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Transmission formalism for supercurrent flow in multiprobe superconductor-semiconductor-superconductor devices

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A theoretical study is given of supercurrent flow in a one-dimensional semiconductor channel coupled to superconductors at both ends. In addition, the channel is coupled to a semiconductor reservoir by means of a junction with variable coupling strength ε. The supercurrent $I(\phi)$ is calculated from the phase-coherent propagation of electronlike and holelike excitations emitted by the superconductor reservoirs, together with electron and hole excitations from the semiconductor reservoir. The effect of temperature and ε on $I(\phi)$ is studied. It is shown that a voltage applied between the semiconductor reservoir and the superconductors modifies the $I(\phi)$ relation, even in the limit $\varepsilon \rightarrow 0$.

In recent years our understanding of electron transport in mesoscopic conductors has greatly improved. It has become clear that at low temperatures electrons can maintain their phase coherence over considerable distances. Recently it has also become possible to study devices through which the electron can travel ballistically, without being scattered by impurities.

A challenge is now presented by the possibility to combine this ballistic, phase-coherent transport with superconductivity. A promising system is a high-mobility two-dimensional electron gas (2DEG), in which a narrow channel is defined (e.g., by means of a quantum point contact[1]), and connected to one or more superconducting electrodes. Very recently it was predicted[2] that the supercurrent in such a device, measured as a function of the width of the channel, should exhibit steps each time an additional one-dimensional channel is opened.

A large volume of literature exists on the theory of supercurrent flow through superconductor–normal-metal–superconductor (S-N-S) or superconductor-semiconductor-superconductor (S-SM-S) systems. A selection is given in Refs. 2–8. In most of these papers the supercurrent is calculated using Green's-function methods.[9] In this paper we will describe the supercurrent flow through a S-SM-S device with the use of a transmission formalism. This formalism has already been applied for the description of electron transport in normal metals and semiconductors.[10,11] It has also been used for the description of transport through superconductor–normal-metal interfaces.[12–14]

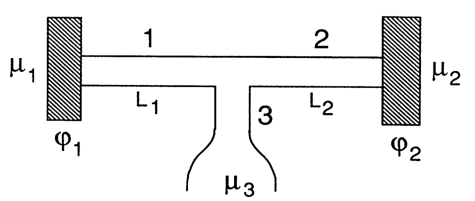

FIG. 1. Layout of the system, which consists of a 1D channel formed by leads 1 and 2, coupled to superconductors 1 and 2. Lead 3 couples the channel to a semiconductor reservoir 3.

The device geometry is illustrated in Fig. 1. Leads 1 and 2 (with lengths L_1 and L_2) form a one-dimensional (1D) channel,[15] which is connected at both ends to superconductors 1 and 2 (sc1 and sc2), with electrochemical potentials μ_1 and μ_2. By means of lead 3, the 1D channel can also be coupled to a wide 2D region, which can be described as a semiconductor reservoir with electrochemical potential μ_3. The transmission and reflection of the electron waves at the junction formed by leads 1, 2, and 3 can be described by an $\underline{\mathbf{S}}$ matrix,[16,17] which relates the amplitudes of the outgoing waves to the amplitudes of the incoming waves at the junction

$$\underline{\mathbf{S}} = \begin{pmatrix} -(a+b) & \varepsilon^{1/2} & \varepsilon^{1/2} \\ \varepsilon^{1/2} & a & b \\ \varepsilon^{1/2} & b & a \end{pmatrix} \tag{1}$$

with $a = 1/2[\sqrt{(1-2\varepsilon)} - 1]$, and $b = 1/2[\sqrt{(1-2\varepsilon)} + 1]$. The parameter ε describes the strength of the coupling between the 1D channel and the semiconductor reservoir. Maximum coupling is achieved for $\varepsilon = \frac{1}{2}$, the channel is completely decoupled when $\varepsilon = 0$.

We use a well-known model[2–6] where the pair potential $\Delta(x) = 0$ in the semiconductor channel, and $\Delta(x) = \Delta_0 \times \exp(i\phi_{1,2})$ in sc1 and sc2, with $\phi_{1,2}$ the superconductor phases. The wave functions can be found from the time-independent Bogoliubov–de Gennes equation[18]

$$E \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} = \begin{pmatrix} H - E_F & \Delta \\ \Delta^* & -(H^* - E_F) \end{pmatrix} \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} . \tag{2}$$

In Eq. (2) $u(x)$ describes the electron wave function, and $v(x)$ describes the hole wave function with an excitation energy E relative to the Fermi energy E_F. There are four types of solutions. In the leads 1, 2, and 3,

$$\begin{pmatrix} u(x) \\ v(x) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \exp(\pm iq^+ x) ,$$

$$\text{with } \hbar q^+ = \sqrt{2m}\sqrt{E_F + E} \tag{3a}$$

with $+(-)$ corresponding to electron excitations which move in the positive(negative) x direction,[19] and

$$\begin{pmatrix} u(x) \\ v(x) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \exp(\pm iq^{-}x), \quad \text{with } \hbar q^{-} = \sqrt{2m}\sqrt{E_F - E} \tag{3b}$$

with $+(-)$ corresponding to hole excitations which move in negative(positive) x direction. In the superconductors the four solutions are[20]

$$\begin{pmatrix} u(x) \\ v(x) \end{pmatrix} = \begin{pmatrix} u_0 \exp(i\phi) \\ v_0 \end{pmatrix} \exp(\pm ik^{+}x), \quad \text{with } \hbar k^{+} = \sqrt{2m}\,[E_F + (E^2 - \Delta_0^2)^{1/2}]^{1/2} \tag{4a}$$

with $+(-)$ corresponding to electronlike excitations which travel in positive(negative) x direction, and

$$\begin{pmatrix} u(x) \\ v(x) \end{pmatrix} = \begin{pmatrix} v_0 exp(i\phi) \\ u_0 \end{pmatrix} \exp(\pm ik^{-}x), \quad \text{with } \hbar k^{-} = \sqrt{2m}\,[E_F - (E^2 - \Delta_0^2)^{1/2}]^{1/2} \tag{4b}$$

with $+(-)$ corresponding to holelike excitations which travel in negative(positive) x direction. The coherence factors are given by

$$u_0 = 1/\sqrt{2}\{1 + [1 - (\Delta_0/E)^2]^{1/2}\}^{1/2},$$

$$v_0 = 1/\sqrt{2}\{1 - [1 - (\Delta_0/E)^2]^{1/2}\}^{1/2}.$$

The formalism we use is based on Büttiker's description of phase-coherent electron transport in multiprobe normal conductors.[11] The physical basis of this description is that at sufficiently low temperatures inelastic processes are negligible in leads 1, 2, and 3, and occur exclusively in the reservoirs. In normal conductors these reservoirs can be thought of as emitters (and absorbers) of electron waves. We extend this description by treating the superconductors as reservoirs also. A semiconductor reservoir can emit (and absorb) electron and hole excitations with energies $E > 0$. However, because no propagating solutions of Eq. (2) exist in the superconductor for $E < \Delta_0$, the superconductors can emit (and absorb) electronlike and holelike excitations for $E > \Delta_0$ only.

An important feature of 1D transport is that the product of one-dimensional density of states $N(E) = 2/\pi(dE/dk)^{-1}$ and the group velocity $v_g(E) = 1/\hbar\,(dE/dk)$ is independent of E and equal to $4/h$ (including both spin directions). This result holds for electron and hole states in the semiconductor, but also for the electronlike and holelike states in the superconductors.[13] It follows that the *particle* current[21] which is emitted by a reservoir per unit energy is equal to $2/h$.

Our description can also be used to describe normal currents. Normal currents can be defined as currents which flow as a result of a difference between μ_1, μ_2, and μ_3. In this paper we focus on the supercurrent which flows between sc1 and sc2, and its dependence on the phase difference $\phi = \phi_1 - \phi_2$. We define the reflection probabilities $R_{nn}(\phi,E)$ as the *charge* currents (in units of e) which flow back into the nth reservoir as a result of the *particle* current which is emitted by the nth reservoir at energy E. Similarly the transmission probabilities $T_{nm}(\phi,E)$ are defined as the ratio of the charge currents which flow into the nth reservoir as a result of the particle currents emitted by the mth reservoir at energy E. Because the reservoirs can emit two types of particles, we add a superscript p, with $p = 1$ for electron (or electronlike) excitations, and $p = 2$ for hole (or holelike) excitations. We now define a charge current density (current per unit energy) $J_n^{\text{ex}}(\phi,E)$ in lead n by

$$J_n^{\text{ex}}(\phi,E) = \frac{2e}{h} \sum_{p=1,2} C_n^p(E) - R_{nn}^p(\phi,E) - \sum_{m \neq n} T_{nm}^p(\phi,E). \tag{5}$$

In Eq. (5) $C_n^p(E)$ indicates the ratio between the charge current (in units of e) and the particle current carried by the excitations which are emitted by the nth reservoir. $C_n^p(E) = -1$ for electrons, 1 for holes, $(v_0^2 - u_0^2)^{-1}$ for electronlike, and $(u_0^2 - v_0^2)^{-1}$ for holelike excitations. $J_n^{\text{ex}}(\phi,E)$ can be considered as the contribution to the supercurrent in lead n from excitations with energy E, under the condition that the occupation probability of the excited states in the reservoirs is *unity*.

We calculate the supercurrent $I_n(\phi,T)$ in lead n $(n = 1,2)$, with $\mu_1 = \mu_2 = \mu_3$. At $T = 0$ no excitations are present, and this seems to prevent the calculation of $I_n(\phi,T=0)$. We can now perform a simple manipulation,[22] and write the supercurrent at a finite T as

$$I_n(\phi,T) = I_n(\phi,T=0) + I_n^{\text{ex}}(\phi,T),$$

with $I_n^{\text{ex}}(\phi,T)$ the supercurrent generated by the excitations at temperature T. Since the occupation of the excited states is given by $f(E,T) = 1/[1 + \exp(E/kT)]$, and $I_n(\phi,T)$ should vanish for $T \rightarrow \infty$, we can write:[23]

$$I_n(\phi,T) = \int_0^\infty [f(E,T) - \tfrac{1}{2}]J_n^{\text{ex}}(\phi,E)dE. \tag{6}$$

Equations (5) and (6) show that the supercurrent is expressed as an incoherent sum of the contributions of excitations at different energies, and from different reservoirs.

We have calculated the energy and phase dependence of $R_{nn}^p(\phi,E)$ and $T_{nm}^p(\phi,E)$ with an algorithm which matches the wave functions (3) and (4) at the superconductor-semiconductor interfaces, taking into account the phase shifts in leads 1 and 2, and matches the wave functions at the junction according to $\underline{\mathbf{S}}$ matrix given in (2).

The fundamental mechanism which makes supercurrent flow possible is that of *coherent Andreev reflection.*[24] An electron in the 1D channel with $E < \Delta_0$ is reflected at the interface with sc2 as a hole, and its wave function can be shown to acquire a phase factor $\exp[i(-\phi_2 - \pi/2)]$. The reflected hole is converted back to an electron at the interface with sc1, and acquires an additional phase factor $\exp[i(\phi_1 - \pi/2)]$. This implies that when $\varepsilon = 0$, bound states are formed in the semiconductor channel.[3–6] The energies of these states (positive states) depend on $\phi = \phi_1 - \phi_2$, and can be written as $E_n^+ = \hbar v_F/2L^*[2\pi(n + \tfrac{1}{2}) + \phi]$, with $L^* = L_1 + L_2 + \hbar v_F/\Delta_0$, the effective length between the superconductors.[25] Similarly, the energy of the negative states, which correspond to electrons traveling in negative x direction (and holes traveling in positive x direction), can be written as

$$E_n^- = \hbar v_F/2L^*[2\pi(n+\tfrac{1}{2})-\phi].$$

For the calculation we have taken[26] $E_F=10\Delta$, $k_FL_1=77.9$, and $k_FL_2=94.9$. For a typical 2DEG in a $GaAs/Al_xGa_{1-x}As$ heterostructure with effective mass $m^*=0.067m_e$, and $E_F=10$ meV, this corresponds to $\Delta=1$ meV, and $L_1+L_2\approx 1.5$ μm. Figure 2 shows the calculated $J_1^{ex}(\phi,E)$ in lead 1 for $\varepsilon=0.2$. For $\phi\neq 0$ and $\phi\neq\pi$, $J_1^{ex}(\phi,E)$ for $E<\Delta_0$ consists of a series of positive and negative peaks, which correspond to the positive and negative states discussed above. For $\phi=0$ the positive and negative states are all at the same energy, and $J_1^{ex}(\phi,E)=0$. When ϕ is increased, the positive and negative states shift away from each other, until at $\phi=\pi$, the states again coincide, resulting in $J_1^{ex}(\phi,E)=0$.

For $E<\Delta_0$, only excitations from the semiconductor reservoir contribute to $J_1^{ex}(\phi,E)$. Due to the coupling to this reservoir, the positive and negative states are broadened in energy. When the coupling strength ε is reduced, the peaks develop into δ functions. It is interesting to note that when $\varepsilon\to 0$, a finite supercurrent can be carried in an infinitesimally narrow energy interval. This should be contrasted with normal current flow between reservoirs, where the current per unit energy is limited to $2e/h$. The major contribution in the regime $E>\Delta_0$ is from the excitations from the superconductors.[27] For $E>\Delta_0$ there is only partial (Andreev) reflection at the S-SM interfaces. Consequently the positive and negative states are broadened substantially for $E>\Delta_0$ (even when $\varepsilon=0$). However, as can be seen in Fig. 2 these states can still give a (positive or negative) contribution to $J_1^{ex}(\phi,E)$.

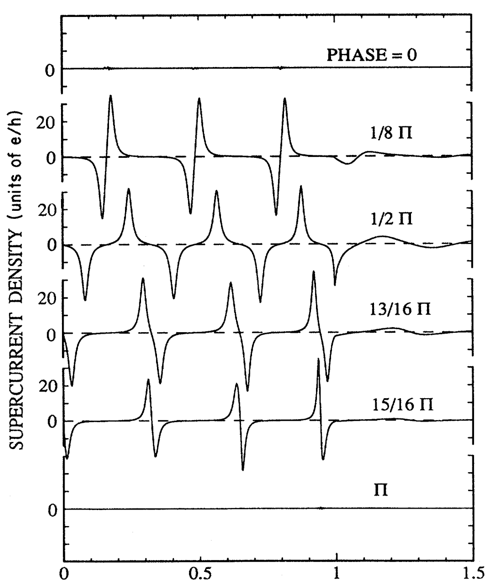

FIG. 2. Supercurrent density $J_1^{ex}(\phi,E)$ in lead 1 due to excitations with E, for a set of phase differences ϕ.

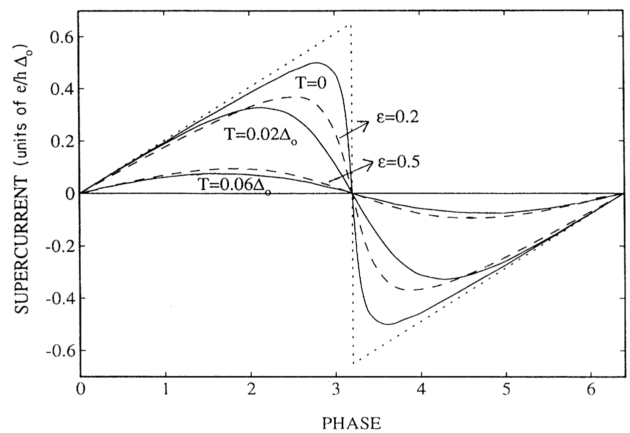

FIG. 3. Supercurrent-phase relation $I_1(\phi,T)$. Solid lines: $I_1(\phi,T)$ calculated for several temperatures at fixed $\varepsilon=0.05$. Dashed lines: $I_1(\phi,T=0)$, calculated for several values of ε. The dotted line shows the $I_1(\phi,T=0)$ relation which can be obtained from elementary considerations (see text).

The supercurrent-phase relation $I_1(\phi,T)$, calculated from Eq. (6) is shown in Fig. 3, for several values of ε and T. For small values of ε, $I_1(\phi,T=0)$ approaches a sawtooth shape (which was also obtained in Refs. 3-5). This supercurrent-phase relation can also be derived in a simple way: The jump in $I_1(\phi,T)$ at $\phi=\pi$ can be explained with Fig. 2, which shows that a negative state shifts below $E=0$, and stops contributing to $I_1(\phi,T)$, whereas a positive state shifts above $E=0$, and starts contributing to $I_1(\phi,T)$. This results in a jump[28] $\Delta I=2ev_F/L^*$.

An increase in ε reduces $I_1(\phi,T)$, and the jump at $\phi=\pi$ is smoothed, until an almost sinusoidal relation is obtained at the maximum coupling $\varepsilon=\frac{1}{2}$. Figure 3 shows that raising T has similar effects. The typical T at which $I_1(\phi,T)$ is suppressed substantially is of the order of the energy spacing between the positive and negative states. With our parameters, this corresponds to about kT

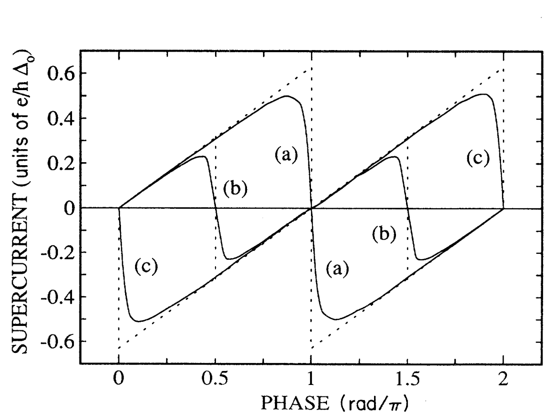

FIG. 4. Supercurrent-phase relation ($\varepsilon=0.05$, $T=0$) for several values of applied voltage V between semiconductor reservoir and superconductors. The dotted lines show the extrapolated results for $\varepsilon\to 0$. (a) $|eV|=n\Delta E$. (b) $|eV|=(n+\frac{1}{4})\Delta E$. (c) $|eV|=(n+\frac{1}{2})\Delta E$.

$\approx 0.03\Delta \approx 0.1$ K.

We now show that $I_1(\phi, T=0)$ can be modified by applying a voltage V between the semiconductor reservoir and the superconductors ($eV = \mu_3 - \mu_1$, $\mu_1 = \mu_2$). As a result electron excitations (for $V<0$) or hole excitations (for $V>0$) are emitted with energies $0 < E < |eV|$. The supercurrent can now be written as[29]

$$I_1(\phi, eV) = I_1(\phi, T=0) + \frac{1}{2}\int_0^{|eV|} J_1^{\mathrm{ex}}(\phi, E)\,dE\,. \quad (7)$$

The results are shown in Fig. 4. The supercurrent-phase relation depends periodically on the applied voltage with the period given by $\Delta E/e$, with $\Delta E = hv_F/2L^*$, the spacing between consecutive positive (or negative) states. The explanation for this effect is that for energies $E < \Delta_0$, the particles from the semiconductor channel are completely reflected at the S-SM interfaces. Therefore the (nonequilibrium) occupation of the states in the channel with $E < \Delta_0$ is completely determined by μ_3, even in the limit $\varepsilon \to 0$. In real systems some inelastic scattering will always be present between the channel and the superconductors. This will tend to equilibrate the states to the electrochemical potential of the superconductors. In real systems the effect will therefore disappear in the limit $\varepsilon \to 0$. The value of ε at which the effect disappears will be a measure of the strength of the inelastic scattering.

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[15] These channels can be defined in a 2DEG by laterally confining the electrons in a wire with a width comparable to the Fermi wavelength (≈ 40 nm).

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[18] P. G. de Gennes, *Superconductivity of Metals and Alloys* (Benjamin, New York, 1966).

[19] The direction of motion of the excitations is determined by their group velocity $v_g = 1/\hbar(dE/dk)$.

[20] The Fermi wavelength λ_F in actual superconductors is typically much smaller than λ_F in the semiconductor. This implies that the 1D semiconductor channel should be matched to a large number of 1D channels in the superconductors. However, we do not expect that this will substantially change the results which are obtained with our one-dimensional model.

[21] The difference between particle (or probability) current and charge current is due to the fact that the electron part $|u(x)|^2$ and hole part $|v(x)|^2$ of the wave function contribute to the particle currents with opposite signs, but to the charge currents with equal signs.

[22] A similar method was used for the calculation of critical currents in Refs. 4 and 25.

[23] In our system the supercurrent is already suppressed for $kT \ll \Delta_0$. We can therefore assume that Δ_0 is independent of T.

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[26] L_1 and L_2 have been chosen slightly different to avoid possible effects related to the symmetry of the system.

[27] Due to the partial reflection at the S-SM interfaces, there is a small contribution to $J_1^{\mathrm{ex}}(\phi, E)$ due to excitations from the semiconductor reservoir with energies slightly larger than Δ_0. This contribution vanishes when $\varepsilon \to 0$.

[28] With $V = \hbar/2e(d\phi/dt)$ one can write the supercurrent due a fully occupied positive or negative state (which contains two electrons) as $I = 4e/\hbar(dE/d\phi) = 2ev_F/L^*$. Equation (6) then yields a jump $\Delta I = 2ev_F/L^*$.

[29] When $V \neq 0$, a normal current flows between the semiconductor reservoir and the superconductors. This current vanishes for $\varepsilon \to 0$.