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Multiscaling in diffusion-limited aggregation

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It is shown using large-scale simulations that the density profile in diffusion-limited aggregation in two dimensions satisfies a scaling form (multiscaling) of the type $g(r,R)=r^{-d+D(r/R)}C(r/R)$, where r is the distance from the origin, R is the radius of gyration, and $C(x)$ is an amplitude. Contrary to standard scaling, which predicts $D(x)$ =const, here $D(x)$ is found to depend continuously on x.

Growth phenomena are a subject of very active current research, due to their widespread occurrence in a variety of different physical systems.[1–3] In particular, considerable attention has been focused on the diffusion-limited-aggregation[4] (DLA) model, which, although simple, appears to exhibit the essential features of many growth phenomena such as electrodeposition,[5] dielectric breakdown,[6] viscous fingers,[7,8] crystal growth in aqueous solution,[9] and neuron growth.[10]

One of the features, well established in DLA, is multifractality.[11–13] This consists of associating a measure to each perimeter site of the aggregate, usually given by the harmonic measure or growth probability p, and in decomposing the total aggregate of radius R in infinitely many fractal sets each characterized by a singularity $\alpha = -\log_{10} p/\log_{10} R$ with fractal dimension usually referred to as $f(\alpha)$.[12] This continuity of fractal dimensions characterizes the class of universality to which the growth model belongs. To calculate or measure this continuity of fractal dimensions, the scaling behavior of the growth probability distribution has to be considered. This quantity is usually difficult to evaluate by means of analytical and numerical calculations or by direct experimental observations.

A different quantity, which is more accessible to calculations and direct experimental observations, is the density profile $g(r,R)$, which is defined as

$$g(r,R)d^dr = dN, \tag{1}$$

where dN is the number of particles in the infinitesimal d-dimensional volume d^dr at distance r from the origin, and the dependence on the total number of particles N is expressed via the radius of gyration $R=R(N)$.

In the theory of DLA it is usually assumed, by analogy with critical phenomena, that $g(r,R)$ satisfies the standard scaling form

$$g(r,R) = r^{-(d-D)}C(r/R), \tag{2}$$

where $C(x)$ is a scaling function and D is the fractal dimension[14] of the aggregate. However, very recently it has been suggested[15] that the scaling structure of DLA clusters is much richer than that allowed by standard scaling and that (2) ought to be replaced by the multiscaling form

$$g(r,R) = r^{-d+D(r/R)}C(r/R). \tag{3}$$

Apparently the two forms may seem to coincide in the limit $R\sim\infty$. This is true if r is finite; however, if both r and R diverge with their ratio $x=r/R$ fixed, the two forms differ considerably. In fact, the scaling form (2) gives $g(r,R)\sim R^{-(d-D)}$, while the multiscaling form gives $g(r,R)\sim R^{-[d-D(x)]}$. Namely, the density profile (3) exhibits a continuity of power-law decay exponents, one for each value of x; consequently there is a different local fractal dimension $D(x)$ for each shell corresponding to a given value of x. The fractal dimension of the infinite aggregate is obtained well inside the frozen region for r finite and $R\to\infty$, namely, for $x=0$, yielding $D(0)=D$.

The scaling ansatz (3) follows from general scaling invariance under the requirement that the transformations form a group,[15] and standard scaling is contained in (3) as a particular case for $D(x)$ =const. It is not possible to use simple scaling arguments to predict the form of $D(x)$ or even to determine if it is a constant or not. This can only be obtained from explicit calculations. In

Ref. 15 it was shown that the data for two-dimensional DLA clusters of Plischke and Racz[16] are consistent with a multiscaling form of the growth probability $P(r,N)$ defined as the probability that the Nth particle to be added to the cluster sticks at a distance r from the origin. Furthermore, the exact solution for a model of spinodal decomposition was found to exhibit a multiscaling form in the time-dependent pair-correlation function,[17] indicating that multiscaling may be a general feature of a large variety of systems.

We emphasize that multiscaling is a distinct phenomenon from multifractality.[18] In the case of multifractality the existence of infinitely many exponents is the consequence of the introduction of an appropriate measure on the fractal. By contrast, by multiscaling we refer to the existence of a continuity of power-law exponents in the pair-correlation function. We stress that multiscaling is more accessible to direct experimental observation. For example, in those cases[5–10] that are believed to be in the same universality class as DLA, an analysis of the density profile should be easier to perform than the growth probability distribution. Consequently the multiscaling spectrum should be easier to detect than the multifractal spectrum. Furthermore, as the result[17] for spinodal decomposition demonstrates, multiscaling is by far more tractable by analytical methods than multifractality. Finally, while numerical calculations to measure multifractality are difficult to perform on large systems, there are no limitations due to the system size in the measurement of multiscaling. In this paper we describe an explicit calculation of $D(x)$ using large two-dimensional DLA clusters. The main result is shown in Fig. 2 and supports the multiscaling picture.

We performed two sets of calculations A and B. In A we grew 2 000 DLA clusters on the square lattice of $N = 10\,000$ particles. In B we grew 200 off-lattice DLA clusters of $N = 100\,000$ particles. The results are not significantly different in the two sets. For each cluster the growth was divided into 20 stages, corresponding to 20 different values of the the radius of gyration R . At each stage of the growth corresponding to a radius R, the cluster was divided into 20 shells of radius $r_n = (n-0.5)R/10$ with integer $n > 2$. Instead of calculating the "mass" in a shell between two consecutive radii (to avoid lattice effects, which are present mainly in the small clusters of the set A), we calculated the mass $M(r_n, R)$ in the shell between r_{n-1} and r_{n+1} corresponding to an average radius r_n and to a value $x_n = (n-0.5)/10$. The relation of the mass $M(r,R)$, where the index n has been suppressed, with the density profile is given by $M(r,R) = 2\pi r g(r,R)\Delta r$, where $\Delta r = R/5$ is the width of the shell. From Eq. (3) we have

$$M(r,R) = A(x) r^{D(x)}, \quad (4)$$

where $A(x) = 2\pi C(x)/5x$. For a fixed value of x the slope of $\log_{10} M$ as function of $\log_{10} r$ gives $D(x)$ (Fig. 1). In Fig. 2 we plot the values of $D(x)$ obtained from the first set of measurements. The values obtained from the second set were not substantially different. For completeness we give the values of $D(x)$ obtained in the two sets.

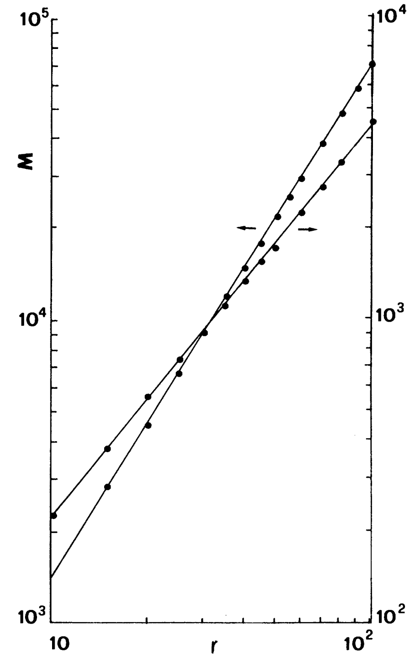

FIG. 1. log-log plot of the mass $M(r,R)$ as function of r for $x = 1.5$ and $x = 1.9$. The slopes give the fractal dimensions.

To check our result and to find the amplitude $A(x)$ we have analyzed the data using the data collapse method. Using the calculated values of $D(x)$ we evaluate $r^{-D(x)}M(r,R) = A(x)$ for different values of r and R. The data collapse for the set of data A is shown in Fig. 3(a). Figure 3(b) shows the results obtained from an attempted data collapse using the standard scaling form (2) with a constant $D(x)$ [$D(x) = 1.701$]. The data collapse is not as good as that obtained using multiscaling.

X	D(X)(A)	D(X)(B)
15	1.67	1.68
25	1.68	1.69
35	1.67	1.69
45	1.68	1.69
55	1.69	1.70
65	1.70	1.70
75	1.70	1.71
85	1.72	1.71
95	1.74	1.72
1.05	1.75	1.73
1.15	1.77	1.75
1.25	1.76	1.76
1.35	1.75	1.76
1.45	1.74	1.74
1.55	1.70	1.70
1.65	1.61	1.64
1.75	1.50	1.53
1.85	1.31	1.32
1.95	1.05	

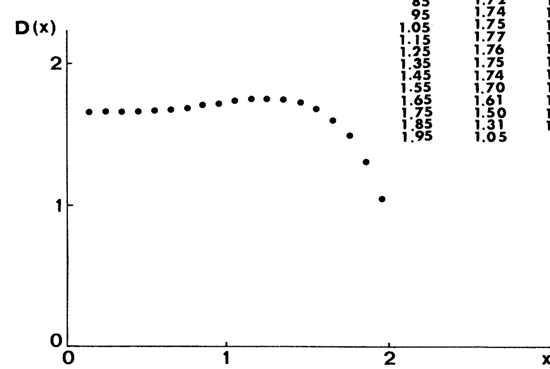

FIG. 2. Plot of $D(x)$ for the set of configurations A. The numerical values of $D(x)$ for the two sets A and B are also given. The uncertainty in $D(x)$ has been estimated ranging from 0.01 for small values of x to 0.02 for large values of x.

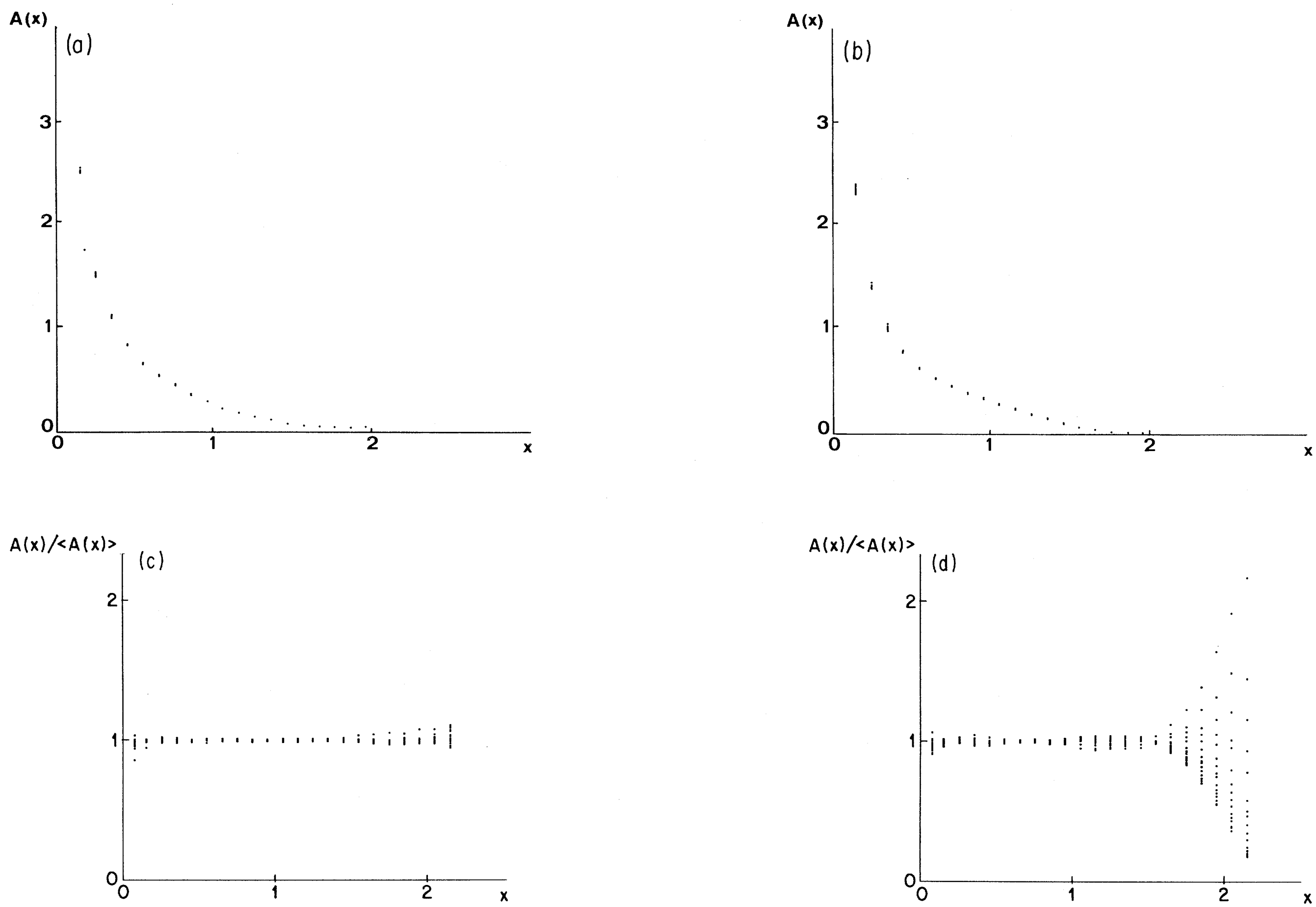

FIG. 3. Data collapse Eq. (4) (a) using multiscaling, (b) using standard scaling, (c) using deviation from the average data collapse $A(x)/<A>$ for multiscaling, and (d) for standard scaling.

To better appreciate the difference in quality of the two data collapses, we have evaluated the following quantity: $A(x)/<A>$, where $<A>$ is the average over R for a fixed x, using the multiscaling form [Fig. 3(c)] and the scaling form [Fig. 3(d)]. By comparing the two figures we see that multiscaling has much less spread and is thus a better data collapse.

Discussion. The form of $D(x)$ shows a slow increase as function of x before it decreases. A simple argument shows, however, that for DLA $D(x)$ must be a nonincreasing function of x. Consider the density $g(r,R_1)$ of a DLA aggregate of radius R_1 at a distance r from the origin. Now let the same aggregate grow to a larger radius $R_2 > R_1$ and consider the density $g(r,R_2)$ at the same distance r from the origin. Due to the irreversibility of the growth process, we must have $g(r,R_1) \leq g(r,R_2)$. From Eq. (3) it follows that

$$r^{D(x_2)-D(x_1)}C(x_2)/C(x_1) \geq 1, \qquad (5)$$

with $x_1 > x_2$, which for large values of r implies $D(x_2) \geq D(x_1)$. Thus $D(x)$ must be a nonincreasing function of x. The numerical finding of a small bump in $D(x)$ might be due to two reasons. One is a numerical inaccuracy, the other is that the asymptotic regime has not been reached yet. The first possibility can be excluded, since the error bars are smaller than the difference $D_{\max} - D(0)$, where $D_{\max}$ is the maximum value of $D(x)$. Moreover, our numerical data are consistent with (5), considering that the numerical values of $D(x)$ and $C(x)$ have been obtained from clusters of the order of 10^5 particles, while in order to see a violation of (5) with those numerical values of $D(x)$ and $C(x)$ the number of particles should be greater than 10^{17}. Therefore the bump must *necessarily* disappear for such large clusters. This is also supported by the value of $D_{\max} - D(0)$ that shows a slight tendency to decrease from the value of ~ 0.10 for the 10 000 particle clusters to the value of ~ 0.08 for the 100 000 particle clusters. Hence, asymptotically we expect $D(x)$ to be roughly constant over a region up to some value $x_0 \sim 1.5$ and then to decrease for larger values of x. The first behavior corresponds to the interior part of DLA where growth almost barely occurs, and it is characterized by the fractal dimension of the infinite aggregate, while the second behavior corresponds to the growing regions, which are characterized by infinitely many fractal dimensions.

In conclusion we have performed large-scale simulations of DLA clusters and found that the density profile obeys a multiscaling form given by Eq. (3).

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