treatment of the fluctuations around this mean field are incapable of restoring the periodicity. This is a nonperturbative effect. In particular, operators which create excitations which change the amount of flux per particle in *even* multiples of the flux quantum are soliton operators which restore the periodicity broken by the mean field. This phenomenon is quite analogous to the role played by solitons in one-dimensional systems. On the other hand, one can take advantage of the periodicity of the CS description to choose the period in which the mean-field theory is simplest. This is the approach that we take in this paper to attack the FQHE. As it will be apparent in the next section, at the mean-field–semiclassical level, this approach reproduces Jain's construction of the FQHE states.

In second-quantized language, what we have proven is the physical equivalence of theories whose dynamics are governed by the actions (in units in which $\hbar=1$) $\mathcal{S}_\theta$ and $\mathcal{S}_{\theta'}$, which are defined by

$$\mathcal{S}_\theta=\int d^3z\left[\psi^*(z)(iD_0+\mu)\psi(z)+\frac{1}{2m}|\mathbf{D}\psi(z)|^2+\frac{\theta}{4}\epsilon_{\mu\nu\lambda}\mathcal{A}^\mu\mathcal{F}^{\nu\lambda}\right]$$
$$-\tfrac{1}{2}\int d^3z\int d^3z'[|\psi(z)|^2-\bar{\rho}]V(|z-z'|)[|\psi(z')|^2-\bar{\rho}]\ , \tag{2.9}$$

where $\bar{\rho}$ is the average particle density, provided that θ and θ' satisfy Eq. (2.8). In Eq. (2.9) $\psi(z)$ is a second-quantized Fermi field, μ is the chemical potential, and D_μ is the covariant derivative which couples the fermions to both the external electromagnetic field A_μ and to the statistical gauge field $\mathcal{A}_\mu$,

$$D_\mu=\partial_\mu+i\frac{e}{c}A_\mu+i\mathcal{A}_\mu\ . \tag{2.10}$$

In particular, a theory of interacting *fermions* (which has $\theta=0$) is equivalent to a *family* of theories of *fermions* with $1/\theta=2\pi\times$even integer. However, the equivalency holds for *arbitrary* values of θ. Hence, our theory will also be applicable to the study of *anyons* in a magnetic field and, hence, to the study of the hierarchy.

In the next section we will develop a semiclassical theory of the FQHE based on an action of the type of Eq. (2.9) with the choice of θ given in Eq. (2.7), i.e., $1/\theta=2\pi(m-1)$, where m is the *odd* integer which appears in the Laughlin wave function. As a by-product, we will also find the FQHE states for anyons.

III. THE SEMICLASSICAL LIMIT AND THE LAUGHLIN GROUND STATE

In this section we will show that the *semiclassical limit* of the theory described by the action S_θ of Eq. (2.9), with $1/\theta=2\pi(m-1)$, yields the same physics as the Laughlin state Ψ of Eq. (1.1). In order to prove this statement we will develop a semiclassical approach to this problem. As a by-product, our formalism provides for a systematic procedure to compute corrections to the Laughlin approximation. This is, to the best of our knowledge, the first formalism for which the Laughlin ansatz arises as the first of a series of approximations.

The action of Eq. (2.9), with the choice $1/\theta=2\pi(m-1)$, governs the dynamics of a system of spinless *fermions* interacting through a pair interaction potential $(V|\mathbf{x}-\mathbf{x}'|)$ coupled to both electromagnetic and statistical gauge fields. The starting point of the semiclassical approximation effectively maps this FQHE problem into an *equivalent* IQHE system much in the same way as in Jain's reinterpretation of the Laughlin wave function. This mapping is made possible by the statistical or Chern-Simons gauge fields which screen out enough of the external magnetic field to the point that the number of flux quanta of the effective magnetic field which is left is an exact factor of the total number of particles. Naturally, this perfect screening is not possible for arbitrary values of the external magnetic field for a fixed number of electrons. The values of the filling fraction for which this perfect screening can be accomplished happens to be the same as the Laughlin sequence $1/m$ and the first level of the hierarchy. For all other cases, there will be some partially filled level left over. However, these quasiparticles are effectively anyons. The conventional hierarchy scheme assumes that these quasiparticles can condense into another sort of FQHE state. This procedure is repeated in an iterative fashion until perfect screening is achieved. We will not further discuss the higher stages of the hierarchy here. In the rest of this section we assume that θ can take an arbitrary value. Thus, our results will include the FQHE for fermions as well as the more general case of anyons.

Consider the quantum partition function for this problem (at $T=0$)

$$Z=\int \mathcal{D}\psi^*\mathcal{D}\psi\mathcal{D}\mathcal{A}_\mu\exp(iS_\theta)\ . \tag{3.1}$$

We will treat this path integral in the semiclassical approximation. In order to do that we will first integrate out the fermions and treat the resulting theory within the saddle-point-expansion (SPA) characteristic of semiclassical approaches to quantum mechanics and quantum field theory.[29,30] In the absence of electron-electron interactions the fermions can be integrated out immediately since the action becomes quadratic in Fermi fields. In the presence of interactions this is no longer possible since the interaction term in the action Eq. (2.9) spoils this feature. However, this problem can be sidestepped by means of a Hubbard-Stratonovich transformation by which we trade a quartic form in fermions for a quadratic one coupled to a new Bose field, in this case the density fluctuation. This procedure will allow us to give a full description of the spectrum of collective modes of the FQHE states. Finally, note that since we are dealing with a gauge theory, a gauge has to be specified in order to make the functional integral well defined. We will assume from now on that a gauge fixing condition has been imposed but, for the moment, we will not make any specific choice of gauge.

Before we proceed to integrate the Fermi degrees of freedom out, we perform the Hubbard-Stratonovich transformation in terms of a scalar Bose field $\lambda(x)$

$$\exp\left[-i\int d^3z\int d^3z'\frac{1}{2}[|\psi(z)|^2-\bar{\rho}]V(z-z')[|\psi(z')|^2-\bar{\rho}]\right] = \mathcal{N}\int \mathcal{D}\lambda \exp\left[i\int d^3z\,\lambda(z)[|\psi(z)|^2-\bar{\rho}]+\frac{i}{2}\int d^3z\int d^3z'\lambda(z)V^{-1}(z-z')\lambda(z')\right], \quad (3.2)$$

where $\mathcal{N}$ is a normalization constant and $V(z-z')$ represents the instantaneous pair interaction, i.e.,

$$V(z-z')=V(|\mathbf{z}-\mathbf{z}'|)\delta(t-t') . \quad (3.3)$$

$V^{-1}(z-z')$ is the inverse of $V(z-z')$ in an operator sense. In this paper we will assume that the physics of the FQHE can be studied in a model system in which the pair potential is reasonably local. In fact it has been argued by Haldane[3] that it is possible to approximate the realistic $1/R$ Coulomb potential by a set of *pseudopotentials* when the states are projected onto a specific set of Landau level basis states (the lowest in his case). Furthermore, Kivelson and Trugman[7] (KT) have shown that the Laughlin wave function is the *exact* ground state for a system with the ultralocal pair potential

$$V_{\mathrm{KT}}(|\mathbf{z}-\mathbf{z}'|)=V_2\nabla_z^2\delta(\mathbf{z}-\mathbf{z}') . \quad (3.4)$$

Since the methods developed in this paper are completely general and, hence, work for any reasonable pair potential, we will not restrict ourselves to any particular form of $V(z)$ beyond the assumption that it is behaves reasonably well. In the case of the Kivelson-Trugman potential, the inverse potential operator $V^{-1}(z-z')$ takes the particularly simple form

$$V_{\mathrm{KT}}^{-1}(z-z')=\frac{1}{V_2}G_0(|\mathbf{z}-\mathbf{z}'|)\delta(t-t') . \quad (3.5)$$

Here $G_0(R)$ is the two-dimensional Coulomb Green function, i.e.,

$$V_{\mathrm{KT}}^{-1}(R)=\frac{1}{2\pi V_2}\ln(R/a) , \quad (3.6)$$

where a is a constant with dimensions of length which represents the range of the interaction.

After the Hubbard-Stratonovich transformation is performed, the partition function Z can be written in the form of a functional integral involving the Fermi fields ψ, the statistical gauge fields $\mathcal{A}_\mu$, and the collective modes λ. The action for the system is now given by

$$S=\int d^3z\left[\psi^*(z)[iD_0+\mu+\lambda(z)]\psi(z) +\frac{1}{2M}|\mathbf{D}\psi(z)|^2+\frac{\theta}{4}\epsilon_{\mu\nu\lambda}\mathcal{A}^\mu\mathcal{F}^{\nu\lambda}-\lambda(z)\bar{\rho}\right] +\tfrac{1}{2}\int d^3z\int d^3z'\lambda(z)V^{-1}(z-z')\lambda(z') . \quad (3.7)$$

The Fermi fields can be integrated out without any difficulty yielding, as usual, a fermion determinant. The resulting partition function can thus be written in terms of an effective action S_{eff} given by

$$S_{\mathrm{eff}}=-i\,\mathrm{tr}\ln\left[iD_0+\mu-\frac{1}{2m}\mathbf{D}^2\right] +S_{\mathrm{CS}}(\mathcal{A}_\mu-\tilde{A}_\mu-\delta_{\mu 0}\lambda)-\int d^3z\,\lambda(z)\bar{\rho} +\frac{1}{2}\int d^3z\int d^3z'\lambda(z)V^{-1}(z-z')\lambda(z') , \quad (3.8)$$

where D_0 and $\mathbf{D}$ are the covariant derivatives of Eq. (2.10) and S_{CS} is the Chern-Simons action Eq. (2.2). The field $\tilde{A}_\mu$ represents a small fluctuating electromagnetic field, with vanishing average everywhere, which will be used to probe the system. The electromagnetic currents will be calculated as first derivatives of Z with respect to $\tilde{A}_\mu$. The full electromagnetic response will be obtained in this way. Notice that we have used the invariance of the measure $\mathcal{D}\mathcal{A}_\mu$ with respect to shifts to move $\tilde{A}_\mu$ out of the covariant derivatives and into the Chern-Simons term S_{CS}.

We are now ready to proceed with the semiclassical approximation. The path integral Z will be approximated by expanding its degrees of freedom around stationary configurations of the effective action S_{eff} in powers of the fluctuations. This is the conventional WKB-like approximation. The classical configurations $[\bar{\mathcal{A}}_\mu(z)=\langle\mathcal{A}_\mu(z)\rangle$ and $\bar{\lambda}(z)=\langle\lambda(z)\rangle]$ can be obtained by demanding the S_{eff} be stationary under small fluctuations. This requirement yields the classical equations of motion

$$\frac{\delta S_{\rm eff}}{\delta \mathcal{A}_\mu(z)}\bigg|_{\bar{\mathcal{A}},\bar{\lambda}}=0,\quad \frac{\delta S_{\rm eff}}{\delta \lambda(z)}\bigg|_{\bar{\mathcal{A}},\bar{\lambda}}=0\ . \tag{3.9}$$

By varying $S_{\rm eff}$ with respect to $\mathcal{A}_\mu(z)$ and $\lambda(z)$ we get

$$\langle j_\mu(z)\rangle_F+\frac{\theta}{2}\epsilon_{\mu\nu\lambda}[\langle \mathcal{F}^{\nu\lambda}(z)\rangle-e\langle F^{\nu\lambda}\rangle]=0\ ,$$
$$\langle j_0(z)\rangle_F-\bar{\rho}+\int d^3z'V^{-1}(z-z')\langle\lambda(z')\rangle=0\ . \tag{3.10}$$

In addition we must fix the particle density to be uniform and equal to $\bar{\rho}$, i.e.,

$$\langle j_0(z)\rangle=\bar{\rho}\ , \tag{3.11}$$

where $\langle j_\mu(z)\rangle_F$ represents the expectation value of the charge and current of the equivalent fermion problem.

These equations have many possible solutions, which include uniform (liquid) state and Wigner crystals. There are also solutions which represent *defects,* such as the *quasihole.*

The solution with uniform particle density, which represents a liquid phase, is only consistent if

$$\bar{\lambda}(z)=0\ . \tag{3.12}$$

In the Wigner crystal state the density has, in addition to $\bar{\rho}$, an oscillatory component. Equation (3.10) requires that $\langle\lambda\rangle$ should also have an oscillatory component as well. In this paper we will only discuss the liquid FQHE states.

If the external electromagnetic fluctuation is assumed to have zero average, the only time-independent uniform solutions of Eq. (3.10) have uniform average statistical flux $\langle\mathcal{B}\rangle$ and vanishing average statistical electric field $\langle\mathcal{E}\rangle$ (unless there is a nonzero current in the ground state) and satisfy

$$\langle\mathcal{B}\rangle=-\frac{\bar{\rho}}{\theta},\quad \langle\mathcal{E}\rangle=0\ . \tag{3.13}$$

Equation (3.13) shows that, for a translationally invariant ground state, the effect of the statistical gauge fields, at the level of the saddle-point approximation, is to *reduce* the effective flux experienced by the fermions. The total effective field is thus reduced from the value of the external field B down to $B_{\rm eff}=B+\langle\mathcal{B}\rangle=B-\bar{\rho}/\theta$. Let us assume that we have a situation in which we are trying to find the ground state of N (interacting) electrons in the presence of an external magnetic field of strength B. We will further assume that the linear size L of the sample is such that a total of N_ϕ quanta of the magnetic flux are piercing the surface. In general, the filling fraction $\nu=N/N_\phi$ is not an integer. Thus, a perturbative approach based on a Slater determinant wave function of the occupied single-particle states does not yield a stable answer. This is so because there is a macroscopic number of essentially degenerate states which will mix with this trial state. On the other hand, a Laughlin state is known to represent a state with an energy gap. Thus, the correlations have removed the massive degeneracy of the free electrons. Furthermore, this gap is not equal to the Landau gap of the noninteracting electrons. Therefore, we can expect our saddle-point expansion to succeed only if the *effective* theory ends up with a nonzero gap.

The uniform effective magnetic fields $B_{\rm eff}$, which solve Eq. (3.13), define a new set of Landau levels. Each level has a degeneracy equal to the total number of *effective* flux quanta $N_{\rm eff}$ and the separation between levels is the effective cyclotron frequency $\omega_c^{\rm eff}=eB_{\rm eff}/Mc$. Similarly, there is an effective cyclotron radius $l_0^{\rm eff}$. It is easy to check that the uniform saddle-point state which satisfies Eq. (3.13) has a gap *only if* the effective field $B_{\rm eff}$ experienced by the N fermions is such that the *fermions fill exactly an integer number p of the effective Landau levels.* This is precisely the point of view advocated by Jain: the FQHE is an IQHE of a system of electrons dressed by an even number of flux quanta. However, this condition cannot be met for arbitrary values of the filling fraction ν at fixed field (or at fixed density). Let $N_\phi^{\rm eff}$ denote the effective number of flux quanta piercing the surface after screening. It is given by

$$2\pi N_\phi^{\rm eff}=2\pi N_\phi-\frac{\bar{\rho}}{\theta}L^2\ . \tag{3.14}$$

Thus, the effective cyclotron frequency $\omega_c^{\rm eff}$ is *reduced* from its free electron value of eB/M_c down to $\omega_c^{\rm eff}=\omega_c(1-\nu/2\pi\theta)$. The effective cyclotron radius is given by $l_0^{\rm eff}=(l_0/\sqrt{1-\nu/2\pi\theta})$, which is larger than the noninteracting value. Therefore, even though the bare Landau levels may be separated by a sizable Landau gap $\hbar\omega_c$, the effective Landau levels have the smaller gap $\hbar\omega_c^{\rm eff}$.

We will now find the solutions to this equation for the cases of fermions, bosons, and, more generally, anyons.

A. Fermions

For the particular case of fermions, Eq. (3.14) reads

$$2\pi N_\phi^{\rm eff}=2\pi N_\phi-2\pi 2sN\ , \tag{3.15}$$

where $2s$ is an even integer. The spectrum supported by this state has an energy gap if the N fermions fill exactly p of the Landau levels created by the effective field $B_{\rm eff}$. In other words, the *effective* filling fraction is $\nu_{\rm eff}\equiv N/N_\phi^{\rm eff}=p$. Using Eq. (3.15), we find that the filling fraction ν and the external magnetic field B must satisfy

$$\frac{N}{p}=\frac{N}{\nu}-2sN\ , \tag{3.16}$$

or, equivalently,

$$\frac{1}{\nu}=\frac{1}{p}+2s\ . \tag{3.17}$$

The filling fraction ν is in general equal to the ratio of two integers, $\nu=n/m$. Equation (3.17) holds if the integers n and m satisfy the equation

$$\frac{m}{n}=\frac{1}{p}+2s\ . \tag{3.18}$$

The effective Landau gap for these solutions is

$$\hbar\omega_c^{\rm eff}=\frac{\hbar\omega_c}{1+2sp}\ , \tag{3.19}$$

which is small if either p or s are large. Thus, the energy to excite a *fermion* is $\hbar\omega_c^{\text{eff}}$ and it is considerably smaller than the bare free-particle value $\hbar\omega_c$.

The states are thus parametrized by two integers p (the number of filled Landau levels of the effective field) and $2s$ (the number of flux quanta carried by each fermion). The Laughlin sequence is an obvious solution of Eq. (3.18) since, for $n=1$ and m odd, Eq. (3.18) yields the unique solution $p=1$ and $2s=m-1$. The effective fermions thus fill up exactly one Landau level and θ has to be chosen to be $1/\theta=2\pi(m-1)$. This is the Jain result. At this mean-filled level the wave function is the Slater determinant for one filled Landau level χ_1 of Eq. (1.3). We will show below that the additional factor $\prod_{i<j}(z_i-z_j)^{m-1}$ is due to fluctuations.

In addition to the Laughlin sequence ($p=1$, $2s=m-1$), Eq. (3.18) has a host of other solutions. For $n<m$, we can use the division algorithm to find a pair of integers r and q $(0\le q<n)$ such that $m=nr+q$. Equation (3.18) can only hold if $r=2s$ and q is a factor of n such that $q/n=1/p$. For instance, the sequence $m=2sn+1$ is a solution if m is an odd integer ($q=1$). Clearly, this case has $p=n$ filled Landau levels. This is the first level of the hierarchy. Here too, our mean-field results yield the same answers as in Jain's approach.[11]

B. Bosons

For the case of bosons, Eq. (3.14) becomes

$$2\pi N_\phi^{\text{eff}}=2\pi N_\phi-2\pi(2s+1)N \; , \tag{3.20}$$

where s is an arbitrary integer. Thus, for the case of bosons, the allowed filling fractions $\nu=n/m$ must satisfy

$$\frac{1}{\nu}=\frac{1}{p}+2s+1 \; . \tag{3.21}$$

In addition to the *even* denominator Laughlin sequence ($n=1$, $m=2s$), we also get a first level of a hierarchy for bosons $m=n(2s+1)+1$.

C. Anyons

Finally, let us discuss the more general case of anyons with $\theta=(1/2\pi)(p/q)$ and statistics $\delta=\pi q/p$ measured relative to fermions. The allowed filling fractions now are given by

$$\frac{1}{\nu}=\frac{1}{r}+2s\frac{1}{2\pi\theta} \; , \tag{3.22}$$

where r is an arbitrary *positive* integer which is equal to the number of filled Landau levels of the effective fermions. For example, if we consider the case of *semions* (i.e., $\delta=\pi/2$), Eq. (3.22) becomes $1/\nu=1/r+2s-\frac{1}{2}$. One particular solution is the choice $r=1$, which corresponds to the lowest effective level being completely filled. This particular case has been discussed elsewhere by one of us within the framework of the theory of the singlet quantum Hall effect.[31] In general, there are many solutions to Eq. (3.22). These solutions are important for the construction of the higher levels of the hierarchy of FQHE states.

The saddle-point approximation yields a very simple spectrum which consists just of the single- and many-particle excitations of fermions in the effective field B_{eff}. The single-particle gap is equal to $\hbar\omega_c^{\text{eff}}$. Notice that, although the Coulomb energy does not enter explicitly in this effective gap, its existence is a consequence of the Coulomb interactions. However, there are many reasons which suggest that this spectrum is not the right one. First of all, the saddle-point approximation neglects the fluctuations about the average field. In Sec. IV we show that fluctuations give rise to a host of collective modes. Also, in the strong-field limit $B\to\infty$, both the bare and the effective Landau gaps diverge. Thus, there should exist other particlelike states whose energy remains finite even in the limit $B\to\infty$. Intuitively, we expect that the energy of this charged excitation should be determined primarily by the interactions. Such solutions *do* exist. They are just Laughlin's quasiparticle and quasihole states.

The saddle-point equations (3.10) have a host of *nonuniform* solutions which have finite energy. They can be viewed as *soliton* or *vortex* solutions. Our construction is very close in spirit to the picture of the quasihole presented by Laughlin.[32] In this paper we will only discuss the qualitative features of these solutions. The details will be discussed elsewhere. For the sake of simplicity we will only discuss the case of fermions in the Laughlin $1/m$ sequence. Recall that this sequence is represented by the solution with $p=1$ and $m=2s+1$. The uniform state was constructed by filling up the lowest effective Landau level. Let us consider the state which results from removing a fermion from the single-particle state, centered around the origin $z=0$ and the lowest angular momentum, and placing it on the first unoccupied angular momentum state. Physically, this new state lies on the outer edge of the system. For a uniform effective field, this states does not exist. But, if the effective field is *increased* at the origin by an amount equal to one flux quantum, the angular momentum of all its eigenstates is raised by one whole unit. The radius R_N of the droplet with N particles swells to a new value $R_N+\delta R$ large enough to include a new cyclotron orbit. Qualitatively, a quasihole localized at z_0 has the *mean-field* wave function $\Psi_h=\Psi_h(z_0,\{z_j\})$ given by

$$\Psi_h(z_0,\{z_j\})=\prod_{i=1}^{N}(z_i-z_0)\chi_1(\{z_j\}) \; , \tag{3.23}$$

where $\chi_1(\{z_j\})$ is the wave function for N fermions occupying the lowest Landau level. Notice that this wave function differs from the Laughlin state for the quasihole by the prefactor $\prod_{i<j}(z_i-z_j)^{m-1}$. Indeed, this prefactor is also missing in the *mean-field* wave function for the ground state. In our picture, both prefactors arise from fluctuations which attach fluxes to the particles. The quasihole wave function of Eq. (3.23) is an approximation valid in the limit $|z_i-z_0|\gg l_0^{\text{eff}}$. From Eq. (3.10) it is easy to see that the mean-field *excitation energy* of the quasihole ε_h is given approximately by the Coulomb energy $V(l_0^{\text{eff}})$.

Finally, let us note some drawbacks of this approach. It is clear that, as it stands, the semiclassical theory described here is not capable of describing states in which the filling fraction ν has a nonzero integer part and some Landau level other than the lowest is partially filled, such as $\nu=k+1/m$ (with k a positive integer). At a qualitative level, there should be no difference between this physical situation and the Laughlin states which have $k=0$. However, it is clear that a wave function which is a product of a Slater determinant of the filled first Landau level and a Laughlin wave function for the fractionally filled second level is not fully antisymmetric. Similarly, even if $\nu<1$, for values of ν which do not satisfy Eq. (3.17), the effective theory has a number p of filled levels but the $(p+1)$ st level remains fractionally filled. To solve both of these situations Jain proposed a generalization of his scheme.[11,33] This generalized approach amounts to a *fractionalization* of the electric charge *by hand* in such a way that the wave function now looks like a product of Slater determinants *raised to fractional powers.* Such wave functions have branch cuts ending at the places where the particles are located. These cuts should not be present in the wave function for *electrons* which should be single valued. This means that the anyons have "to sit on top of each other." Thus, these anyons are not part of the physical spectrum, i.e., they are *confined* into bound states. Wen and Blok[33] have recently proposed a description of the hierarchy states based on a bosonic Ginzburg-Landau description with explicit charge fractionalization and were able to reproduce Jain's results in that framework. However, for the charge fractionalization to be physically meaningful it must be the result of the dynamics of the problem. It remains unclear to us whether the generalized approaches are a deep statement on the nature of the quantum states of these systems or just an algorithm to generate wave functions with the desired properties. This issue is also closely related to the more general question of the dynamical separation of spin and charge in strongly correlated Fermi systems. We will not discuss these issues in this paper.

In this section we showed that, in the semiclassical or mean-field approximation, the FQHE ground state of a gas of fermions in a uniform magnetic field is equivalent to an IQHE state for fermions bound to an even number of flux quanta in the presence of a partially screened external magnetic field. In the next section we discuss the excitation spectrum in the semiclassical limit. This will require the consideration of the zero-point motion (Gaussian fluctuations) around the mean-field state.

IV. THE EXCITATION SPECTRUM IN THE SEMICLASSICAL LIMIT

In this section we consider the role of the Gaussian fluctuations around the *uniform* classical solutions discussed in Sec. III. This is equivalent to a WKB approximation of the functional integral. We begin by considering the effective action of Eq. (3.8). This approximation works provided that the external field is *finite.* In the limit $B\to\infty$, nonuniform "solitonlike" solutions of the saddle-point equations are responsible for the dominant fluctuation effects. In that limit, only the states in the lowest Landau level can participate of the dynamics. However, if B is large but finite the Gaussian fluctuations are still important. We will see below that the collective modes and the quantum numbers of the excitations are the same in both limits although the dependence of various effective parameters, such as gaps and propagation speeds, may be different. In a sense, these two limits are similar to the strong- and weak-coupling limits of the spin-wave dynamics in a Hubbard model: the low-energy spin degrees of freedom are described by a nonlinear sigma model in both cases but the spin rigidity and the spin-wave velocity are very different. In this paper we only discuss the Gaussian approximation.

In Sec. III we showed that the saddle-point approximation of Eq. (3.9) has a uniform liquidlike solution of Eqs. (3.12) and (3.13). We will not consider the problem of the relative stability of the liquid and Wigner crystal states. Let $\mathcal{A}_\mu(x)$ and $\lambda(x)$ denote the *fluctuations* of the statistical vector potential $\mathcal{A}_\mu$ and of the collective mode $\lambda(x)$, respectively, i.e., we set $\mathcal{A}_\mu\to\langle\mathcal{A}_\mu\rangle+\mathcal{A}_\mu$ and $\lambda\to\langle\lambda\rangle+\lambda$. The effective action of Eq. (3.8) can be expanded in a series in powers of the fluctuations. We will be interested only in keeping just up to quadratic terms in the fluctuations. In the language of Feynman diagrams, we are summing up all the one-loop bubble contributions. Thus, the effective action at the quadratic level involves the linear response kernels [evaluated in the random-phase approximation (RPA)] for a system of fermions in an external static uniform magnetic field B_{eff} with an integer number of filled effective Landau levels. As usual, the linear terms are canceled if the saddle-point equations are satisfied. It will be convenient to shift the component $\mathcal{A}_0$ of the statistical vector potential by $\mathcal{A}_0\to\mathcal{A}_0+\lambda$. In this way, the collective mode λ disappeared from the fermion determinant. Naturally, this means that the Chern-Simons piece of the action now has the form $S_{\text{CS}}(\mathcal{A}_\mu-\tilde{\mathcal{A}}_\mu-\delta_{\mu0}\lambda)$.

At the quadratic (Gaussian) level the effective action has the form

$$S^{(2)}=\frac{1}{2}\int d^3x\, d^3y\,\mathcal{A}_\mu(x)\Pi^{\mu\nu}(x,y)\mathcal{A}_\nu(y)+S_{\text{CS}}(\mathcal{A}_\mu-\tilde{\mathcal{A}}_\mu-\delta_{\mu0}\lambda)+S_\lambda(\lambda)\ , \tag{4.1}$$

where $S_\lambda(\lambda)$ is the piece of the action which depends on the Hubbard-Stratonovich fields and thus carries the information about electron-electron interactions. $S_\lambda(\lambda)$ is given by

$$S_\lambda(\lambda)=\frac{1}{2}\int d^3z\int d^3z'\lambda(z)V^{-1}(z-z')\lambda(z')\ . \tag{4.2}$$

The effective action $S^{(2)}$ is a quadratic functional of the field λ. Thus it is possible to carry out the functional integral over λ first and to obtain an effective action for the statistical gauge fields alone. As a matter of fact, λ plays a role very similar to the component $\mathcal{A}_0$ of the statistical gauge field. This is natural since they both couple to the particle density. By inspecting the Chern-Simons term in Eq. (4.1), it is easy to see that the total λ-dependent contribution $S^{(2)}(\lambda)$ to the action that we get from that term has the form

$$S^{(2)}(\lambda)=-\int d^3x\,\lambda(x)\theta[\mathcal{B}(x)-\tilde{B}(x)]+S_\lambda(\lambda)\ . \tag{4.3}$$

The integration over the field λ is straightforward. The resulting effective action S_{eff} is

$$S_{\text{eff}}=\frac{1}{2}\int d^3x\,d^3y\,\mathcal{A}_\mu(x)\Pi^{\mu\nu}(x,y)\mathcal{A}_\nu(y)+\frac{\theta}{4}S_{\text{CS}}(\mathcal{A}_\mu-\tilde{A}_\mu)-\frac{\theta^2}{2}\int d^3z\,d^3z'[\mathcal{B}(z)-\tilde{B}(z)]V(z-z')[\mathcal{B}(z')-\tilde{B}(z')]\ . \tag{4.4}$$

Notice that the only approximation used in this formula is in the first term which follows from the expansion of the fermion determinant in powers of $\mathcal{A}_\mu$. The second term is exact.

The tensor $\Pi_{\mu\nu}$ is the *polarization tensor* of the equivalent fermion problem at the mean-field level and it can be obtained by differentiating the fermion determinant

$$Z_0[A_\mu]\equiv\det\left[iD_0+\mu+\lambda+\frac{1}{2M}\mathbf{D}^2\right]=Z[0]\exp\left[\frac{i}{2}\int d^Dx\int d^Dy\;A_\mu(x)\Pi_{\mu\nu}(x,y)A_\nu(y)+\dots\right]\ . \tag{4.5}$$

The tensor $\Pi_{\mu\nu}$ *should not* be confused with the true *electromagnetic* polarization tensor which measures the response of the whole system to a weak electromagnetic field. We will come back to this issue in the next section.

It is straightforward to derive an expression for $\Pi_{\mu\nu}$ in terms of the one-particle Green functions $G(x,y)$

$$G(x,y)=\left\langle x\left|\frac{1}{iD_0+\mu+(1/2M)\mathbf{D}^2[\langle A\rangle]}\right|y\right\rangle\ . \tag{4.6}$$

The components of the polarization tensor $\Pi_{\mu\nu}(x,y)$ are ($\hbar=1$)

$$\Pi_{00}(x,y)=iG(x,y)G(y,x)\ , \tag{4.7a}$$

$$\Pi_{0j}(x,y)=\frac{1}{2M}[G(x,y)D_j^yG(y,x)-G(y,x)D_j^{y\dagger}G(x,y)] \tag{4.7b}$$

$$\Pi_{j0}(x,y)=+\frac{1}{2M}[-G(x,y)D_j^{x\dagger}G(y,x)+G(y,x)D_j^xG(x,y)] \tag{4.7c}$$

$$\begin{aligned}\Pi_{jk}(x,y)=&\frac{i}{M}\delta(x-y)\delta_{jk}G(x,y)-\frac{i}{4M^2}[D_j^xG(x,y)][D_k^yG(y,x)]-\frac{i}{4M^2}[D_j^{x\dagger}G(y,x)][D_k^{y\dagger}G(x,y)]\\&+\frac{i}{4M^2}G(y,x)[D_j^xD_k^{y\dagger}G(x,y)]+\frac{i}{4M^2}[D_j^{x\dagger}D_k^yG(y,x)]G(x,y)\ .\end{aligned} \tag{4.7d}$$

It can be shown that the effective action of Eq. (4.5) is gauge invariant and that, in consequence, $\Pi_{\mu\nu}$ is *transverse* i.e.,

$$\partial_\mu^x\Pi_{\mu\nu}(x,y)=0\ . \tag{4.8}$$

In Appendix B we give detailed expressions for the various components of $\Pi_{\mu\nu}$. In particular we show that the action is gauge invariant and that $\Pi_{\mu\nu}$ is transverse, albeit only in a weak sense i.e., as a distribution. Equation (4.7) agrees with the results of Randjbar-Daemi, Salam, and Strathdee,[34] but disagrees with the calculation of $\Pi_{\mu\nu}$ by Chen *et al.*[19]

We will now extract from the effective action S_{eff} the spectrum of low-lying collective excitations and the low-energy response of the system. We will restrict our discussion to the long-wavelength modes. In a separate paper we discuss the spectrum of magnetorotons as well as the number of distinct stable collective modes.[35] The first step is to obtain an expression for the effective action valid in the long-wavelength, low-energy, limit. The gradient expansion of $\Pi_{\mu\nu}$ derived in Appendix B yields the result

$$S_{\text{eff}}=\int d^3z\left[\frac{\epsilon}{2}\mathcal{E}^2-\frac{\chi}{2}\mathcal{B}^2\right]+\frac{\sigma_{xy}^0}{4}S_{\text{CS}}(\mathcal{A}_\mu)+\frac{\theta}{4}S_{\text{CS}}(\mathcal{A}_\mu-\tilde{A}_\mu)-\frac{\theta}{2}\int d^3z\,d^3z'[\mathcal{B}(z)-\tilde{B}(z)]V(z-z')[\mathcal{B}(z')-\tilde{B}(z')]\ . \tag{4.9}$$

The coefficients ϵ, χ, and σ_{xy}^0 are found to be given by

$$\epsilon=\frac{pM}{2\pi B_{\text{eff}}}=\frac{M}{B}\frac{1}{1-\nu/2\pi\theta},\quad \chi=\frac{p^2}{2\pi M},\quad \sigma_{xy}^0=\frac{p}{2\pi}\ , \tag{4.10}$$

where $B_{\text{eff}}=B-\rho/\theta$, ϵ, χ, and σ_{xy}^0 are, respectively, the (static) dielectric constant, diamagnetic susceptibility, and Hall conductance of the equivalent fermion problem which has p filled Landau levels in the effective field B_{eff} and M is the electron bare mass. Once again, it is important to stress that these quantities are *not* the actual responses of the system to a weak electromagnetic field, such as $\tilde{A}_\mu$, which we will calculate below. In fact, σ_{xy}^0 is the *integer* Hall conductance of the system of fermions bound to an even number of flux quanta, in agreement with Jain's interpretation. The effective action of Eq. (4.9) determines the response of the ground state of the system to slowly varying external fields. The FQHE states are incompressible and, hence, do not have any low-energy modes. Thus, the effective action of Eq. (4.9) does not describe the dynamics of the collective modes. In contrast, the action of Eq. (4.4) does contain information about collective modes.

Since we want to use the effective action of Eq. (4.9) to determine the low-energy electromagnetic response of the system, it is important to know how much the results of this semiclassical theory are affected by non-Gaussian corrections. By dimensional analysis and gauge invariance we expect that the *exact* low-energy effective action should have the form of Eq. (4.9), but with renormalized coefficients. Indeed, higher-order perturbative corrections, as well as nonperturbative effects due to other saddle points, contribute in a significant way to the actual values of ϵ and χ. However, the induced Hall conductance σ_{xy}^0 does not get corrected to any order in perturbative theory. This is so because, at least for a system with a gap and in the thermodynamic limit, σ_{xy}^0 is determined by a topological invariant,[36,37] the first Chern character $\mathcal{C}$. In the problem at hand, we have $\mathcal{C}=p$. Thus, the Hall conductance that we will calculate below is actually exact.

We will now discuss the spectrum of collective excitations (within this Gaussian approximation) and compute the Hall conductance. In another publication we will present a semiclassical construction of the fractionally charged quasiparticles and of the spectrum of collective modes.[35]

A. Spectrum of collective excitations

The effective action of Eq. (4.4) gives a good description of the behavior of the collective modes. Here we will restrict our discussion to the hydrodynamic regime $\mathbf{Q}\to 0$. In this limit, only the long-distance behavior of the pair potential should matter. Although formally inconsistent with the gradient expansion, we will keep the full form of the potential $V(z)$.

We have calculated[35] the electromagnetic response functions of the system from the propagator of the statistical gauge field $\mathcal{A}_\mu$ determined by the effective action of Eq. (4.4). We find that the Gaussian fluctuations of the statistical gauge field represent the magnetoplasmon of the FQHE states. The dispersion curve $\mathcal{Q}_0(\mathbf{Q})$ for this mode is

$$\mathcal{Q}_0(\mathbf{Q})=\left[\omega_c^2+[\alpha\theta^2 M\tilde{V}(\mathbf{Q})+\beta]\omega_c^2\frac{\mathbf{Q}^2}{B_{\text{eff}}}\right]^{1/2}, \quad (4.11)$$

where the dimensionless coefficients α and β are given by

$$\alpha=\frac{2\pi p}{(2\pi\theta+p)^2},$$
$$\beta=\frac{2p}{\theta^2}\frac{[(p/2\pi)^2+3(p/2\pi)\theta]}{(1+p/2\pi\theta)^2[(1+p/2\pi\theta)^2-4]}. \quad (4.12)$$

If $\tilde{V}(\mathbf{Q})|\mathbf{Q}|^2$ vanishes in the limit $\mathbf{Q}\to 0$, we find that this collective mode has a zero-momentum gap $\mathcal{Q}_0(0)=\omega_c$. Thus, this collective mode has to be identified as an *inter*-Landau level mode, or cyclotron resonance mode, whose existence is guaranteed by Kohn's theorem.[38] Recently Lee and Zhang[28] studied the bosonic representation of this theory. Their results also indicate that the Gaussian fluctuations represent the cyclotron mode. It is also interesting to remark that for a $1/r$ potential, the dispersion curve becomes linear at small momentum, i.e., $\mathcal{Q}_0\approx\mathcal{Q}_0(0)+\text{const}|Q|$, in agreement with the results of Kallin and Halperin.[39]

B. Hall conductance

We now demonstrate that this state does exhibit the fractional quantum Hall effect. The simplest way to do that is calculate the Hall conductance of the whole system. Let us return to the effective action of Eq. (4.9). The *electromagnetic* response can be determined by calculating the effective action for the external electromagnetic field. At the Gaussian level, this effective action is obtained by integrating out the fluctuations of the statistical vector potential $\mathcal{A}_\mu$. Since we are only interested in the leading long-distance behavior, it is sufficient to keep only those terms in Eq. (4.9) which have the smallest number of derivatives, i.e., we can approximate S_{eff} by

$$S_{\text{eff}}\approx\frac{\sigma_{xy}^0}{4}S_{\text{CS}}(\mathcal{A}_\mu)+\frac{\theta}{4}S_{\text{CS}}(\mathcal{A}_\mu-\tilde{A}_\mu). \quad (4.13)$$

Upon integrating over the statistical vector potentials we find that the effective action for the electromagnetic fields is

$$S_{\text{eff}}^{\text{em}}(\tilde{A}_\mu)\approx\frac{\theta_{\text{eff}}}{4}S_{\text{CS}}(\tilde{A}_\mu), \quad (4.14)$$

where θ_{eff} is given by

$$\frac{1}{\theta_{\text{eff}}}=\frac{1}{\theta}+\frac{1}{\sigma_{xy}^0}, \quad (4.15)$$

i.e., the Chern-Simons coupling constants are added "in parallel."

The values of θ and σ_{xy}^0 determined above yield the result

$$\theta_{\text{eff}}=\frac{\nu}{2\pi}, \quad (4.16)$$

where ν is the filling fraction. The electromagnetic current J_μ induced in the system is obtained by differentiating the effective action $S_{\text{eff}}^{\text{em}}(\tilde{A}_\mu)$ with respect to the electromagnetic gauge field. The current is

$$J_\mu=\frac{\theta_{\text{eff}}}{2}\epsilon_{\mu\nu\lambda}\tilde{F}^{\nu\lambda}. \quad (4.17)$$

Thus, if a weak external electric field $\tilde{E}_j$ is applied, the induced current is

$$J_k = \frac{\theta_{\text{eff}}}{2} \epsilon_{kl} \tilde{E}_l \ . \tag{4.18}$$

We can then identify the coefficient θ_{eff} with the *actual* Hall conductance of the system σ_{xy} and get

$$\sigma_{xy} \equiv \theta_{\text{eff}} = \frac{\nu}{2\pi} \ , \tag{4.19}$$

which is a *fractional* multiple of e^2/h (in units in which $e=\hbar=1$). Notice that the coefficient σ_{xy}^{eff} of the effective action represents the integer Hall effect of the bound states and it is different from the true Hall conductance σ_{xy}. Thus, the uniform states exhibit a fractional quantum Hall effect with the correct value of the Hall conductance.

V. CONCLUSIONS AND DISCUSSION

In this paper we have presented a theory of the FQHE based on a second-quantized fermion path-integral approach. We have shown that the problem of interacting electrons moving on a plane in the presence of an external magnetic field is equivalent to a family of systems of fermions bound to an even number of fluxes and that this theory has the fermion coupled to a Chern-Simons gauge field with Chern-Simons coupling constant $\theta=(1/2\pi\times 2n)$. The semiclassical approximation of this system has solutions which describe incompressible liquid states, Wigner crystals, and solitonlike defects. The liquid states belong to the Laughlin sequence and to the first level of the hierarchy. We also give a brief description of the FQHE for bosons and anyons in this picture. The semiclassical spectrum of collective modes of the FQHE states has a gap to all excitations. We derive an effective action for the Gaussian fluctuations in the hydrodynamic regime. The dispersion curve for the magnetoplasmon is calculated in the low-momentum limit. The fractionally quantized Hall conductance is calculated and argued to be exact in this approximation. In two appendixes, we presented a proof of the statement that "an even number of flux quanta does not change the theory" and presented an explicit derivation of the polarization tensor in the integer Hall regime. This tensor is shown to be transverse.

There are many open questions that deserve attention. The semiclassical calculation presented here yields a good description of the uniform FQHE ground state, of its low-energy electromagnetic response, and of the magnetoplasmons. However, the saddle-point equations also have solitonlike solutions which represent quasiholes, quasielectrons, and possibly, quasielectron-quasihole bound states. The vortexlike excitations are responsible for other collective modes, such as the magnetophonons. We will discuss these issues elsewhere.

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APPENDIX A

We prove our assertion that both theories yield the same physical amplitudes (i.e., that they are equivalent) by computing the same (arbitrary) amplitude in both schemes. We follow here the methods of Ref. 40. We find it simpler to do the calculation in path-integral language. The path integral is constructed in the usual manner.[41] The transition amplitude for the system to evolve from some arbitrary initial state $|\Psi_i\rangle$ in the remote past, to some other arbitrary final state $|\Psi_f\rangle$ in the remote future can be written in terms of a Feynman path integral involving arbitrary states at intermediate times. The initial and final states represent identical fermions and hence they are completely antisymmetric under particle exchanges. We can restrict ourselves for the sake of simplicity to the case in which the initial state is the same as the final state. The generalization is trivial. Also it will be sufficient to choose any particular basis states, such as

$$|\Psi_i, -\infty\rangle = \sum_P (-1)^P |\mathbf{x}_{P_1}, \ldots, \mathbf{x}_{P_N}\rangle \ , \tag{A1}$$

where P are all possible permutations. The diagonal matrix element of the evolution (S-matrix) operator has the form

$$\langle f|S|i\rangle = \text{const} \int_{\mathbf{x}_1,\ldots,\mathbf{x}_N} \sum_P (-1)^P \langle \mathbf{x}_{P_1}, \ldots, \mathbf{x}_{P_N}; +\infty | \mathbf{x}_1, \ldots, \mathbf{x}_N; -\infty\rangle \ , \tag{A2}$$

so that we can concentrate on processes involving unsymmetrized initial and final states which differ only by the permutation of a subset of the particle coordinates.

The Feynman path integral for the inner product

$$\langle \mathbf{x}_{P_1}, \ldots, \mathbf{x}_{P_N}; +\infty | \mathbf{x}_1, \ldots, \mathbf{x}_N; -\infty\rangle$$

is

$$\langle \mathbf{x}_{P_1}, \ldots, \mathbf{x}_{P_N}; +\infty | \mathbf{x}_1, \ldots, \mathbf{x}_N; -\infty\rangle = \int \mathcal{D}\mathbf{z}_1[t] \ldots \mathcal{D}\mathbf{z}_N[t] e^{(i/\hbar) S[\mathbf{z}_1(t), \ldots, \mathbf{z}_N(t)]} \ , \tag{A3}$$

with the boundary conditions

$$\lim_{t\to-\infty} \mathbf{z}_j(t)=\mathbf{x}_j \ ,$$

$$\lim_{t\to+\infty} \mathbf{z}_j(t)=\mathbf{x}_{P_j} \ . \qquad \text{(A4)}$$

The action in Eq. (A3) is the standard action for nonrelativistic quantum mechanics of particles coupled to the electromagnetic gauge field

$$S=\int_{-\infty}^{+\infty} dt \sum_{j=1}^{N} \left[\frac{m}{2} \left[\frac{d\mathbf{z}_j}{dt} \right]^2 + \frac{e}{c}\frac{dz_j^\mu}{dt}(t) A_\mu(\mathbf{z}_j(t)) - \sum_{i<j} V(|\mathbf{z}_i(t)-\mathbf{z}_j(t)|) \right] , \qquad \text{(A5)}$$

where the second term in the integrand of Eq. (2.12) is the shorthand for

$$\frac{e}{c}\frac{dz_j^\mu}{dt}(t)A_\mu(\mathbf{z}_j(t)) \equiv \left[\frac{e}{c}\frac{d\mathbf{z}_j}{dt}(t)\cdot \mathbf{A}(\mathbf{z}_j(t))+eA_0(\mathbf{z}_j(t)) \right] . \qquad \text{(A6)}$$

Let us consider now the same amplitude of Eq. (A3) but for the system coupled to the Chern-Simons gauge field. This amplitude has a path-integral representation analogous to that of Eq. (A3). However, the action, which we denote by S_{new}, now has two additional terms: one to describe the coupling to the statistical vector potentials $\mathcal{A}_\mu$ and another, the Chern-Simons S_{CS} term of Eq. (2.2). The action S_{new} is given by

$$S_{\text{new}}=S+\int_{-\infty}^{+\infty} dt \frac{d\mathbf{z}_j^\mu}{dt}\mathcal{A}_\mu(\mathbf{z}_j(t))+S_{\text{CS}} \ , \qquad \text{(A7)}$$

with the same boundary conditions as in Eq. (A4) and S is the action of Eq. (A5). Note that all the terms involving vector potentials of either type, both in S and in S_{new}, can be written in the form of a line integral of the type $\int_{\Gamma_j} dz_j^\mu \mathcal{A}_\mu(\mathbf{z}_j)$, i.e., the line integral of the statistical vector potential along the set $\Gamma=\cup_{j=1}^N \Gamma_j$ of the world lines of the particles. It is convenient to define the *current* $\mathcal{J}^\mu(x)$, which is three-vector of unit length tangent to the worldlines and takes a nonzero value *only* on the world lines

$$\mathcal{J}_0(\mathbf{z},t)=\sum_{j=1}^{N} \delta(\mathbf{z}_j(t)-\mathbf{z}(t)) \ ,$$

$$\mathcal{J}(\mathbf{z},t)=\sum_{j=1}^{N} \delta(\mathbf{z}_j(t)-\mathbf{z}(t))\frac{d\mathbf{z}_j}{dt} \ . \qquad \text{(A8)}$$

In terms of $\mathcal{J}^\mu$, the second term in Eq. (A7) takes the simpler form $\int d^3z \, \mathcal{J}^\mu \mathcal{A}_\mu$.

It is possible to perform the functional integral over the statistical gauge fields exactly. These fields enter in only two terms of S_{new}: the current term and the Chern-Simons term. Thus, the average over all configurations of the statistical gauge fields has the form

$$\left\langle \exp\left[i\int d^3z \, \mathcal{J}_\mu(z)\mathcal{A}^\mu(z) \right] \right\rangle_{\text{CS}} \equiv \exp(iI[\mathcal{J}_\mu]) \ , \qquad \text{(A9)}$$

where the notation $\langle \mathcal{O} \rangle_{\text{CS}}$ indicates the average of the operator $\mathcal{O}$ over the statistical gauge fields with only a Chern-Simons action. The result, in Euclidean space (imaginary time) is

$$\left\langle \exp\left[i\int d^3z \, \mathcal{J}^\mu \mathcal{A}_\mu \right] \right\rangle_{\text{CS}} = \exp\left[-\frac{i}{2}\int d^3z \int d^3z' \mathcal{J}^\mu(z)\mathcal{J}^\nu(z')G_{\mu\nu}(z,z') \right] , \qquad \text{(A10)}$$

where $G_{\mu\nu}(z,z')$ is the Green function

$$G_{\mu\nu}(z,z')=\frac{1}{\theta}G_0(z,z')\epsilon_{\mu\nu\lambda}\partial_\lambda \delta(z-z') \ , \qquad \text{(A11)}$$

and G_0 is the Coulomb Green function

$$-\partial^2 G_0(z,z')=\delta^{(3)}(z-z') \ . \qquad \text{(A12)}$$

By direct substitution of Eq. (A12) into Eq. (A11), we can write the exponent $I[\mathcal{J}_\mu]$ of the right-hand side (rhs) of Eq. (A11) in the form

$$I[\mathcal{J}_\mu]=\frac{1}{2\theta}\int d^3z \, d^3z' \mathcal{J}_\mu(z)\mathcal{J}_\nu(z')G_0(z,z')\epsilon^{\mu\nu\lambda}\partial_\lambda \delta(z-z') \ . \qquad \text{(A13)}$$

The substitution of the definition of $\mathcal{J}_\mu$ into Eq. (A13) allows us to write $I[\mathcal{J}_\mu]$ entirely in terms of line integrals

$$I[\mathcal{J}_\mu]=\frac{1}{2\theta}\sum_{j,k=1}^{N} \oint_{\Gamma_j} dz_\mu^j \oint_{\Gamma_k} dz_\nu^k G_0(z_\mu^j,z_\nu^k)\epsilon^{\mu\nu\lambda} \times \partial_\lambda \delta(z_j-z_k) \ . \qquad \text{(A14)}$$

In practice it is easier to think in terms of the set Γ of all the world lines. The trajectories in Eq. (A14) are closed due to the boundary conditions used. The trajectories are just a collection of loops which close on the imaginary-time direction. We can picture them as a set of loops closing on the periodic direction of a cylinder (imaginary time). Any change in the final state, i.e., a permutation, results in another set of trajectories in which initial and final coordinates are tied together in a different way. We can think of the world lines as a set of *wires* which are *braided* in all possible ways.

We are now going to show that the integrals in Eq. (A14) are related to a topological invariant of the set of wires (i.e., world lines) which is known as the *linking number.* We follow here the arguments first presented by Polyakov[42] and by Witten.[43] More specifically, what we are going to show is that the integrand on the rhs of Eq. (A14) is given by the expression

$$I[\mathcal{J}_\mu]=\frac{\nu_\Gamma}{2\theta} \ , \qquad \text{(A15)}$$

where ν_Γ is the linking number of the configuration of world lines. We begin our argument by first comparing the values of the amplitudes for two different permutations P and P', which differ by a finite number of exchanges. Any exchange is equivalent to the knotting of pairs of world lines, which may or may not have been previously knotted. In particular, consider the simple but generic case of a pair of unknotted world lines which become a *single* knot upon exchange. In this case we have two separate wires which, upon exchange, become a single self-linking wire. This self-linking property is topologically stable because the wires wrap around the cylinder. Otherwise this is not a well-defined concept since it can be undone by a smooth deformation, unless a careful definition is given for the short-distance properties of the trajectories, i.e., the knot has to be "framed."[43] These issues will not matter to our discussion since they only may give rise to an anomalous (or fractional) "spin."[42]

By making further use of the magnetostatic analogy, we can now regard $\mathcal{J}_\mu$ as a current in three-dimensional Euclidean space and use it to evaluate the expressions in Eqs. (A10)–(A14). Let $\mathcal{C}_\mu$ be a vector field related to $\mathcal{J}_\mu$ by the equation ("Ampère's Law")

$$\nabla\times\mathcal{C}=\mathcal{J}\ , \tag{A16}$$

such that

$$\nabla\cdot\mathcal{C}=0\ . \tag{A17}$$

We can readily find the solution of Eq. (A16), subject to the constraint Eq. (A17). Let ϕ_μ be a "vector" potential in the "Coulomb gauge," i.e.,

$$\mathcal{C}_\mu=\epsilon_{\mu\nu\lambda}\partial_\nu\phi_\lambda,\quad \partial_\mu\phi_\mu=0\ . \tag{A18}$$

Hence,

$$\mathcal{J}_\mu=-\partial^2\phi_\mu\ . \tag{A19}$$

ϕ_μ can be solved in terms of the Green function $G_0(z,z')$

$$\phi_\mu(z)=\int d^3w\ G_0(z,w)\mathcal{J}_\mu(w)\ . \tag{A20}$$

Thus, the field $\mathcal{C}_\mu$ is given by

$$\mathcal{C}_\mu(z)=\int d^3w\ \epsilon_{\mu\nu\lambda}\partial_\nu G_0(z,w)\mathcal{J}_\lambda(w)\ . \tag{A21}$$

By substituting Eq. (A21) back into Eq. (A13), we find that $I[\mathcal{J}_\mu]$ takes the simpler form

$$I[\mathcal{J}_\mu]=\frac{1}{2\theta}\int d^3z\ \mathcal{C}_\mu(z)\mathcal{J}_\mu(z)\ . \tag{A22}$$

Now, since the currents $\mathcal{J}_\mu$ are nonzero only on the world lines, we can rewrite the volume integral in Eq. (A22) in the form of a line integral over the configuration Γ. The set of closed loops Γ are the boundary of the surface Σ, $\Gamma=\partial\Sigma$. We can then apply Stokes' theorem to get the result

$$\begin{aligned} I[\mathcal{J}_\mu]&=\frac{1}{2\theta}\oint_{\Gamma=\partial\Sigma}dz_\mu\mathcal{C}_\mu(z)\\ &=\frac{1}{2\theta}\int_\Sigma d\sigma\ n_\mu(\nabla\times\mathcal{C})_\mu=\frac{1}{2\theta}\int_\Sigma d\sigma\ n_\mu\mathcal{J}_\mu\ , \end{aligned}\tag{A23}$$

where n_μ is a vector field normal to the surface Σ. The integral $\int_\Sigma d\sigma\ n_\mu\mathcal{J}_\mu$ is an integer which counts the number of times the current $\mathcal{J}_\mu$ pierces the surface Σ. Thus, it is also equal to the number of knots of the curve, Γ, i.e., the linking number ν_Γ. Hence, Eq. (A15) is proven.

The result of Eq. (A15) has very important consequences. It means that the average over the statistical gauge fields Eq. (A10) has the simple form

$$\left\langle\exp\left[i\int d^3z\ \mathcal{J}_\mu(z)\mathcal{A}^\mu(z)\right]\right\rangle_{\rm CS}=\exp\left[i\frac{\nu_\Gamma}{2\theta}\right]\ . \tag{A24}$$

We are now in position to ask when the two systems are equivalent. In other words, when is the average over the statistical field equal to one or, at least, independent of the linking number of the trajectory? In the first case we would have proven that the amplitude in the system with the statistical gauge fields is exactly equal to the same amplitude calculated in their absence, whereas in the second case all amplitudes differ by a constant phase factor, which can be dropped. By inspecting Eq. (A24) we see that the amplitude is equal to one if

$$\frac{1}{2\theta}=2\pi n\ , \tag{A25}$$

where n is an arbitrary integer. Let us now recall[21] the relation between the statistical angle δ and the Chern-Simons coupling constant θ, $\delta=1/2\theta$. Equation (A25) implies the validity of the condition $\theta=(1/2\pi)/(1/2n)$ which we argued for at the beginning of this section. In short, *attaching an even number of flux quanta to each particle leaves the system unaltered.* This is the main result that we wanted to prove.

APPENDIX B

In this appendix we show some details of the calculation of the polarization tensor $\Pi_{\mu\nu}$. The components of $\Pi_{\mu\nu}$ are ($\hbar=1$)

$$\Pi_{00}(x,y)=iG(x,y)G(y,x)\ , \tag{B1a}$$

$$\Pi_{0j}(x,y)=\frac{1}{2M}[G(x,y)D_j^yG(y,x)-G(y,x)D_j^{y\dagger}G(x,y)]\ , \tag{B1b}$$

$$\Pi_{j0}(x,y)=\frac{1}{2M}[-G(x,y)D_j^{x\dagger}G(y,x)+G(y,x)D_j^xG(x,y)]\ , \tag{B1c}$$

$$\Pi_{jk}(x,y)=+\frac{i}{M}\delta^3(x-y)\delta_{jk}G(x,y)-\frac{i}{4M^2}[D_j^xG(x,y)][D_k^yG(y,x)]-\frac{i}{4M^2}[D_j^{x\dagger}G(y,x)][D_k^{y\dagger}G(x,y)]$$
$$+\frac{i}{4M^2}G(y,x)[D_j^{x\dagger}D_k^{y\dagger}G(x,y)]+\frac{i}{4M^2}[D_j^{x\dagger}D_k^yG(y,x)]G(x,y)\ , \tag{B1d}$$

where $G(x,y)$ is the one-particle fermion Green function whose equation of motion is

$$(iD_0+\mu-h[\langle\mathcal{A}\rangle])_xG(x,y)=\delta^3(x-y)\ . \tag{B2}$$

We work in the Landau gauge in which we can express the vector potential and the Landau wave functions as follows:

$$\mathcal{A}_1(x)=-\mathcal{B}_{\text{eff}}x_2,\quad \mathcal{A}_2(x)=0\ , \tag{B3}$$

$$\varphi_{mk}(x)=\left[\frac{\sqrt{\mathcal{B}_{\text{eff}}}}{2^m m!\sqrt{\pi}}\right]^{1/2}e^{ikx_1}\exp\left[-\frac{1}{2}\left[\sqrt{\mathcal{B}_{\text{eff}}}x_2-\frac{k}{\sqrt{\mathcal{B}_{\text{eff}}}}\right]^2\right]H_m\left[\sqrt{\mathcal{B}_{\text{eff}}}x_2-\frac{k}{\sqrt{\mathcal{B}_{\text{eff}}}}\right]\ , \tag{B4}$$

where H_m is the Hermite polynomial. We can write the one-particle fermion propagator in terms of the Landau wave functions:

$$iG(x,y)=+\Theta(x_0-y_0)\sum_{m=p}^{\infty}\int\frac{dk}{2\pi}e^{-i\omega_m(x_0-y_0)}\varphi_{mk}(\mathbf{x})\varphi_{mk}^*(\mathbf{y})-\Theta(y_0-x_0)\sum_{m=0}^{p-1}\int\frac{dk}{2\pi}e^{-i\omega_m(x_0-y_0)}\varphi_{mk}(\mathbf{x})\varphi_{mk}^*(\mathbf{y})\ , \tag{B5}$$

and then substitute this expression into the equations for the components of $\Pi_{\mu\nu}$. Here the energy ω_m is given by $\omega_m=(\mathcal{B}_{\text{eff}}/M)(m+\frac{1}{2})$.

The Fourier transform of the polarization tensor is

$$\Pi_{\mu\nu}(Q,P)=\int d^3x\,d^3y\,e^{i(Q_0x_0-\mathbf{Qx})}e^{i(P_0y_0-\mathbf{Py})}\Pi_{\mu\nu}(x,y)\ . \tag{B6}$$

We should stress here that, except for $\Pi_{00}(x,y)$ all the other components of $\Pi_{\mu\nu}(x,y)$ are not rotational invariants, therefore we cannot choose any particular direction for the momentum transfer. In order to compute the expression (B6) and given the form of the components of the polarization tensor [Eq. (B1)] we need to calculate terms of the form

$$T_{1,2}(Q,P)=\int d^3x\,d^3y\,e^{i(Q_0x_0-\mathbf{Qx})}e^{i(P_0y_0-\mathbf{Py})}O_1(x,y)G(x,y)O_2(x,y)G(y,x)\ , \tag{B7}$$

where $O_1(x,y)$ and $O_2(x,y)$ are the identity in the case of Π_{00}, a covariant derivative for Π_{0j}, Π_{j0}, and the first terms of Π_{ij}, or a product of two of them for the last two terms of Π_{ij}. Equation (B7) can be written as follows:

$$T_{1,2}(Q,P)=\int d^3x\,d^3y\,e^{i(Q_0x_0-\mathbf{Qx})}e^{i(P_0x_0-\mathbf{Px})}i\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}e^{-i\omega(x_0-y_0)}$$
$$\times\sum_{m=p}^{\infty}\int\frac{dk}{2\pi}\sum_{m'=0}^{p-1}\int\frac{dk'}{2\pi}\left[\frac{O_1(x,y)[\varphi_{mk}(\mathbf{x})\varphi_{mk}^*(\mathbf{y})]O_2(x,y)[\varphi_{m'k'}(\mathbf{y})\varphi_{m'k'}^*(\mathbf{x})]}{\omega-(\omega_m-\omega_{m'})+i\eta}\right.$$
$$\left.-\frac{O_1(x,y)[\varphi_{m'k'}(\mathbf{x})\varphi_{m'k'}^*(\mathbf{y})]O_2(x,y)[\varphi_{mk}(\mathbf{y})\varphi_{mk}^*(\mathbf{x})]}{\omega+(\omega_m-\omega_{m'})-i\eta}\right]\ , \tag{B8}$$

where we have used the integral representation of the theta functions.

We will calculate the first component of the $\Pi_{01}(Q,P)$ as an example; all the other components can be calculated in a similar way. In this case we have to take $O_1(x,y)=1$ and $O_2(x,y)=D_1^{(y)}$. Therefore

$$T_{1,2}(Q,P)=\frac{i}{2M}\int d^3x\,d^3y\,e^{i(Q_0x_0-\mathbf{Qx})}e^{i(P_0x_0-\mathbf{Px})}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}e^{-i\omega(x_0-y_0)}$$
$$+\sum_{m=p}^{\infty}\int\frac{dk}{2\pi}\sum_{m'=0}^{p-1}\int\frac{dk'}{2\pi}\left[\frac{[\varphi_{mk}(\mathbf{x})\varphi_{mk}^*(\mathbf{y})]D_1^{(y)}(\varphi_{m'k'}(\mathbf{y})\varphi_{m'k'}^*(\mathbf{x}))}{\omega-(\omega_m-\omega_{m'})+i\eta}\right.$$
$$\left.-\frac{[\varphi_{m'k'}(\mathbf{x})\varphi_{m'k'}^*(\mathbf{y})]D_1^{(y)}(\varphi_{mk}(\mathbf{y})\varphi_{mk}^*(\mathbf{x}))}{\omega+(\omega_m-\omega_{m'})-i\eta}\right]\ . \tag{B9}$$

Using the expressions of the Landau wave functions (B4), the definition of the covariant derivative $(D_1^{(y)}=\partial_1^{(y)}-i\mathcal{B}_{\text{eff}}y_2)$, and integrating over x_2 and y_2 first, we obtain

$$T_{1,2}(Q,P)=(2\pi)^3\delta^3(Q+P)\frac{\mathcal{B}_{\rm eff}}{4M(2\pi)}e^{-\bar{Q}^2}\sum_{m=p}^{\infty}\sum_{m'=0}^{p-1}\left[\frac{Q_1L_{m'}^{m-m'}(\bar{Q}^2)[\bar{Q}^2-(m-m')]+iQ_2F_{m,m'}(\bar{Q}^2)}{[Q_0-(\omega_m-\omega_{m'})+i\eta]}\right.$$
$$\left.-\frac{Q_1L_{m'}^{m-m'}(\bar{Q}^2)[\bar{Q}^2+(m-m')]+iQ_2F_{m,m'}(\bar{Q}^2)}{[Q_0+(\omega_m-\omega_{m'})-i\eta]}\right], \quad \text{(B10)}$$

where

$$F_{m,m'}(\bar{Q}^2)=\bar{Q}^2[L_{m'}^{m-m'}(\bar{Q}^2)+2L_{m'-1}^{m-m'+1}(\bar{Q}^2)(1-\delta_{m',0})]-(m-m')L_{m'}^{m-m'}(\bar{Q}^2) \quad \text{(B11)}$$

and $\bar{Q}^2=\mathbf{Q}^2/2\mathcal{B}_{\rm eff}$.

From equation (B10) we can write

$$T_{1,2}(Q,P)=(2\pi)^3\delta^3(Q+P)\tilde{T}_{1,2}(Q) , \quad \text{(B12)}$$

which has to be satisfied by all the terms in $\Pi_{\mu\nu}$ since it is translationally invariant. After computing all the terms in a similar way, we write $\Pi_{\mu\nu}(Q,P)$ in the form

$$\Pi_{\mu\nu}(Q,P)=(2\pi)^3\delta^3(Q+P)\Pi_{\mu\nu}(Q) \quad \text{(B13)}$$

and get the following expressions for $\Pi_{\mu\nu}(Q)$:

$$\Pi_{00}(Q)=\mathbf{Q}^2\Pi_0(Q) , \quad \text{(B14a)}$$

$$\Pi_{0j}(Q)=Q_0Q_j\Pi_0(Q)+i\epsilon_{jk}Q_k\Pi_1(Q) , \quad \text{(B14b)}$$

$$\Pi_{j0}(Q)=Q_0Q_j\Pi_0(Q)-i\epsilon_{jk}Q_k\Pi_1(Q) , \quad \text{(B14c)}$$

$$\Pi_{ij}(Q)=Q_0^2\delta_{ij}\Pi_0(Q)-iQ_0\epsilon_{ij}\Pi_1(Q)+(\mathbf{Q}^2\delta_{ij}-Q_iQ_j)\Pi_2(Q)+\delta_{ij}\Pi_3(\mathbf{Q}) . \quad \text{(B14d)}$$

The functions Π_0, Π_1, Π_2, and Π_3 are given by

$$\Pi_0(Q_0,\mathbf{Q})=-\frac{\mathcal{B}_{\rm eff}}{(2\pi)M}e^{-\bar{Q}^2}\sum_{m=p}^{\infty}\sum_{m'=0}^{p-1}\frac{(m-m')}{Q_0^2-(\omega_m-\omega_{m'})^2}\frac{m'!}{m!}\bar{Q}^{2(m-m'-1)}[L_{m'}^{m-m'}(\bar{Q}^2)]^2 , \quad \text{(B15)}$$

$$\Pi_1(Q_0,\mathbf{Q})=\frac{\mathcal{B}_{\rm eff}^2}{(2\pi)M^2}e^{-\bar{Q}^2}\sum_{m=p}^{\infty}\sum_{m'=0}^{p-1}\frac{(m-m')}{Q_0^2-(\omega_m-\omega_{m'})^2}\frac{m'!}{m!}\bar{Q}^{2(m-m'-1)}L_{m'}^{m-m'}(\bar{Q}^2)$$
$$\times\{\bar{Q}^2[L_{m'}^{m-m'}(\bar{Q}^2)+2L_{m'-1}^{m-m'+1}(\bar{Q}^2)(1-\delta_{m',0})]-(m-m')L_{m'}^{m-m'}(\bar{Q}^2)\} , \quad \text{(B16)}$$

$$\Pi_2(Q_0,\mathbf{Q})=-\frac{\mathcal{B}_{\rm eff}^2}{(2\pi)2M^3}e^{-\bar{Q}^2}\sum_{m=p}^{\infty}\sum_{m'=0}^{p-1}\frac{(m-m')}{Q_0^2-(\omega_m-\omega_{m'})^2}\frac{m'!}{m!}\bar{Q}^{2(m-m'-1)}$$
$$\times[L_{m'}^{m-m'}(\bar{Q}^2)+2L_{m'-1}^{m-m'+1}(\bar{Q}^2)(1-\delta_{m',0})]$$
$$\times\{\bar{Q}^2[L_{m'}^{m-m'}(\bar{Q}^2)+2L_{m'-1}^{m-m'+1}(\bar{Q}^2)(1-\delta_{m',0})]$$
$$-2(m-m')L_{m'}^{m-m'}(\bar{Q}^2)\} , \quad \text{(B17)}$$

$$\Pi_3(\mathbf{Q})=\frac{\mathcal{B}_{\rm eff}}{(2\pi)M}e^{-\bar{Q}^2}\sum_{m=p}^{\infty}\sum_{m'=0}^{p-1}\frac{m'!}{m!}\bar{Q}^{2(m-m'-1)}[L_{m'}^{m-m'}(\bar{Q}^2)]^2(m-m')-p\frac{\mathcal{B}_{\rm eff}}{(2\pi)M} . \quad \text{(B18)}$$

With the form of the $\Pi_{\mu\nu}(Q)$ given in Eqs. (B14), it is not clear that the polarization tensor is transverse. To clarify this point we should analyze the last term in $\Pi_{ij}(Q)$. If we split the sum in Π_3 as a term with m and m' such that $m-m'=1$ (i.e., $m=p$, $m'=p-1$), and another term with $m-m'\neq 1$ (notice that this last term is a series in powers of $\bar{Q}^2$ and it vanishes when $\bar{Q}^2=0$), we obtain

$$\Pi_3(\mathbf{Q})=\frac{\mathcal{B}_{\rm eff}}{(2\pi)M}e^{-\bar{Q}^2}\sum_{m=p}^{\infty}\sum_{m'=0}^{p-1}(1-\delta_{m-m',1})\frac{m'!}{m!}\bar{Q}^{2(m-m'-1)}[L_{m'}^{m-m'}(\bar{Q}^2)]^2(m-m')-p\frac{\mathcal{B}_{\rm eff}}{(2\pi)M}+p\frac{\mathcal{B}_{\rm eff}}{(2\pi)M} . \quad \text{(B19)}$$

This shows that the density term cancels and that $\Pi_3(\mathbf{Q})$ is a regular function of $\mathbf{Q}$ and independent of Q_0. The term in the action where $\Pi_3(\mathbf{Q})$ enters is

$$\int dQ_0d^2\mathbf{Q}\mathcal{A}_i(Q_0,\mathbf{Q})\Pi_3(\mathbf{Q})\mathcal{A}_i(-Q_0,-\mathbf{Q}) . \quad \text{(B20)}$$

We will only consider the class of configurations of the statistical gauge field $\mathcal{A}_\mu$ which vanish sufficiently fast at large frequencies Q_0 so that the integration contour implied in Eq. (B20) can be closed in the complex frequency plane without enclosing any singularities. Thus the integrand is a holomorphic function of Q_0 with singularities at most at infinity. Cauchy's theorem tells us that the integral vanishes for such configurations. In a general case, these restrictions on the class of configurations may appear to be too restrictive. However, only *sufficiently regular* configurations contribute in the semiclassical limit. Therefore we can drop the last term in expression (B14d) since it does not contribute to the action. It is now clear that the polarization tensor is transverse.

Finally, let us derive the gradient expansion for $\Pi_{\mu\nu}(Q)$. At $Q=0$ the functions Π_0, Π_1, Π_2 take the values

$$\Pi_0(0)=\left[\frac{1}{2\pi}\right]\frac{pM}{\mathcal{B}_{\rm eff}}\ , \tag{B21a}$$

$$\Pi_1(0)=\left[\frac{1}{2\pi}\right]p\ , \tag{B21b}$$

$$\Pi_2(0)=-\left[\frac{1}{2\pi}\right]\frac{p^2}{M}\ . \tag{B21c}$$

Thus, the small-momentum and low-frequency limit of the action is

$$\begin{aligned} S&=\tfrac{1}{2}\int d^3Q\,\mathcal{A}_\mu(Q)\Pi^{\mu\nu}(Q)\mathcal{A}_\nu(-Q)\\ &\approx\frac{\Pi_0(0)}{2}\int d^3Q\,\mathcal{A}_\mu(Q)[(Q^2\eta^{\mu\nu}-Q^\mu Q^\nu)-\delta_i^\mu\delta_j^\nu(\mathbf{Q}^2\delta_{ij}-Q_iQ_j)]\mathcal{A}_\nu(-Q)\\ &\quad+\frac{\Pi_1(0)}{2}\int d^3Q\,\mathcal{A}_\mu(Q)(i\epsilon^{\mu\nu\lambda}Q_\lambda)\mathcal{A}_\nu(-Q)+\frac{\Pi_2(0)}{2}\int d^3Q\,\mathcal{A}_j(Q)(\mathbf{Q}^2\delta_{jk}-Q_jQ_k)\mathcal{A}_k(-Q)\ , \end{aligned} \tag{B22}$$

where $\eta^{00}=1$ and $\eta^{ii}=-1$ and $\eta^{\mu\nu}=0$ if $\mu\neq\nu$. In coordinate space we get the gradient expansion of the effective action:

$$S\approx\tfrac{1}{2}\int d^3x\,(\epsilon\mathcal{E}^2-\chi\mathcal{B}^2+\sigma_{xy}^{(0)}\epsilon_{\mu\nu\lambda}\mathcal{A}^\mu\partial^\nu\mathcal{A}^\lambda)\ . \tag{B23}$$

The coefficients ϵ, χ, and $\sigma_{xy}^{(0)}$ are related with Π_0, Π_1, and Π_2 through

$$\epsilon=\Pi_0(0)\ , \tag{B24a}$$

$$\sigma_{xy}^{(0)}=\Pi_1(0)\ , \tag{B24b}$$

$$\chi=-\Pi_2(0)\ . \tag{B24c}$$

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