Dipolar electric field induced by a vortex moving in an anisotropic superconductor

Mark W. Coffey and Zhidong Hao
Ames Laboratory and Department of Physics and Astronomy, Iowa State University, Ames, Iowa 50011
(Received 26 December 1990)

The dipolar electric field induced by an isolated vortex aligned along a principal axis and moving at constant velocity in an anisotropic type-II superconductor is investigated. It is shown that the standard dipolar-field case can be extended by the inclusion of a mass-anisotropy parameter β. It is shown that the streamlines of the electric field can be easily calculated analytically. In addition, an inertial mass tensor per unit length of vortex is found.

In Ref. 1 the Bardeen-Stephen model[2–7] of viscous flux motion in a type-II superconductor was extended to the anisotropic case. The total rate of energy dissipation was calculated and, from it, the viscosity tensor, in terms of a phenomenological effective-mass tensor and normal-state conductivity tensor. In this paper we further investigate the dipolar electric field induced by the motion of a single vortex moving at constant velocity in an anisotropic superconductor. We show that the standard dipolar-field case can be extended by the inclusion of a mass-anisotropy parameter β. We show that the streamlines of the electric field can be easily calculated analytically. We also calculate the surface-charge density at the vortex-core boundary in this local model of a type-II superconductor. In addition, by computing the electric-field energy per unit length of vortex, we find a vortex inertial-mass tensor per unit length.

For comparison purposes we first recall a few selected facts concerning the electric field of a point dipole. Locating the dipole at the origin, the electric field is given by[8–11]

$$\mathbf{E}(\mathbf{r})=4\pi\left[\frac{3\mathbf{r}\cdot\mathbf{p}}{r^5}\mathbf{r}-\frac{\mathbf{p}}{r^3}\right], \tag{1}$$

where $\mathbf{p}$ is the dipole moment. We wish to consider the electric field in the xy plane. We take $z=0$ and set $\mathbf{p}=p(\cos\phi,\sin\phi)$. Then the equation of the electric-field streamlines (lines of force) is

$$\frac{dy}{dx}=\frac{E_y(x,y)}{E_x(x,y)}=\frac{-x^2\tan\phi+3xy+2y^2\tan\phi}{2x^2+3xy\tan\phi-y^2}. \tag{2}$$

In the anisotropic model,[1] the vortex core is taken to be elliptical in shape and its boundary is specified by the equation

$$\frac{x^2}{\xi_x^2}+\frac{y^2}{\xi_y^2}=1, \tag{3}$$

where the coherence lengths ξ_i, $i=1,2\underline{=}x,y$, are related to the effective masses m_i by $\xi_i=\xi/\sqrt{m_i}$. (Further information on the anisotropic Ginzburg-Landau theory may be found, e.g., in Ref. 1.)

The electric field induced outside the core of an isolated vortex oriented along the z axis (a principal axis) and moving with constant velocity $\mathbf{V}=V(\cos\theta,\sin\theta)$ in an anisotropic type-II superconductor may be written as[1]

$$\mathbf{e}_s(\mathbf{r})=\frac{\hbar\sqrt{m_x m_y}\,V}{2e(m_x x^2+m_y y^2)^2}\times[\hat{\mathbf{x}}(m_x x^2\sin\theta-2m_x xy\cos\theta-m_y y^2\sin\theta)+\hat{\mathbf{y}}(m_x x^2\cos\theta+2m_y xy\sin\theta-m_y y^2\cos\theta)], \tag{4}$$

where m_x,m_y are effective masses. Inside the vortex core, the field is uniform:

$$\mathbf{e}_c(\mathbf{r})=\frac{\hbar V}{2e\xi_x\xi_y}(-\hat{\mathbf{x}}\sin\theta+\hat{\mathbf{y}}\cos\theta). \tag{5}$$

These expressions for the electric field are expected to hold for low field ($H\ll H_{c2}$) when the Ginzburg-Landau parameter $\kappa=\lambda/\xi$ is large and only for temperatures close to the transition temperature.[1] Furthermore, the quasistatic approximation to the superfluid velocity was made to obtain Eqs. (4) and (5), along with the use of the asymptotic forms of modified Bessel functions K_ν.[1]

The electrostatic potential, defined by $\mathbf{e}=-\nabla\Phi$, can be readily found from Eqs. (4) and (5). Inside the core region we have

$$\Phi_c(\mathbf{r})=-\frac{\hbar V}{2e\xi_x\xi_y}(y\cos\theta-x\sin\theta), \quad \frac{x^2}{\xi_x^2}+\frac{y^2}{\xi_y^2}\le 1, \tag{6a}$$

while outside the core we have

$$\Phi_s(\mathbf{r})=-\frac{\hbar\sqrt{m_x m_y}\,V}{2e}\frac{(y\cos\theta-x\sin\theta)}{(m_x x^2+m_y y^2)}, \quad \frac{x^2}{\xi_x^2}+\frac{y^2}{\xi_y^2}\ge 1. \tag{6b}$$

Equations (6) extend the electrostatic potential of Ref. 7 to the anisotropic case. The potential Φ is continuous across the vortex boundary, implying that its tangential derivative is continuous there. However, the normal derivative $\partial\Phi/\partial n$ is not continuous across the boundary, and the corresponding surface-charge density will be calculated later on. The equipotential curves $\Phi=\text{const}$ are everywhere locally orthogonal to the electric-field streamlines, the calculation of which a large part of this paper is devoted. Inside the vortex core there are straight-line equipotentials. By Eq. (6b), outside the core, the equipotentials are given by a family of ellipses. Specifically, if we define the constant $c_\Phi \equiv -\hbar\sqrt{m_x m_y}V/2e\Phi_s$, we then have the equation of the equipotentials there as

$$m_x(x-x_c)^2+m_y(y-y_c)^2 = \frac{c_\Phi^2}{4}\left[\frac{1}{m_x}\sin^2\theta+\frac{1}{m_y}\cos^2\theta\right]. \quad (6c)$$

The centers of the ellipses,

$$(x_c,y_c)=\left[-\frac{c_\Phi}{2m_x}\sin\theta,\frac{c_\Phi}{2m_y}\cos\theta\right],$$

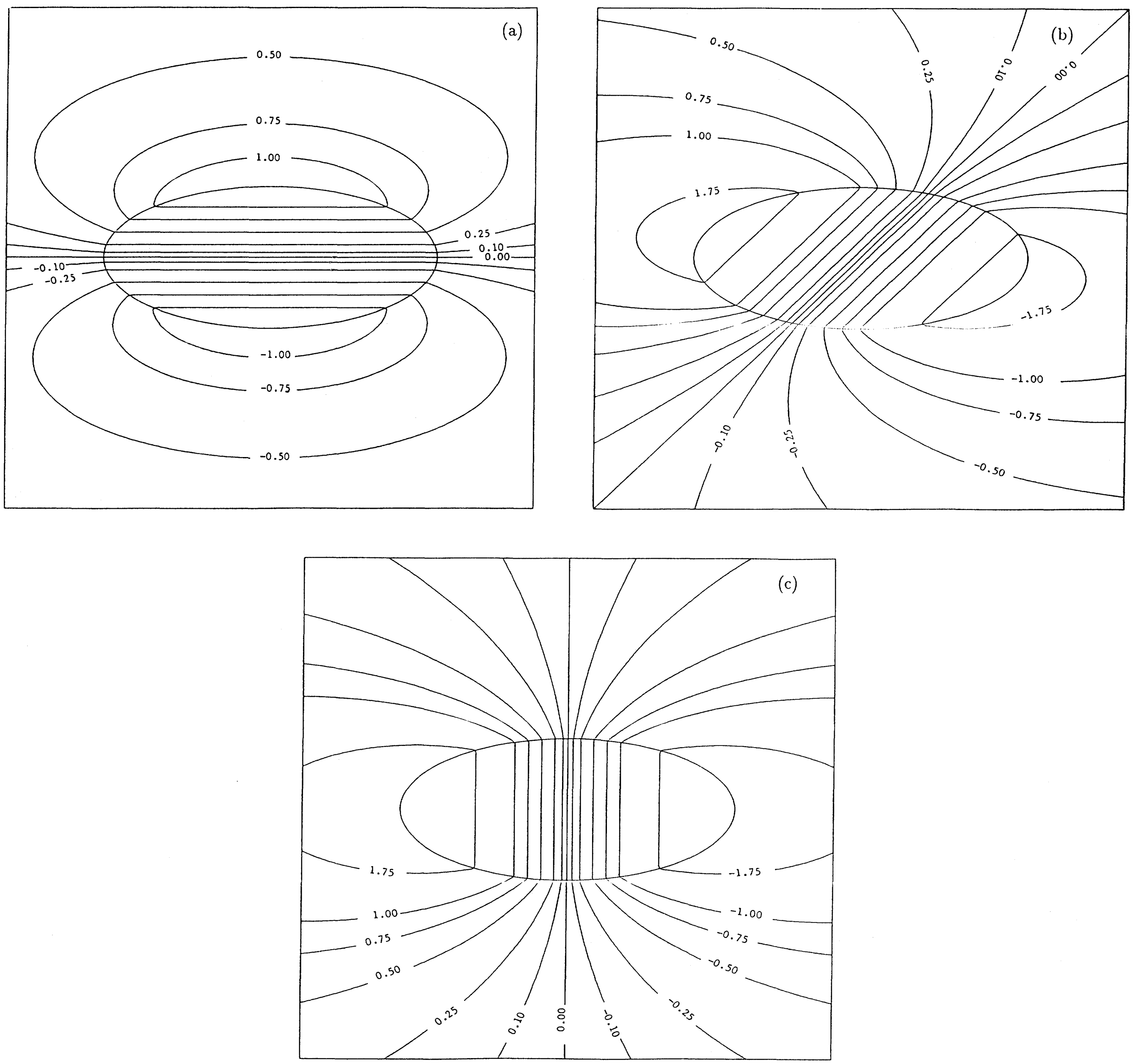

FIG. 1. Equipotentials for vortex motion along (a) the x axis, (b) the line $x=y$, and (c) the y axis, for a fixed mass-anisotropy ratio $\beta=5$. The horizontal axis is the x axis, the vertical axis is the y axis, and the z axis points out of the page. The equipotentials are straight lines within the vortex core, whose boundary is also shown and given by the ellipses in Eq. (6c) outside the core.

vary with the values of the potential. In Fig. 1 the equipotentials have been plotted for the three directions of vortex motion, $\theta=0$, $\pi/4$, and $\pi/2$, with the mass ratio $m_y/m_x=5$.

Inside the vortex core, the electric-field streamlines are simply straight lines. Outside the core, the equation analogous to (2) for the streamlines is

$$\frac{dy}{dx}=\frac{e_{sy}(x,y)}{e_{sx}(x,y)}=\frac{x^2+2\beta xy\tan\theta-\beta y^2}{x^2\tan\theta-2xy-\beta y^2\tan\theta}, \quad (7)$$

where β is the effective-mass-anisotropy ratio m_y/m_x. Comparing Eq. (7) to Eq. (2), we can note that vortex motion in the direction **V** corresponds to an induced dipole with axis perpendicular to **V** ($\phi=\theta+\pi/2$). The isotropic case $\beta=1$ was treated by Bardeen and Stephen.[2]

The first-order, nonlinear differential equation (7) will be discussed from several mathematical points of view. Although Eq. (7) is nonlinear, its special form makes it elementary, in the sense that the solution may be reduced to a quadrature. General equations of the form (7) have been well studied in nonlinear systems. For example, by elimination of the independent variable, it arises in the investigation of the phase space of a two-dimensional (2D) autonomous system. Equation (7) is especially important in giving the local behavior around the critical point (at the origin) for such a system where the leading terms in the functions e_{sy} and e_{sx} are quadratic.[12]

The right-hand side of Eq. (7), e_{sy}/e_{sx}, is a homogeneous function of x and y of degree zero. Then Eq. (7) has an integrating factor $(xe_{sy}-ye_{sx})^{-1}$, and with the aid of Euler's theorem, its solution may be reduced to a quadrature.[12–15] By using the transformation

$$y(x)=xv(x), \quad (8)$$

we have

$$\ln x=-\int\frac{e_{sx}(1,v)}{-e_{sy}(1,v)+ve_{sx}(1,v)}dv. \quad (9)$$

The transformation (8) results from other methods of solution, as seen below, and an equivalent form of Eq. (9) will also appear.

It is useful in the following to consider the special solutions of Eq. (7) given by $y=\alpha x$; i.e., the function v of Eq. (8) is taken to be a constant α to be determined. These solutions, if continued into the origin (within the vortex core), would be tangential to the streamlines there. With this form of the solution, we find that α must satisfy the cubic equation

$$\beta\alpha^3\tan\theta+(2-\beta)\alpha^2+(2\beta-1)\alpha\tan\theta+1=0. \quad (10)$$

Although the discriminant of this cubic equation can be calculated and the solution of Eq. (10) given explicitly, we will not do so here; the expression for the discriminant is sufficiently complicated to be unilluminating by itself. However, on physical grounds, we expect the general cubic equation (10) to have one real root and two complex conjugate roots. The solution $y=\alpha x$, with α real, gives the separatrix of the streamlines: Each streamline forms a closed path on one side or the other of this line. This can be seen in the plots of the streamlines for $\theta=0$, $\pi/4$, and $\pi/2$, with $\beta=5$, given in Ref. 1. For the case that $\tan\theta=0$, the cubic equation in (10) degenerates into a simple quadratic equation. In this case of vortex motion in the x direction, we have the slope of the separatrices as $\alpha_\pm=\pm1/\sqrt{\beta-2}$.

The right-hand side of Eq. (7) is a function of y/x alone. This means that the differential equation for the streamlines is invariant under the simple stretching or perspective transformation $y\to y'=ay$, $x\to x'=ax$. We have here a simple example of a group invariance of a differential equation.[15–19] This implies that a similarity variable exists, namely, the function $v=y/x$ of Eq. (8). The particular importance of the differential equation for the function v,

$$x\frac{dv}{dx}=\frac{1+(2\beta-1)v\tan\theta+(2-\beta)v^2+\beta v^3\tan\theta}{-(\beta v^2\tan\theta+2v-\tan\theta)}, \quad (11)$$

is that it is separable.

Integrating Eq. (11) for arbitrary direction of vortex motion θ, we have

$$\ln x+\text{const}=-\tfrac{1}{3}\ln[-\beta\tan\theta v^3+(\beta-2)v^2+(1-2\beta)\tan\theta v-1]-\tfrac{2}{3}(\beta+1)\int\frac{(v-\tan\theta)dv}{\beta v^3\tan\theta+(2-\beta)v^2+(2\beta-1)v\tan\theta+1}. \quad (12)$$

In principle, the integral in Eq. (12) can always be done. Solving the cubic equation (10), the denominator of the integrand can be factored, resulting in a sum of elementary integrations. Denoting the roots of the cubic equation (10) by α_i, $i=1,2,3$, the solution of Eq. (7) can be put in the form

$$(y-\alpha_1x)^{\nu_1}(y-\alpha_2x)^{\nu_2}(y-\alpha_3x)^{\nu_3}=\text{const}, \quad (13)$$

where the constants ν_i, $i=1,2,3$ are such that $\nu_1+\nu_2+\nu_3=-1$ and are related to α_i by[12]

$$\nu_i=\frac{-\beta\alpha_i^2\tan\theta-2\alpha_i+\tan\theta}{3\beta\alpha_i^2\tan\theta+2(2-\beta)\alpha_i+(2\beta-1)\tan\theta} \quad (i=1,2,3). \quad (14)$$

Equation (13) gives the general form of the electric-field streamlines or lines of force due to an isolated vortex moving in an anisotropic type-II superconductor.

By the construction in Ref. 1, the electric field [Eqs. (4) and (5)] satisfies the condition that the tangential component is continuous across the vortex-core boundary. However, the normal component is discontinuous, with an associated surface-charge density σ. We can find σ from the relation $\sigma=(\mathbf{e}_s-\mathbf{e}_c)\cdot\hat{\mathbf{n}}/4\pi$, where $\hat{\mathbf{n}}$ is the unit outward normal vector:

$$\hat{\mathbf{n}}=\frac{x\hat{\mathbf{x}}+\beta y\hat{\mathbf{y}}}{(x^2+\beta^2 y^2)^{1/2}} . \tag{15}$$

It is easily checked that $\hat{\mathbf{n}}$ is orthogonal to the superfluid velocity $\mathbf{v}_s$ (Ref. 1) at the core boundary. With the use of Eq. (15), the surface-charge density is given as

$$\sigma=-\frac{H_{c2\parallel z}V}{2\pi c}\,\frac{\sin(\psi-\theta)(\cos^2\psi+\beta^2\sin^2\psi)^{1/2}}{\cos^2\psi+\beta\sin^2\psi} , \tag{16}$$

where $H_{c2\parallel z}=\phi_0/2\pi\xi_x\xi_y$ is the upper critical field for an applied field oriented along the z axis and ψ is the angle between the position vector on the vortex-core boundary and the x axis. This surface-charge density results from the use of a local model for the superconductor. That is, the charge density would not exist solely on the vortex-core boundary, but would be smeared out over a finite distance in a nonlocal theory, as pointed out by Bardeen and Stephen. We have plotted the dimensionless surface-charge density $(2\pi c/H_{c2\parallel z}V)\sigma$ as a function of the angle ψ around the vortex core in Fig. 2. Figures 2(a)–2(c) correspond to vortex motion given by $\theta=0$, $\pi/4$, and $\pi/2$, respectively. For each figure the mass ratio β takes the values 1, 5, and 25, giving the three curves displayed. For vortex motion along the y axis, the isotropic case in Fig. 2(c) shows the simple $\cos\phi$ dependence obtained by Bardeen and Stephen.[2] However, for general $\beta\neq 1$, these figures show that anisotropy introduces additional local maxima and minima in the surface-charge density.

We conclude by finding a vortex inertial-mass tensor per unit length μ_{ij} associated with the electric field. For an expression for the inertial mass of a vortex in the isotropic case, obtained by means of time-dependent Ginzburg-Landau theory, we mention Ref. 20 by Suhl. In our simple approach, we do not consider any variation in the order parameter which could contribute to the inertial mass. (We recall that we are discussing only the high-temperature case $|T-T_c|\ll T_c$, where the order parameter is small.) For a counterpart of an inertial mass per unit length for a Josephson vortex, we mention Ref. 21.

We equate the vortex kinetic energy per unit length $\frac{1}{2}\mu_{ij}V_iV_j$ to the electric-field energy $\varepsilon_{1f}=\int e^2 d^2x/8\pi$ per unit length produced by the vortex motion. Writing $\varepsilon_{1f}=\varepsilon_{1fc}+\varepsilon_{1fs}$ as a sum of field contributions within and without the core region, respectively, we have[1]

$$\varepsilon_{1fc}=\frac{\phi_0 H_{c2\parallel z}}{16\pi c^2}V^2 , \tag{17a}$$

$$\varepsilon_{1fs}=\frac{\phi_0 H_{c2\parallel z}}{32\pi c^2}\left[V_1^2\left[1+\frac{1}{\beta}\right]+V_2^2(1+\beta)\right] . \tag{17b}$$

Then we find the inertial-mass tensor to be diagonal, $\mu_{ij}=\mu_i\delta_{ij}$, $i,j=1,2=x,y$, with components

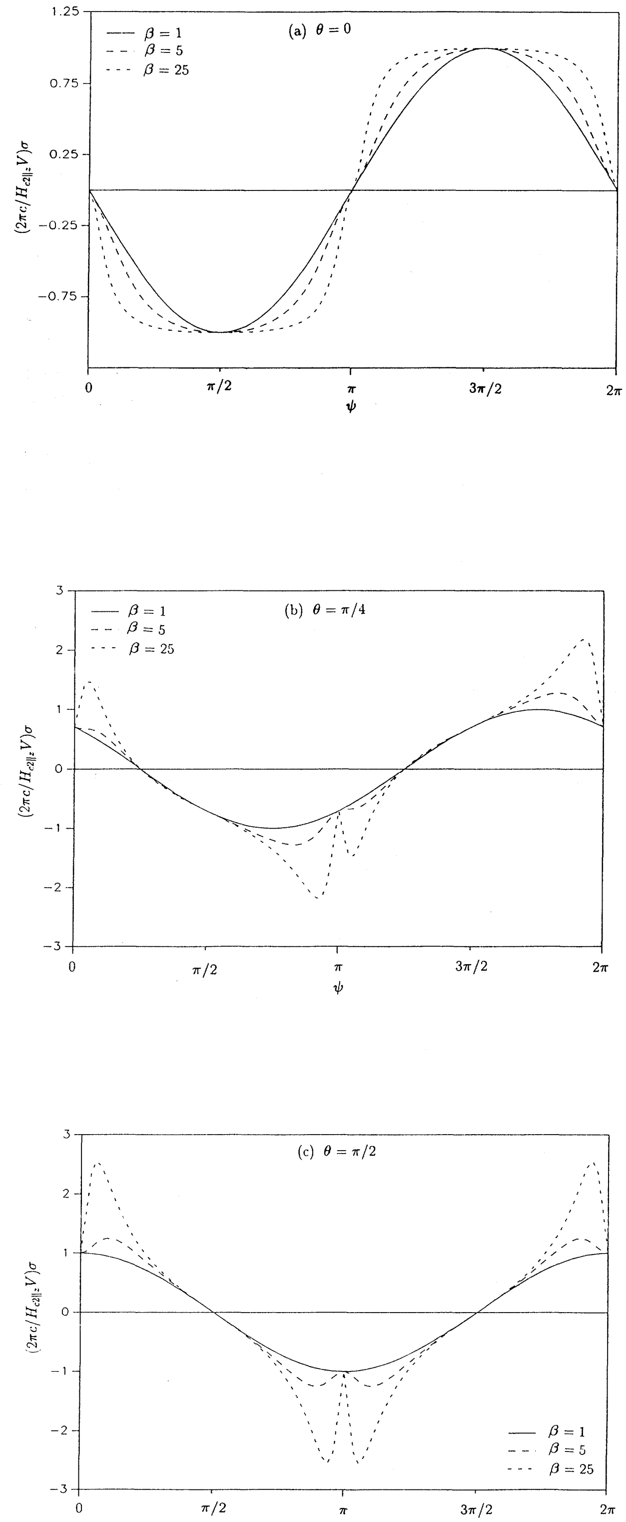

FIG. 2. The dimensionless surface-charge density $(2\pi c/H_{c2\parallel z}V)\sigma$ from Eq. (16) is plotted as a function of the angle ψ around the vortex core, for differing values of the mass-anisotropy ratio β. Parts (a)–(c) correspond to the direction of vortex motion given by $\theta=0$, $\pi/4$, and $\pi/2$, respectively.

$$\mu_x = \frac{\phi_0 H_{c2\|z}}{16\pi c^2}\left[3+\frac{1}{\beta}\right],$$
$$\mu_y = \frac{\phi_0 H_{c2\|z}}{16\pi c^2}(3+\beta). \tag{18}$$

It can be shown that the same expression (18) for the inertial mass tensor is obtained if the expression for the electric field involving modified Bessel functions is used. It is seen that anisotropy in the effective-mass tensor (with components m_i, $i=1,2,3=x,y,z$) is reflected in anisotropy in the inertial-mass tensor. The inertial-mass-anisotropy ratio $\tilde{\beta}=\mu_y/\mu_x=(3+\beta)/(3+1/\beta)$ ranges from 1 in the isotropic case to $\beta/3$ in the large β case. The vortex mass per unit length vanishes at the transition temperature T_c, where $H_{c2\|z}$ vanishes.

To estimate the size of the inertial mass for high-T_c materials, we can consider the limiting isotropic case, $\beta \to 1$. As a function of temperature, we have that

$$\mu(T)=\frac{\phi_0 H_{c2}(T)}{4\pi c^2}. \tag{19}$$

Assuming a $1-t^2$ dependence for $H_{c2}(T)$, where $t \equiv T/T_c$ is the reduced temperature, if we take $H_{c2}(0)=10^2$ T and $t=0.99$, then we obtain an inertial mass per unit length of vortex on the order of several hundred electron masses. Because of the dependence on H_{c2}, for conventional superconductors (for which H_{c2} is much smaller), μ is on the order of the electron mass. It is also interesting to compare the inertial mass (19) with the rest mass per unit length defined by

$$\mu_0(T)=\frac{\varepsilon_1(T)}{c^2}=\frac{\phi_0 H_{c1}(T)}{4\pi c^2}, \tag{20}$$

where ε_1 is the energy per length of a vortex at rest and H_{c1} is the lower critical field.[3,4,22] Then the ratio $\mu_0/\mu=H_{c1}/H_{c2}$ is very small for high-T_c materials.

Finally, we make a brief comparison to Suhl's work.[20] Our inertial mass is most closely related to his electromagnetic mass $\mu_{\rm em}=\xi^2 H_c^2 \lambda^2/4c^2\lambda_d^2$, where λ_d is a shielding length for the electrostatic potential and H_c is the bulk thermodynamic critical field. Replacing λ_d by the coherence length ξ (Ref. 20) and using the relation $4\pi\lambda^2 H_c^2=\phi_0 H_{c2}$ (Ref. 3, p. 129), we have $\mu_{\rm em}=\mu/4$. Using the limiting form of the viscosity[1] as $\beta\to 1$, $\eta=\phi_0 H_{c2}\sigma_n/c^2$, where σ_n is the normal-state conductivity, we may estimate the vortex relation time $\tau=\mu/\eta$. We find that $\tau=1/\pi\sigma_n$, which is to be compared with Suhl's $\mu_{\rm em}/\eta=\lambda^2/3\pi\kappa^2\lambda_d^2\sigma_n$. Therefore, these simple estimates show that our vortex mass per unit length and relaxation time are on the order of Suhl's results when the shielding distance λ_d is taken to be the coherence length ξ.

We recall that in the above we have taken the vortex to be aligned along the z axis. Corresponding results for a vortex along the x or y axis may be obtained by cyclic permutation $(x\to y\to z\to x)$. Again, we point out that the results presented here are expected to be valid only for temperatures close to the transition temperature, for low magnetic field $(H \ll H_{c2})$, and in the high-κ limit. The result (18) for the inertial mass has recently been extended to lower temperatures for a discrete 3D superconductor in the case that the vortex is oriented parallel to the superconducting layers of a Josephson-coupled layer model.[23] In this work[23] the authors account for dimensional crossover in the vortex structure and discuss the importance of the inertial mass in the dynamics of high-temperature superconductors.

Ames Laboratory is operated for the U.S. Department of Energy by Iowa State University under Contract No. W-7405-Eng-82. This work was supported by the Director for Energy Research, Office of Basic Energy Sciences. We thank J. R. Clem and V. G. Kogan for helpful discussions.

[1]Z. Hao and J. R. Clem, IEEE Trans. Magn. **MAG-27**, 1086 (1991).
[2]J. Bardeen and M. J. Stephen, Phys. Rev. A **140**, 1197 (1965).
[3]M. Tinkham, *Introduction to Superconductivity* (McGraw-Hill, New York, 1980).
[4]R. P. Heubener, *Magnetic Flux Structures in Superconductors* (Springer-Verlag, New York, 1979).
[5]Y. B. Kim, C. F. Hempstead, and A. R. Strnad, Phys. Rev. A **139**, 1163 (1965).
[6]P. Nozières and W. F. Vinen, Philos. Mag. **14**, 667 (1966).
[7]A. G. Van Vijfeijken and A. K. Niessen, Phys. Lett. **16**, 23 (1965); Philips Res. Rep. **20**, 505 (1965).
[8]W. Panofsky and M. Phillips, *Classical Electricity and Magnetism* (Addison-Wesley, Reading, MA, 1962).
[9]J. R. Reitz and F. J. Milford, *Foundations of Electromagnetic Theory* (Addison-Wesley, Reading, MA, 1967).
[10]J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1975).
[11]P. Lorrain and D. Corson, *Electromagnetic Fields and Waves* (Freeman, San Francisco, 1970).
[12]T. V. Davies and E. M. James, *Nonlinear Differential Equations* (Addison-Wesley, Reading, MA, 1966).
[13]H. T. Davis, *Introduction to Nonlinear Differential and Integral Equations* (U.S. Atomic Energy Commission, Washington, D.C., 1960).
[14]C. M. Bender and S. A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers* (McGraw-Hill, New York, 1978).
[15]W. F. Ames, *Nonlinear Ordinary Differential Equations in Transport Processes* (Academic, New York, 1968).
[16]G. W. Bluman and J. D. Cole, *Similarity Methods for Differential Equations* (Springer, New York, 1974).
[17]A. Cohen, *An Introduction to the Lie Theory of One-Parameter Groups with Applications to the Solution of Differential Equations* (Heath, Boston, 1911).
[18]L. V. Ovsiannikov, *Group Analysis of Differential Equations* (Academic, New York, 1982).
[19]J. M. Page, *Ordinary Differential Equations with an Introduction to Lie's Theory of Group of One Parameter* (Macmillan, New York, 1897).
[20]H. Suhl, Phys. Rev. Lett. **14**, 226 (1965).
[21]P. Lebwohl and M. J. Stephen, Phys. Rev. **163**, 376 (1967).
[22]A. L. Fetter and P. C. Hohenberg, in *Superconductivity,* edited by R. D. Parks, (Dekker, New York, 1969), p. 817.
[23]M. W. Coffey and J. R. Clem, Phys. Rev. B (to be published).