Quasiparticle inelastic lifetimes in disordered superconducting films

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We present a comprehensive theory of quasiparticle inelastic lifetimes close to equilibrium in disordered superconducting films. The lifetime due to Coulomb interactions in two-dimensional (2D) films is calculated in analogy to a recent 3D calculation. For the phonon contribution, we extend the results of Reizer and Sergeyev for clean 3D systems to 2D, and determine the leading disorder renormalization. We then construct an interpolation formula for films of finite thickness that correctly reproduces the 2D and 3D limits. We give a general discussion of our results, and compare them with experiments. Agreement with measurements on thick films is good, while in the 2D limit nonequilibrium effects are likely to play an important role.

I. INTRODUCTION

The inelastic lifetime τ_{in} of electronic excitations in disordered metallic systems has received much attention in recent years. Schmid[1] was the first to realize that in a three-dimensional (3D) disordered normal metal, diffusive electron dynamics leads to an anomalous enhancement of the Coulomb contribution to the one-particle scattering rate $\Gamma_{in}=1/2\tau_{in}$. This enhancement of Γ_{in} (i.e., the imaginary part of the one-particle self-energy Σ) has the same mathematical and physical origin as the "correlation gap" in the density of states (i.e., the real part of Σ) discovered later.[2] In 2D metals the corresponding effects are so strong that they lead to divergences in a perturbative treatment.[3] For Γ_{in} this divergency has been remedied by means of a self-consistent treatment.[4] If considered together with the phonon contribution to Γ_{in}, the result is in good agreement[5] with experiment.

In a superconductor the situation is more complex because of the appearance of the gap. The self-energy Σ is customarily separated into the normal self-energy or renormalization function Z and the anomalous self-energy W. A second normal self-energy piece is constant in a clean superconductor and can be neglected.[6] This second piece is actually the one which, in a disordered normal metal, produces the correlation gap and enhancement of Γ_{in}. In a disordered superconductor, it has been shown[7,8] that it is crucial to retain this piece, which was denoted by Y in Ref. 7. We will call Y the correlation-gap piece of the self-energy. Its real part leads to a strong disorder dependence of the transition temperature, and an analysis of this effect showed that it accounts well for the experimentally observed degradation of T_c.[9]

The imaginary parts of the self-energy pieces in a superconductor determine the quasiparticle inelastic lifetime τ_{in}, which is analogous to the single-particle lifetime in a normal metal. The physical meaning of τ_{in} is that of the lifetime of a single quasiparticle injected into an energy eigenstate above the gap. It is the one-quasiparticle limit of the energy relaxation time in a nonequilibrium situation.[10] The presence of a finite τ_{in} smooths out the various singularities one encounters in BCS theory, e.g., the square-root singularity in the density of states (DOS). This allows for a direct observation of τ_{in} in a tunneling experiment. In a clean superconductor, the dominant contribution to Γ_{in} is due to thermal phonons. This has been calculated[11] and is well confirmed by experiment.[12] In contrast, contributions due to the Coulomb interaction[13] (i.e., virtual photons) as well as real photons[14] are negligible. This is no longer the case in disordered superconductors. Experiments[15–17] show a Γ_{in} which is strongly enhanced by disorder, and it has been suggested[15] that Schmid's mechanism of diffusion enhancement of the Coulomb contribution to Γ_{in} is effective here as well. A recent calculation[18] for 3D systems showed that this is indeed the case and that, for typical parameter values, one should expect Coulomb and phonon contributions to Γ_{in} to be equally important in a disordered superconductor.

It is the purpose of the present paper to present a comprehensive theory for Γ_{in} in disordered superconductors which treats all contributions which are known to be important on equal footing. In Refs. 7 and 9 it was shown that in addition to the disorder-generated self-energy contribution Y, there are many different disorder renormalizations of the conventional contributions Z and W. This previous work was restricted to calculating the real parts of the self-energies due to the absence of a theory for the dynamically screened Coulomb interaction in disordered superconductors. This situation has now changed, as Ref. 19 provided a gauge-invariant expression for the Coulomb propagator in the presence of disorder. We can thus calculate the respective imaginary parts as well. In this sense the present paper completes the program which was started in Ref. 7. Since many experiments are performed on films whose dimensionality is in between 2D and 3D,[20] we treat both the 2D and 3D cases and construct an interpolation formula between these two limits.

Our paper is organized as follows. In Sec. II we present the general formalism and introduce our notation. In Sec. III we calculate the Coulomb contribution to Γ_{in} for the 2D case. (For 3D this was done in Ref. 18.) In Sec. IV we study the phonon contributions. For clean superconductors we consider the 2D analogon to the 3D calculation by Reizer and Sergeyev.[21] We then proceed to calculate the disorder renormalization for both the 2D and 3D cases, including the contribution from the correlation-gap self-energy piece Y. In Sec. V we study the crossover between 2D and 3D behavior and give the final result for the leading contribution to Γ_{in} in a film of finite thickness. Finally, in Sec. VI we discuss our results and compare with experiments. A short report of this work has been published elsewhere.[22]

II. GENERAL FORMALISM

A. Self-energy

Our starting point is the strong-coupling theory for disordered superconductors developed in Ref. 7. The model and general methods we use are the same as in Ref. 7, and so we will be very brief. The only difference is that here we use the dynamically screened Coulomb interaction which was calculated in Ref. 19, while Ref. 7 used a static interaction. Apart from this generalization, we consider the various self-energy parts as derived there. Reference 7 then proceeded to calculate the real part of the self-energy in order to determine the transition temperature. Here we calculate the respective imaginary parts which determine Γ_{in}.

As usual, we split the self-energy into a normal and an anomalous piece, which we denote by S and W, respectively. Both the Coulomb interaction and the phonon-exchange contribute to S and W. For the Coulomb contributions, we obtain[7]

$$W_C(\epsilon,i\omega)=-T\sum_{i\omega'}\sum_{\mathbf{q}}\int d\epsilon'\, R_C^F(\mathbf{q},\epsilon-\epsilon')\mathcal{F}(\epsilon',i\omega')V_C(\mathbf{q},i\omega-i\omega') , \tag{2.1}$$

$$S_C^F(\epsilon,i\omega)=-T\sum_{i\omega'}\sum_{\mathbf{q}}\int d\epsilon'\, R_C^F(\mathbf{q},\epsilon-\epsilon')\mathcal{G}(\epsilon',i\omega')V_C(\mathbf{q},i\omega-i\omega') , \tag{2.2a}$$

and

$$S_C^H(\epsilon,i\omega)=S_C^H(\epsilon)=2T\sum_{i\omega'}\sum_{\mathbf{q}}\int d\epsilon'\, R_C^H(\mathbf{q},\epsilon-\epsilon')\mathcal{G}(\epsilon',i\omega')V_C(\mathbf{q},i0) . \tag{2.2b}$$

Here V_C denotes the dynamically screened Coulomb interaction, which will be given explicitly in Sec. III. $\mathcal{G}$ and $\mathcal{F}$ are the normal and anomalous Green's functions, respectively, and $i\omega$ and $i\omega'$ are fermionic Matsubara frequencies. The superscripts H and F denote direct and exchange contributions, respectively, in the self-consistent Hartree-Fock scheme of Ref. 7. Note that there is no direct contribution to W due to particle-number conservation. Note also that the (ensemble-averaged) self-energies are energy rather than wave-vector dependent, since we are dealing with disordered systems. The functions R_C^F and R_C^H can be written as integrals over the phase-space density-correlation function for noninteracting electrons. The disorder information for our problem is contained in these functions as well as in the interactions. The corresponding phonon contributions read[7]

$$W_{ph}(\epsilon,i\omega)=-T\sum_{i\omega'}\sum_{\mathbf{q},b}\int d\epsilon'\, R_b^F(\mathbf{q},\epsilon-\epsilon')\mathcal{F}(\epsilon',i\omega')D_b(\mathbf{q},i\omega-i\omega') , \tag{2.3}$$

$$S_{ph}^F(\epsilon,i\omega)=-T\sum_{i\omega'}\sum_{\mathbf{q},b}\int d\epsilon'\, R_b^F(\mathbf{q},\epsilon-\epsilon')\mathcal{G}(\epsilon',i\omega')D_b(\mathbf{q},i\omega-i\omega') , \tag{2.4a}$$

and

$$S_{ph}^H(\epsilon,i\omega)=S_{ph}^H(\epsilon)=2T\sum_{i\omega'}\sum_{\mathbf{q},b}\int d\epsilon'\, R_b^H(\mathbf{q},\epsilon-\epsilon')\mathcal{G}(\epsilon',i\omega')D_b(\mathbf{q},i0) . \tag{2.4b}$$

Here $D_b(\mathbf{q},i\Omega)$ is the phonon propagator for phonons with wave vector $\mathbf{q}$, (bosonic) Matsubara frequency $i\Omega$, and polarization b ($b=L,T$ for longitudinal and transverse phonons, respectively). The functions R_b^F and R_b^H are again certain correlation functions for noninteracting electrons. A representation of $R_{b,c}^{H,F}$ in terms of the phase-space Kubo function can be found in Ref. 7, and we will quote explicit results below where we need them.

We further follow Ref. 7 in decomposing the normal self-energy into parts which are odd and even, respectively, with respect to the frequency,

$$S(\epsilon,i\omega)=i\omega[1-Z(\epsilon,i\omega)]+Y(\epsilon,i\omega) , \tag{2.5}$$

where both Z and Y are even functions of $i\omega$. The Green's functions then can be written as

$$\mathcal{G}(\epsilon,i\omega)=\frac{i\omega Z(\epsilon,i\omega)+\epsilon+Y(\epsilon,i\omega)}{[i\omega Z(\epsilon,i\omega)]^2-[\epsilon+Y(\epsilon,i\omega)]^2-W(\epsilon,i\omega)^2} , \tag{2.6a}$$

$$\mathcal{F}(\epsilon,i\omega)=\frac{-W(\epsilon,i\omega)}{[i\omega Z(\epsilon,i\omega)]^2-[\epsilon+Y(\epsilon,i\omega)]^2-W(\epsilon,i\omega)^2} . \tag{2.6b}$$

Equation (2.1)–(2.6) form a closed set of equations which can be solved for the self-energy.

B. Quasiparticle decay rate

We now express the width $\Gamma\equiv\Gamma_{\rm in}$ of the resonance in the Green's functions in terms of the real and imaginary parts of the self-energy.[11] From here on we denote real parts by ${\rm Re}Z\equiv Z'$, etc., and imaginary parts by ${\rm Im}Z\equiv Z''$, etc. We find, from Eq. (2.6),

$$\Gamma(\omega)=\omega Z''(\omega)/Z'-W''(\omega)\Delta/\omega Z'-Y''(\omega)(\omega+Y')/\omega Z'^2 . \tag{2.7}$$

Here we have analytically continued to real frequencies and taken the on-shell values. We have also used $W'(\omega)/Z'(\omega)\approx\Delta$, with Δ the energy gap, and $Z'(\omega)\approx Z'(0)\equiv Z'$.

In Ref. 7, $Y(\epsilon,i\omega)$ was expanded in a Taylor series with respect to ϵ. This was necessary in order to perform the energy integrations in the expressions for the self-energy. This expansion eliminated the delicate frequency dependence of Y, which in Sec. IV we will find to be crucial for the evaluation of Γ. We therefore keep both the energy and frequency dependence of the self-energy, but confine ourselves to lowest-order perturbation theory around the BCS state with respect to Coulomb and electron-phonon interactions. That is, we replace the Green's functions in Eqs. (2.1)–(2.4) by BCS Green's functions. Consistently, we also neglect Y in the Coulomb propagator. Then the integrals can be done. In Sec. III C and in the Appendix, we will show that going beyond lowest order by including Γ self-consistently in the Green's functions has negligible effects for realistic values of the parameters.

We now proceed to calculate the Coulomb contributions to the imaginary parts of the self-energy in Sec. III and then the phonon contributions in Sec. IV.

III. COULOMB CONTRIBUTIONS

A. General expression

We start with the normal self-energy. S_C^H [Eq. (2.2b)] is frequency independent and therefore real and does not contribute to Γ. In Eq. (2.2a) we perform the frequency summation by using the spectral representation of $\mathcal{G}$ and V_C, respectively. We obtain

$$S_C^F(\epsilon,i\omega)=\sum_{\mathbf{q}}\int d\epsilon'\int\frac{dx}{\pi}V_c''(\mathbf{q},x)\int\frac{dy}{\pi}G''(\epsilon',y)R_C^F(\mathbf{q},\epsilon-\epsilon')[n(x)-f(y)+1]\frac{i\omega+x+y}{(i\omega)^2-(x+y)^2} , \tag{3.1}$$

where $n(x)$ and $f(x)$ are the Bose and Fermi distributions, and G'' and V_C'' denote the spectral functions for $\mathcal{G}$ and V_C, respectively. We now split $S(\epsilon,i\omega)$ into Y and Z according to Eq. (2.5). We obtain, for the imaginary parts,

$$\omega Z_C''(\epsilon,\omega)=\frac{1}{2}\sum_{\mathbf{q}}\int\frac{d\epsilon'}{\pi N_F}\int\frac{dx}{\pi}V_C''(\mathbf{q},x)\int dy\, G''(\epsilon',y)\delta(\omega-x-y)[n(x)-f(y)+1][\Phi''(\mathbf{q},\epsilon-\epsilon')+\Phi''(\mathbf{q},\epsilon+\epsilon')] , \tag{3.2a}$$

$$Y_C''(\epsilon,\omega)=-\frac{1}{2}\sum_{\mathbf{q}}\int\frac{d\epsilon'}{\pi N_F}\int\frac{dx}{\pi}V_C''(\mathbf{q},x)\int dy\, G''(\epsilon',y)\delta(\omega-x-y)[n(x)-f(y)+1][\Phi''(\mathbf{q},\epsilon-\epsilon')-\Phi''(\mathbf{q},\epsilon+\epsilon')] . \tag{3.2b}$$

$\Phi''=\pi N_F R_C^F$ is the spectrum of Kubo's density-density correlation function[23] for a fictitious system of noninteracting electrons with the same disorder as the superconductor. For a clean system with small momentum transfer q, the spectrum is white. For $D=2,3$ one has

$$\Phi''(\mathbf{q},\epsilon)=(N_F/E_F)(k_F/2q)(\pi/2)^{D-2}\quad(\text{clean}) . \tag{3.3a}$$

For diffusive quasiparticle dynamics, Φ'' is given by a diffusion pole,

$$\Phi''(\mathbf{q},\epsilon)=\frac{N_F Dq^2}{(Dq^2)^2+\epsilon^2}\quad(\text{diffusive}) , \tag{3.3b}$$

with diffusion constant D. We shall choose units such that $\hbar=k_B=1$. Here N_F is the density of states per spin at the Fermi level. For $D=2,3$ we have $N_F=(m/2\pi)(k_F/\pi)^{D-2}$. D is related to the (extrapolated) normal-state residual resisitivity ρ by an Einstein relation $\rho=1/2N_F De^2$, where e is the electron charge. ρ or D characterizes the amount of disorder in the system and will serve as our disorder parameter.

In the same manner, we evaluate the imaginary part of Eq. (2.1). We then use the result together with Eq. (3.2) in Eq. (2.7) to obtain the Coulomb contribution to the scattering rate, Γ_C. We find

$$\Gamma_C(\omega)=\frac{1}{Z'}\sum_{\mathbf{q}}\int\frac{d\epsilon'}{\pi N_F}\int\frac{dx}{\pi}V_C''(\mathbf{q},x)\int dy[n(x)-f(y)+1]\delta(\omega-x-y)$$
$$\times\left[G''(\epsilon',y)\frac{1}{2}[\Phi''(\mathbf{q},\omega-\epsilon')+\Phi''(\mathbf{q},\omega+\epsilon')]-\frac{\Delta}{\omega}\Phi''(\mathbf{q},\omega-\epsilon')F''(\epsilon',y)\right]$$
$$+\left[\frac{\omega+Y'}{\omega Z'^2}\right]\left[\sum_q\int\frac{d\epsilon'}{\pi N_F}\int\frac{dx}{\pi}V_C''(\mathbf{q},x)\int dy[n(x)-f(y)+1]\delta(\omega-x-y)\right.$$
$$\left.\times\frac{1}{2}G''(\epsilon',y)[\Phi''(\mathbf{q},\omega-\epsilon')-\Phi''(\mathbf{q},\omega+\epsilon')]\right]. \quad (3.4)$$

In order to evaluate Eq. (3.4) further, we need an explicit expression for the Coulomb propagator.

B. Coulomb propagator

The Coulomb interaction can be written as $V_C(\mathbf{q},\omega)=v(\mathbf{q})/\epsilon(\mathbf{q},\omega)$, where $\epsilon(\mathbf{q},\omega)$ is the longitudinal dielectric function and $v(\mathbf{q})$ is the bare Coulomb interaction. We write $\epsilon(\mathbf{q},\omega)=1+v(\mathbf{q})2C(\mathbf{q},\omega)$, where $2C$ is the screened density response function (this defines C). Then the imaginary part of V_C is

$$V_C''(\mathbf{q},\omega)=\frac{-2C''(\mathbf{q},\omega)}{[1/v(\mathbf{q})+2C'(\mathbf{q},\omega)]^2+[2C''(\mathbf{q},\omega)]^2}. \quad (3.5a)$$

The function $C(\mathbf{q},\omega)$ has been calculated in Ref. 19 for arbitrarily disordered superconductors. This calculation, which followed the conserving approximation scheme of Prange,[24] produces a manifestly gauge-invariant result. It has the structure

$$C(\mathbf{q},\omega)=B(\mathbf{q},\omega)+B_c(\mathbf{q},\omega). \quad (3.5b)$$

Here B is the density response function in pair approximation,[13] while B_c contains the collective excitation (the Anderson-Bogoliubov mode) which restores gauge invariance. As has been discussed before,[18,19] B_c is important only for wave numbers $q<1/\xi$, with ξ the superconducting coherence length. For a calculation of Γ_C, which is given by an integral over all wave numbers [Eq. (3.4)], we therefore can neglect B_c as long as $k_F\xi\gg 1$.[25] For a conventional superconductor with a realistic amount of disorder, this is always the case. With the same accuracy, we can replace $B'(\mathbf{q},\omega)$ by N_F. The imaginary part $B''(\mathbf{q},\omega)=B_r''(\mathbf{q},\omega)+B_s''(\mathbf{q},\omega)$ is conveniently split into a "recombination" (r) and a "scattering" (s) piece, which correspond to the screening of the Coulomb interaction by the virtual breaking of Cooper pairs and by the scattering of thermally excited quasiparticles, respectively. They have been calculated in Ref. 18, with the result

$$B_r''(\mathbf{q},\omega)=\Phi''(\mathbf{q},0)\Theta(\omega-2\Delta)\left[(\omega+2\Delta)E(\alpha)-\frac{4\Delta\omega}{\omega+2\Delta}K(\alpha)\right], \quad (3.6a)$$

$$B_s''(\mathbf{q},\omega)=\Phi''(\mathbf{q},0)\begin{cases}e^{-\Delta/T}T(1-e^{-\omega/T}) & \text{for } \omega\ll\Delta,\\ \Delta(1+T/\omega)K_1(\Delta/T) & \text{for } \omega\geq 2\Delta.\end{cases} \quad (3.6b)$$

Here $\alpha=(\omega-2\Delta)/(\omega+2\Delta)$, E and K are complete elliptical integrals, and K_1 is a Bessel function. Equation (3.6a) is valid for $Dq^2\gg[\omega(\omega-2\Delta)]^{1/2}$. Since the relevant frequency scale in Eq. (3.4) is $\omega\simeq\Delta$, this is again consistent with the requirement $q\xi\gg 1$. Finally, we make the perfect screening approximation; i.e., we neglect $1/v(q)$ in Eq. (3.5a). Then we have

$$V_C''(\mathbf{q},\omega)=-\frac{1}{2N_F}\frac{B''(\mathbf{q},\omega)}{N_F}\left[1+\left[\frac{B''(\mathbf{q},\omega)}{N_F}\right]^2\right]^{-1}, \quad (3.7)$$

with $B''=B_r''+B_s''$ given by Eqs. (3.6). We note that because of the perfect screening approximation, the result for V_c'' [Eq. (3.7)] does not depend on the dimensionality (except through D and N_F). However, the term $(B''/N_F)^2$ in the denominator, which can be dropped in $D=3$,[18] has to be kept in the 2D calculation in order to avoid a spurious logarithmic singularity.

C. Lifetimes in thin films

We now return to the evaluation of Eq. (3.4) for thin films. The second term in Eq. (3.4) contains a difference of two density spectra. In the clean limit, it therefore vanishes according to Eq. (3.3a). For diffusive quasiparticle dynamics [Eq. (3.3b)], this term contributes only for wave numbers $q\lesssim\sqrt{\Delta/D}\simeq 1/\xi$. Neglecting the second term therefore has the same range of validity as the pair approximation for the Coulomb propagator.

We note that, diagrammatically, the use of Eq. (3.3b) in Eq. (3.4) corresponds to a renormalization of the Coulomb vertex by means of diffusion ladder diagrams [diffusion-propagator renormalization (DPR)].[7] Within DPR the contribution of Y'' to the scattering rate is thus negligible compared to that of $\omega Z''$ for the reasons explained above. The same is true if the vertex is renormalized by means of the maximally crossed ladder diagrams or cooperons [Cooper-propagator renormalization (CPR)].[7] The CPR can be realized within the present correlation function technique by using the symmetry properties of the Kubo function and the projection techniques which were explained in detail in Ref. 7. This way we obtain all diffusion enhanced contributions to Γ_C. We conclude that in order to calculate the leading (i.e., diffusion-enhanced) contributions to Γ_C, we can neglect Y''. Furthermore, an inspection of the integrals in the first term in Eq. (3.4) shows that CPR leads to a result which is smaller than the one resulting from DPR by a factor of $1/k_F\xi$. Equation (3.3b) is therefore sufficient in order to obtain the leading-order contribution. Equation (3.4) then can be written as

$$\Gamma_C(\omega)=-\frac{1}{Z'}\sum_{\mathbf{q}}\int\frac{d\epsilon'}{\pi N_F}\Phi''(\mathbf{q},\epsilon'-\omega)\int_0^\infty\frac{dx}{\pi}[f(x+\omega)+n(x)]V_C''(\mathbf{q},x)\left[G''(\epsilon',x+\omega)+\frac{\Delta}{\omega}F''(\epsilon',x+\omega)\right]+(\omega\to-\omega)\ , \tag{3.8}$$

where $(\omega\to-\omega)$ denotes a term different from the first one by the sign of ω. For $(\Delta/\omega)\to 0$, Eq. (3.8) yields the well-known normal-metal results.[1]

For 3D systems, Eq. (3.8) reproduces the known clean-limit results.[13,14] For 3D disordered superconductors at nonzero temperature, it reproduces the results of Ref. 18.[26] We now first consider the case of zero temperature. At $T=0$ the only possible scattering process is by a quasiparticle breaking a Cooper pair. For such processes to occur, the quasiparticle must have at least an energy of 3Δ. We find, for $(\omega-3\Delta)/(\omega+3\Delta)\ll 1$,

$$\tau_C^{-1}(\omega)=\frac{\pi}{4Z'}\Theta(\omega-3\Delta)\left[\frac{\Delta}{E_F}\right]^{1/2}(3\hat{\rho})^{3/2}\frac{\pi^{3/2}-(\omega/\Delta)^{3/2}}{\pi^2-(\omega/\Delta)^2}\times\left[[(\omega-2\Delta)^2-\Delta^2]^{1/2}-\frac{\Delta^2}{\omega}\ln\left[\frac{[(\omega-2\Delta)^2-\Delta^2]^{1/2}+\omega-2\Delta}{\Delta}\right]\right] . \tag{3.9}$$

Here $\hat{\rho}=\rho/\rho_M$, with $\rho=1/2e^2ND_F$ the resisitivity and $\rho_M=\hbar 3\pi^2/e^2k_F$ the Mott number. The corresponding rate in 2D is given by

$$\tau_C^{-1}(\omega)=\Theta(\omega-3\Delta)\frac{\Delta}{2E_FZ'}\ln(4E_F/\pi\Delta)\left[[(\omega-2\Delta)^2-\Delta^2)]^{1/2}-\frac{\Delta^2}{\omega}\ln\left[\frac{[(\omega-2\Delta)^2-\Delta^2)]^{1/2}+\omega-2\Delta}{\Delta}\right]\right]\quad\text{(clean)}\ , \tag{3.10a}$$

$$\tau_C^{-1}(\omega)=\frac{1}{Z'}\Theta(\omega-3\Delta)\frac{1}{4\omega+4\pi\Delta}\hat{R}_\square\left[[(\omega-2\Delta)^2-\Delta^2)]^{1/2}-\frac{\Delta^2}{\omega}\ln\left[\frac{[(\omega-2\Delta)^2-\Delta^2)]^{1/2}+\omega-2\Delta}{\Delta}\right]\right]\quad\text{(diffusive)}\ . \tag{3.10b}$$

Here $\hat{R}_\square=R_\square/R_M$, where $R_\square$ is the sheet resistance per square, $R_M=\hbar/e^2=4108\Omega/\square$ is the Mott number, and E_F is the Fermi energy. We see that the rates rise as $\sqrt{\omega-3\Delta}$ in the above limit.[27]

At nonzero temperature, we follow the standard procedure[11] (cf. also Ref. 18) to separate τ^{-1} into recombination and scattering contributions, respectively. For a quasiparticle at the gap edge, we have a lifetime $\tau=1/2\Gamma(\Delta)$. For 2D clean systems, we find, to leading order in T/Δ,

$$\tau_{c,r}^{-1}=\left[\frac{\Delta}{Z'}\right]\left[\frac{\Delta}{E_F}\right]\left[\frac{T}{\Delta}\right]e^{-2\Delta/T}\ln\left[\frac{4E_F}{\pi\Delta}\right]\ , \tag{3.11a}$$

$$\tau_{c,s}^{-1}=\left[\frac{\Delta}{Z'}\right]\left[\frac{\Delta}{E_F}\right]\left[\frac{\pi T}{2\Delta}\right]^{1/2}e^{-\Delta/T}\ln\left[\frac{4E_F}{\pi\Delta}\right]\ , \tag{3.11b}$$

and for diffusive quasiparticle dynamics,

$$\tau_{c,r}^{-1}=\left[\frac{1}{2(1+\pi)}\right]\left[\frac{\Delta}{Z'}\right]\hat{R}_\square\left[\frac{T}{\Delta}\right]e^{-2\Delta/T}\ , \tag{3.12a}$$

$$\tau_{c,s}^{-1}=\frac{1}{2(1+\pi)}\left[\frac{\Delta}{Z'}\right]\hat{R}_\square\left[\frac{\pi T}{2\Delta}\right]^{1/2}e^{-\Delta/T}\ . \tag{3.12b}$$

Comparison with the corresponding 3D results shows that the 2D rate is roughly of the same order and has the same temperature dependence as the 3D result, but has a different disorder dependence. The disorder dependences in 2D and 3D are the same as in normal metals.

One might find surprising the fact that the leading temperature dependence of the rate is the same in 2D and 3D, since it is in sharp contrast to the case of normal metals.[1,3] Further, comparing the above results for clean and dirty superconductors, one sees that the temperature dependence of the rate is also independent of disorder. The reason for both of these results is that superconductivity introduces Δ as a new energy scale. The leading temperature dependence is determined by the frequency dependence of V_C'' in the vicinity of the gap, which, to leading order, is constant, independent of disorder and dimensionality.

The temperature dependence is dominated by the exponential factor, which inevitably arises within perturbation theory due to the singularity in the BCS DOS. In view of the treatment of the 2D normal-metal case,[4] one might wonder whether a similar procedure would not qualitatively change the perturbative result in a superconductor. The basic physical idea is that the inelastic scattering will smooth out the singularity in the BCS DOS and should therefore be included self-consistently on the right-hand side of Eq. (3.4). We have performed such a calculation and describe it in the Appendix. We find that (i) self-consistency cannot lead to $\Gamma(T=0)\neq 0$, and (ii) at $T>0$ the effects are too small to be of any practical importance. The latter statement holds for amounts of disorder which can be described by the diffusion pole approximation and are thus within the region of validity of the present theory. We conclude that our perturbative results are sufficient to describe the Coulomb scattering rate.

IV. PHONON CONTRIBUTIONS

A. General expressions

We now turn our attention to the electron-phonon contributions to the scattering rate. The general features of the calculation are analogous to the previous section, and so we can be brief. The direct self-energy $S^H_{\rm ph}$ [Eq. (2.4b)] does not contribute, and $S^F_{\rm ph}$ [Eq. (2.4a)] is separated into Y and $i\omega Z$ again. We obtain

$$\omega Z''_{\rm ph}(\epsilon,\omega)=\int d\epsilon' \int d\nu [n(\nu)+f(\nu+\omega)]G''(\epsilon',\omega+\nu)[\alpha^2F^F(\epsilon-\epsilon',\nu)+\alpha^2F^F(\epsilon+\epsilon',\nu)] \ , \tag{4.1a}$$

$$Y''_{\rm ph}(\epsilon,\omega)=-\int d\epsilon' \int d\nu [n(\nu)+f(\nu+\omega)]G''(\epsilon',\omega+\nu)[\alpha^2F^F(\epsilon-\epsilon',\nu)-\alpha^2F^F(\epsilon+\epsilon',\nu)] \ , \tag{4.1b}$$

where

$$\alpha^2F^{F,H}(\epsilon-\epsilon',\nu)=\sum_{\mathbf{q},b} R_b^{F,H}(\mathbf{q},\epsilon-\epsilon')B_b(\mathbf{q},\nu) \tag{4.2}$$

is the generalized Eliashberg function for exchange and direct processes, respectively.[7] $B_b(\mathbf{q},\nu)$ is the phonon spectrum, in terms of which the phonon propagator reads

$$D_b(\mathbf{q},i\Omega)=2\int d\nu \frac{\nu}{(i\Omega)^2-\nu^2}B_b(\mathbf{q},\nu) \ . \tag{4.3}$$

Performing the ϵ' integral in Eqs. (4.1), we obtain

$$\omega Z''_{\rm ph}(\epsilon,\omega)=\pi\int d\nu \frac{\nu+\omega}{[(\nu+\omega)^2-\Delta^2]^{1/2}}[f(\nu+\omega)+n(\nu)] \times\{\alpha^2F^F(\epsilon-[(\nu+\omega)^2-\Delta^2]^{1/2},\nu)+\alpha^2F^F(\epsilon+[(\nu+\omega)^2-\Delta^2]^{1/2},\nu)\}\Theta((\omega+\nu)^2-\Delta^2) \ , \tag{4.4a}$$

$$Y''_{\rm ph}(\epsilon,\omega)=-\pi\int d\nu [f(\nu+\omega)+n(\nu)] \times\{\alpha^2F^F(\epsilon-[(\nu+\omega)^2-\Delta^2]^{1/2},\nu)-\alpha^2F^F(\epsilon+[(\nu+\omega)^2-\Delta^2]^{1/2},\nu)\}\Theta((\omega+\nu)^2-\Delta^2) \ . \tag{4.4b}$$

For $\Delta=0$ the first terms in the curly brackets in Eqs. (4.4a) and (4.4b) cancel each other and the normal-metal result is recovered.[21,28]

B. Conventional contribution in $D=2$

In order to facilitate comparison of our theory with previous work, we neglect Y'' for the time being and combine the imaginary part of the anomalous self-energy, $W''_{\rm ph}$ [Eq. (2.3)] with Eq. (4.4a) according to Eq. (2.7). This way we obtain a phonon contribution to the quasiparticle decay rate which we call $\Gamma^{(1)}_{\rm ph}$:

$$\Gamma^{(1)}_{\rm ph}(\omega)=\frac{\pi}{Z'}\int_0^\infty d\nu\, \Theta((\nu+\omega)^2-\Delta^2)\frac{\nu+\omega-\Delta^2/\omega}{[(\nu+\omega)^2-\Delta^2]^{1/2}}[f(\nu+\omega)+n(\nu)] \times[\alpha^2F^F(\omega-[(\nu+\omega)^2-\Delta^2]^{1/2},\nu)+\alpha^2F^F(\omega+[(\nu+\omega)^2-\Delta^2]^{1/2},\nu)]+(\omega\to-\omega) \ . \tag{4.5}$$

Note that this expression is analogous to Eq. (3.8) for the Coulomb scattering rate. $\alpha^2F^F(\epsilon,\omega)$ is an electronic stress-correlation function for which DPR does not yield a diffusion enhancement.[7,28] This is due to the electron's ability to maintain local charge neutrality and shows, e.g., in Pippard's results for the sound attenuation.[29] α^2F^F can be calculated diagrammatically[28] or by solving a Boltzmann equation in collision time approximation, and the explicit result for $D=3$ can be found in Ref. 28 or 7. If we use it to calculate $\Gamma_{\rm ph}$, we recover the result of Reizer and Sergeyev.[21] Before we turn to CPR for the electron-phonon vertex, we derive the corresponding result for $D=2$. We write the 2D expression for α^2F^F with diffusion ladder renormalization as[5]

$$\alpha^2F^F(\epsilon-\epsilon',\nu)\approx\alpha^2F^F(\nu)=\frac{\nu^2 l}{\pi m}\sum_b \frac{d_b}{c_b^3}f_b\left[\frac{\nu l}{c_b}\right] , \tag{4.6}$$

where

$$f_T(x)=\frac{8}{x^4}\left[1+\frac{x^2}{2}-(1+x^2)^{1/2}\right], \tag{4.7a}$$

$$f_L(x)=2\left[\frac{1}{(1+x^2)^{1/2}-1}-\frac{2}{x^2}\right]. \tag{4.7b}$$

Here $l=2D/v_F$ is the elastic mean free path and c_b the sound velocity. We define the generalized electron-phonon coupling constant to be $d_b=k_F^{D+1}/16\pi\rho_i c_b$ for $D=2,3$ with ρ_i the D-dimensional ion density. Using Eq. (4.5), we find the 2D results analogous to Refs. 21 and 11 for the scattering and recombination rate at the gap edge, viz.,

$$\tau_{\mathrm{ph},s}^{(1)-1}=2\sqrt{2}\frac{\Delta}{E_F}\left[\frac{T}{\Delta}\right]^{5/2}\frac{\Delta}{Z'}\begin{cases} d_L\dfrac{v_F^2}{c_L^2}\Gamma(\tfrac{5}{2})\zeta(\tfrac{5}{2}) & \text{for } \dfrac{\pi T}{E_F}\dfrac{v_F}{c_b}\gg\hat{R}_\square , \\ d_T\dfrac{v_F^3}{c_T^3}[1+(c_T/c_L)^4]\dfrac{\pi}{2\hat{R}_\square}\Gamma(\tfrac{7}{2})\zeta(\tfrac{7}{2})\dfrac{T}{E_F} & \text{for } \dfrac{\pi T}{E_F}\dfrac{v_F}{c_b}\ll\hat{R}_\square , \end{cases} \tag{4.8a}$$

$$\tau_{\mathrm{ph},r}^{(1)-1}=8\pi\frac{\Delta}{Z'}\alpha^2F^F(2\Delta)\left[\frac{\pi T}{2\Delta}\right]^{1/2}e^{-\Delta/T}. \tag{4.8b}$$

$$\alpha^2F^F(2\Delta)=\begin{cases} 2\dfrac{\Delta}{E_F}\dfrac{d_L v_F^2}{\pi c_L^2} & \text{for } \left[\dfrac{v_F}{c_b}\right]\left[\dfrac{2\pi\Delta}{E_F}\right]\gg\hat{R}_\square , \\ \dfrac{2v_F^3}{\hat{R}_\square}\left[\dfrac{\Delta}{E_F}\right]^2\dfrac{d_T}{c_T^3}[1+(c_T/c_L)^4] & \text{for } \left[\dfrac{v_F}{c_b}\right]\left[\dfrac{2\pi\Delta}{E_F}\right]\ll\hat{R}_\square . \end{cases} \tag{4.9}$$

For $T=0$ we find

$$\tau_{\mathrm{ph}}^{-1}(\omega)=\frac{\Delta}{Z'}4\frac{\Delta}{E_F}\Theta(\omega-\Delta)\begin{cases} d_L\dfrac{v_F^2}{c_L^2}\left([(\omega/\Delta)^2-1]^{1/2}[(\omega/2\Delta)+\Delta/\omega]-\tfrac{3}{2}\ln\{[(\omega/\Delta)^2-1]^{1/2}+\omega/\Delta\}\right) \\ \text{for } \left[\dfrac{\pi v_F}{c_b}\right]\left[\dfrac{\omega-\Delta}{E_F}\right]\gg\hat{R}_\square , \\ d_T\dfrac{v_F^3}{c_T^3}[1+(c_T/c_L)^4]\dfrac{\pi}{2\hat{R}_\square}\dfrac{\Delta}{E_F}g(\omega/\Delta) \quad \text{for } \left[\dfrac{\pi v_F}{c_b}\right]\left[\dfrac{\omega-\Delta}{E_F}\right]\ll\hat{R}_\square , \end{cases} \tag{4.10a}$$

where

$$g(x)=\frac{1}{3}(x^2-1)^{3/2}+\frac{5}{2}(x^2-1)^{1/2}+\frac{1}{2x}[1+(2x)^2]\ln[x+(x^2-1)^{1/2}]. \tag{4.10b}$$

Γ denotes the gamma function and ζ the Riemann zeta function. We also have considered CPR of the electron-phonon vertex in $\Gamma_{\mathrm{ph}}^{(1)}$. For the same technical reasons as in the Coulomb case, the resulting contribution to τ_{ph}^{-1} is much smaller than Eq. (4.8) and can thus be ignored.

C. Effects of the self-energy contribution Y

We now return our attention to the contribution of the correlation-gap piece $Y(\epsilon,\omega)$ to the scattering rate, Eq. (4.4b). First, we note that if we use DPR for the stress-correlation functions, we have $\alpha^2F^F(\epsilon,\nu)\approx\alpha^2F^F(\nu)$ for $\epsilon\tau\ll1$, where τ is the elastic collision time, and therefore Y''_{ph} is negligible compared to $\omega Z''_{\mathrm{ph}}$. The evaluation of the maximally crossed ladder renormalization shows that backscattering events do lead to diffusion enhancement and result in a nonanalytic behavior of the stress-correlation function. The physical reason is that as the quasiparticles become less mobile, they become more rigid to shear deformations. This is well known, and it has interesting consequences for the sound attenuation in disordered normal metals.[30] The relevant transverse stress-correlation function with CPR in 3D is given by

$$R_T^F(\mathbf{q},\Omega)=R_T^0(\mathbf{q},\Omega)-R_T(0,0)\left[\frac{3\sqrt{6}}{16}\frac{\sqrt{\Omega\tau}}{(E_F\tau)^2}\right], \tag{4.11a}$$

where

$$R_T(0,0)=\frac{2}{15\pi^2}c_T\left[\frac{v_F}{c_T}\right]^3\frac{m^3}{\rho_i}(E_F\tau)\frac{1}{\pi N_F} \tag{4.11b}$$

and

$$R_T^0(\mathbf{q},\Omega)=\frac{5}{2}R_T(0,0)\frac{2x^3+3x-3(1+x^2)\tan^{-1}x}{x^5}\quad (x=ql)\ . \tag{4.12}$$

Via Eq. (4.2), this gives the dominant contribution to $Y''_{\rm ph}$. The longitudinal phonon contribution is related to the transverse one at zero wave number by $R_L=R_T\frac{4}{3}(c_T/c_L)^2$.

We rewrite Eq. (4.4b) in the quasiparticle representation.[11] We obtain three terms for Y:

$$\begin{aligned}Y''(\epsilon,\omega)=\frac{-\pi}{1-f(\omega)}\sum_{\mathbf{q},b}\Bigg[&\int_{\Delta-\omega}^{\infty}d\nu\, B_b(\mathbf{q},\nu)n(\nu)[1-f(\nu+\omega)]\{R_b^F(\mathbf{q},\epsilon-[(\nu+\omega)^2-\Delta^2]^{1/2})-R_b^F(\mathbf{q},\epsilon+[(\nu+\omega)^2-\Delta^2]^{1/2})\}\\ &+\int_{\Delta+\omega}^{\infty}d\nu\, B_b(\mathbf{q},\nu)[n(\nu)+1]f(\nu-\omega)\{R_b^F(\mathbf{q},\epsilon-[(\nu-\omega)^2-\Delta^2]^{1/2})-R_b^F(\mathbf{q},\epsilon+[(\nu-\omega)^2-\Delta^2]^{1/2})\}\\ &+\int_0^{\omega-\Delta}d\nu\, B_b(\mathbf{q},\nu)[n(\nu)+1][1-f(\omega-\nu)]\{R_b^F(\mathbf{q},\epsilon-[(\nu-\omega)^2-\Delta^2]^{1/2})-R_b^F(\mathbf{q},\epsilon\\ &+[(\nu-\omega)^2-\Delta^2]^{1/2})\}\Bigg]\ .\end{aligned} \tag{4.13}$$

Substituting Eq. (4.11) for R_b into the above equation, we calculate the on-shell contribution of Y'' for a quasiparticle at the gap edge. Using a Debye spectrum $B_b(\mathbf{q},\nu)=\frac{1}{2}qc_b\delta(\nu-c_bq)\Theta(\omega_D-\nu)$, Eq. (4.13) yields, for $T/\Delta\ll 1$,

$$Y''(\Delta)=-\Delta\sqrt{\hat\rho}d_T\frac{\sqrt{6}}{5\sqrt{\pi}}\frac{v_F^4}{c_T^4}\left[\frac{\Delta}{E_F}\right]^{7/2}\left[\frac{T}{\Delta}\right]^{3/2}\left[1+\frac{4}{3}(c_T/c_L)^5\right]\left[\exp(-\Delta/T)+\frac{\Gamma(\frac{9}{2})\zeta(\frac{9}{2})}{4\sqrt{\pi}}(T/\Delta)^3\right]\ , \tag{4.14}$$

while, for $T=0$,

$$Y''(\omega)=-\Delta\sqrt{\hat\rho}d_T\frac{\sqrt{3}}{20\pi}\frac{v_F^4}{c_T^4}\left[\frac{\Delta}{E_F}\right]^{7/2}\Theta(\omega-\Delta)\left[\frac{\omega}{\Delta}\right]^{9/2}f(\omega/\Delta)\left[1+\frac{4}{3}(c_T/c_L)^5\right]\ , \tag{4.15a}$$

where

$$f(x)=\frac{1}{2x^2}\left[1+\frac{3}{4x^2}\right]\ln(x^2-\sqrt{x-1})+\frac{1}{2}\left[1+\frac{3}{4x^2}\right](1-1/x^2)^{1/2}-\left[\frac{1}{60}\right](1-1/x^2)^{3/2}(27+8/x^2)\ . \tag{4.15b}$$

In deriving Eqs. (4.14) and (4.15), we have assumed $\Delta\tau\simeq l^2/\xi^2\ll 1$. The results therefore hold only in the dirty limit. In the opposite limit, $l\gtrsim\xi$, an additional factor of $\hat\rho$ appears. We also note that we have kept both the scattering and recombination contribution in Eq. (4.14), even though the latter is exponentially small at low temperatures. This is to demonstrate that in the disordered case the scattering contribution, which has a power-law temperature dependence, is dominant to much higher temperatures than in a clean superconductor. We will further discuss this important point in Sec. VI.

The corresponding real part of Y was found in Ref. 7 to vary linearly in resisitivity and is given by $Y'=6\Delta[(4\lambda-3\mu)/\pi]\hat\rho$, where λ is the electron-phonon coupling constant and μ is the effective Coulomb potential[31] for the corresponding clean metal. We denote the contribution of Y to the quasiparticle decay rate by $\tau_{\rm ph}^{(2)-1}$ and find, for a quasiparticle at the gap edge in $D=3$,

$$\begin{aligned}\tau_{\rm ph}^{(2)-1}=\frac{\Delta}{Z'^2}\left[(\hat\rho)^{1/2}+6\frac{4\lambda-3\mu}{\pi}(\hat\rho)^{3/2}\right]d_T\frac{2\sqrt{6}}{5\sqrt{\pi}}\frac{v_F^4}{c_T^4}\left[\frac{\Delta}{E_F}\right]^{7/2}\left[\frac{T}{\Delta}\right]^{3/2}\\ \times\left[\exp(-\Delta/T)+\frac{\Gamma(\frac{9}{2})\zeta(\frac{9}{2})}{4\sqrt{\pi}}(T/\Delta)^3\right]\left[1+\frac{4}{3}(c_T/c_L)^5\right]\quad (D=3)\ .\end{aligned} \tag{4.16}$$

We now repeat the above calculation for two-dimensional thin films. We will need the corresponding transverse stress-correlation function with CPR in 2D, which is given by

$$R_T^F(\mathbf{q},\Omega)=R_T^0(\mathbf{q},\Omega)+R_T(0,0)\frac{1}{2\pi E_F\tau}\ln(1/\Omega\tau)\ , \tag{4.17a}$$

where

$$R_T(0,0)=\frac{4ld_T}{mc_T} \tag{4.17b}$$

and

$$R_T^0(\mathbf{q},\Omega)=R_T(0,0)\frac{8}{x^4}\left[1+\frac{x^2}{2}-(1+x^2)^{1/2}\right]\ , \tag{4.17c}$$

where $x=ql$. The calculation for $Y''(\Delta)$ is straightforward, and we obtain

$$Y''(\Delta)=\frac{-2\Delta}{\sqrt{2\pi}}d_T\left[\frac{v_F}{c_T}\right]^3\left[\frac{\Delta}{E_F}\right]^2\left[\frac{T}{\Delta}\right]^{3/2}[1+(c_T/c_L)^4]\left[e^{-\Delta/T}+\frac{\Gamma(\frac{7}{2})\zeta(\frac{7}{2})}{2\sqrt{\pi}}\left[\frac{T}{\Delta}\right]^2\right]\ , \tag{4.18}$$

and for $T=0$,

$$Y''(\omega)=\frac{-\Delta}{4\pi}d_T\left[\frac{v_F}{c_T}\right]^3\left[\frac{\omega}{\Delta}\right]^4\left[\frac{\Delta}{E_F}\right]^2\Theta(\omega-\Delta)h(\Delta/\omega)[1+(c_T/c_L)^4]\ , \tag{4.19a}$$

where

$$h(x)=\frac{(1-x^2)^{1/2}}{2}\left[1+\frac{x^2}{4}\right]-\frac{5}{12}(1-x^2)^{3/2}-\frac{x^2}{2}\left[1-\frac{x^2}{4}\right]\ln\left[\frac{1}{x}+\left[\frac{1}{x^2}-1\right]^{1/2}\right]\ . \tag{4.19b}$$

We next have to calculate the real part of Y in $D=2$. In Ref. 7 the corresponding 3D result was obtained by expanding $Y(\epsilon,\omega)$ in a Taylor series in ϵ. In 2D this procedure produces a spurious infrared singularity in the direct Eliashberg function α^2F^H and the direct Coulomb kernel denoted by U_c^Y in Ref. 7. The technical reason is that, as mentioned before, the Taylor expansion suppresses the ω dependence of Y. Inspection shows that as a result of the ω dependence, the divergence is cut off by the gap, as was the case in the 2D calculation of Γ_C. We therefore can still follow Ref. 7 in calculating Y' if we use the inverse coherence length ξ^{-1} as a lower limit in the wave-number integral. Adding the Coulomb contribution to Y', we finally find

$$Y'=\Delta\hat{R}_\square\ln(2k_F\xi)\left[\frac{c_Lk_F}{8\pi\omega_D}\lambda\left\{1-(2/\pi)\sin^{-1}\left[1-\frac{1}{2}\left[\frac{\omega_D}{k_Fc_L}\right]^2\right]\right\}-\frac{1}{\pi^2}\right]\ . \tag{4.20}$$

We then obtain $\tau_{\rm ph}^{(2)-1}$ in 2D in the form

$$\tau_{\rm ph}^{(2)-1}=\Delta(1+Y'/\Delta)d_T\left[\frac{v_F}{c_T}\right]^3\frac{4}{\sqrt{2\pi}Z'^2}\left[\frac{\Delta}{E_F}\right]^2\left[\frac{T}{\Delta}\right]^{3/2}[1+(c_T/c_L)^4]\left[e^{-\Delta/T}+\frac{\Gamma(\frac{7}{2})\zeta(\frac{7}{2})}{2\sqrt{\pi}}\left[\frac{T}{D}\right]^2\right]\quad (D=2)\ , \tag{4.21}$$

with Y' from Eq. (4.20).

V. LIFETIMES IN FILMS OF FINITE THICKNESS

As the sample width decreases, the interactions governing the transport properties change from 3D to 2D behavior. The effect of this crossover on the DOS and conductivity has been studied in dirty normal metals.[32] As Altshuler, Aronov, and Zyiuzin have pointed out,[32,33] the Coulomb interaction in a normal metal is effectively 2D for sample widths $L_z \ll [D/\max(\omega,\tau_\phi^{-1})]^{1/2}$ where ω is the characteristic energy transfer and τ_ϕ^{-1} is the rate at which quasiparticles lose their phase coherence. Note that the phase coherence rate is temperature dependent. Since most tunneling experiments on superconducting films are performed somewhere in the transition region between 2D and 3D behavior, we will now give an interpolation formula for the leading contribution to the scattering rates for films of arbitrary thickness which recovers the previous expressions in the appropriate limits. We first examine the leading Coulomb contribution to τ^{-1}. Breaking up the sum over wave numbers in Eq. (3.8) into an integral over q in the x-y plane and a sum over quantized wave numbers in the z direction, we find that to leading order in $T/\Delta \ll 1$ we can express the inverse lifetime in the following form:

$$\tau_{c,s}^{-1}=\frac{3\pi^3}{4(1+\pi)}\hat{\rho}(\Delta/E_F)(\xi_0/L_z)(\Delta/Z')(\pi T/2\Delta)^{1/2}e^{-\Delta/T}\left[\frac{4\phi-(4/\pi)_\theta}{\pi-1}\right]\ , \tag{5.1}$$

where

$$\tan\theta = \frac{\tan[(L_z/2\xi)\sqrt{3/2\pi}]}{\tanh[(L_z/2\xi)\sqrt{3/2\pi}]} \tag{5.2a}$$

and

$$\tan\phi = \frac{\tan[(L_z/2\xi)\sqrt{3/2}]}{\tanh[(L_z/2\xi)\sqrt{3/2}]} \ . \tag{5.2b}$$

Here $\xi=\sqrt{\xi_0 l}$ is the dirty-limit coherence length and $\xi_0=v_F/\pi\Delta$ is the Pippard coherence length. For $L_z \ll \xi$ the term in square brackets approaches unity and the 2D result [Eq. (3.12b)] is recovered. For $L_z \gg \xi$ we obtain Eq. (11) of Refs. 18.[26] Thus we find that the criterion for two-dimensional behavior is given by $L_z \ll \xi$, which is temperature independent. A crossover from 3D to 2D behavior thus can be driven by either a decrease in sample width or an increase in disorder for all temperatures below T_c. This is consistent with the normal-metal criterion mentioned above since in a superconductor the characteristic energy transfer is given by the gap. Rewriting Eqs. (5.2) in terms of $\hat{R}_\square$, we recover the second term of Eq. (1a) and Eq. (1c) of Ref. 22.

An inspection of Eqs. (4.16) and (4.21) shows that the dominant contribution to the quasiparticle relaxation rate is given by scattering rather than by recombination for all temperatures both in 3D and 2D. This is in contrast to the case of clean superconductors, where a crossover from scattering-dominated relaxation to recombination-dominated relaxation occurs at $T\simeq 0.1\Delta$ in 3D and $T\simeq 0.3\Delta$ in 2D. Therefore, we focus on the leading-order contribution to $\tau_{\rm ph}^{(2)-1}$ from scattering relaxation processes only.

Following the same procedure for the leading phonon contribution to Y'', we find that

$$Y''(\Delta) = \frac{R_T(0,0)}{16\pi^2 c_T^2 D^{3/2} N_F L_z}\left[\frac{\sinh(x)+\sin(x)}{\cosh(x)-\cos(x)}\right]\left[T^{7/2}\Gamma(\tfrac{7}{2})\zeta(\tfrac{7}{2}) + \frac{L_z}{\pi c_T} T^{9/2}\Gamma(\tfrac{9}{2})\zeta(\tfrac{9}{2})\right], \tag{5.3}$$

where $x=(L_z/\xi)\sqrt{3/2\pi}$. For Y'' the criterion for 3D disorder behavior is the same as the Coulomb case, namely, $L_z \gg \xi$, while the 3D temperature dependence will be seen when $L_z T/c_T \gg 1$. In the 3D limit, Eq. (5.3) recovers the leading term in Eq. (4.14), while the 2D limit recovers the leading term in Eq. (4.18) if we identify $mk_F L_z/2\pi^2$ as the quasi-2D density of states.

Since the real part of the self-energy piece Y and the stress-correlation function R_b^F are not strongly frequency dependent, the relevant criterion for three-dimensional behavior for Y and R_b^F is that $2\pi/k_F L_z \ll 1$, which is usually the case.[34] Therefore, for practical purposes we can use Y' in the 3D limit and $R_T(0,0)$ given by Eq. (4.11b), which then leads to the following expression for $\tau_{\rm PH}^{(2)-1}$:

$$\tau_{\rm PH}^{(2)-1} = \frac{\Delta}{Z'^2}\left[1+6\frac{4\lambda-3\mu}{\pi}\hat{\rho}\right]\frac{9\pi^2}{5}\sqrt{6\hat{\rho}}\lambda\left[\frac{c_L}{c_T}\right]^4\left[\frac{\xi_0}{L_z}\right]\left[\frac{\Delta}{E_F}\right]^{3/2}$$
$$\times\left[\frac{\sinh(x)+\sin(x)}{\cosh(x)-\cos(x)}\right]\Gamma(\tfrac{7}{2})\zeta(\tfrac{7}{2})\left[\frac{T}{\Delta}\right]^{7/2}\left[\frac{\Delta}{\omega_D}\right]^2\left[1+\frac{T}{\Delta}\frac{L_z}{\xi_0}\frac{v_F}{c_T}\frac{7}{2\pi^2}\frac{\zeta(\frac{9}{2})}{\zeta(\frac{7}{2})}\right], \tag{5.4}$$

where $x=(L_z/\xi)\sqrt{3/2\pi}$, and we have rewritten d_T in terms of λ, the clean-limit 3D electron-phonon coupling parameter.[28] If we set $\mu=\frac{1}{2}$ in a perfect screening approximation, we recover the first term of Eq. (1a) and Eq. (1b) of Ref. 22.

The leading-order contribution to the scattering rate in a disordered superconducting film of thickness L_z is given by the sum of Eqs. (5.4) and (5.1).

VI. DISCUSSION

In summary, we have found that the Coulomb contribution to the scattering rate is enhanced in the same manner as in the normal-metal case, namely, a factor of $(\Delta/E_F)^{d-1}$ in the clean case becomes $\hat{\rho}^{3/2}\sqrt{\Delta/E_F}$ in 3D and $\hat{R}_\square$ in 2D, which leads to a net enhancement of up to 10^3. Further, we find that the divergence encountered in the two-dimensional dirty normal metal is cut off by the gap in the superconducting case. Technically, this is due to the diffusion pole of the density-correlation function being cut off in frequency since the gap is a lower bound of the energy transfer for a quasiparticle at the gap edge.

For the phonon contribution to the rate, we find that the conventional contribution decreases with increasing disorder in 2D as it does in 3D. Moreover, we find that, with increasing disorder, interference effects enhance an additional contribution from the correlation gap piece Y of the self-energy. This is the leading phonon contribution to the rate in both 3D and 2D. Contrary to clean systems, the contribution to the rate from phonons due to scattering processes is always dominant over the contribution from recombination processes. Thus, unlike many physical transport quantities in the superconductors, the main contribution to the rate from phonons has a power-law temperature dependence rather than an exponential one. For large disorder the leading contribution to the full rate then results from a competition between the correlation-gap contribution Y'' and the Coulomb contribution.

For large disorder we find that in both 2D and 3D the Coulomb and correlation-gap contributions to the scattering rate have the same resisitivity exponent. An estimate of the prefactor shows that the correlation-gap contribution is dominant over the Coulomb contribution for strongly coupled superconductors for all temperatures. Further, even for materials which are not strongly coupled, the power-law temperature-dependent correlation-gap contribution is dominant over the exponentially temperature-dependent Coulomb contribution at sufficiently low temperatures.

The Coulomb contribution can be isolated in thin films by the application of a magnetic field to suppress localization effects. One must keep in mind that the magnetic field will act as an ergodic pair breaker and fits to the broadened DOS must include the effects of both the magnetic field and inelastic lifetime. One can distinguish the two pair breakers by noting that any nonzero inelastic-scattering rate will render a superconductor gapless, while the strength of an ergodic pair breaker must exceed a certain threshold in order to cause gaplessness.[35]

Unfortunately, there does not exist a large body of experimental data concerning the inelastic-scattering rate in either weakly or strongly coupled disordered superconductors. The most extensive experimental investigation to date has been carried out by Pyun and Lemberger.[17] They measured Γ through the broadening of the DOS in disordered InO_x thin films whose dimensionality lies between 2D and 3D. By varying the thickness and disorder of their films, they were able to obtain a disorder- and temperature-dependent scattering rate.

Using Eqs. (5.1) and (5.4), we have compared our theory with the experimental values. Thereby we are faced with the problem that many of the parameters that enter into these expressions are not known for indium oxide. The degree of disorder in these films is controlled by the amount of oxygen present, and it is expected that the parameters are affected as the oxygen content; thus the electron concentration is varied from film to film. We use parameters as given by the experimentalists if possible, and for the unknown parameters we choose values corresponding to clean indium. Thus we set $k_F = 2.3\ \mathrm{nm}^{-1}$, as given in Ref. 17, and set $\lambda = 0.8$, $c_L = 2460$ m/s, $c_T = 710$ m/s, and $\omega_D = 108$ K. In order to have only one disorder parameter ρ, we choose the ratio of E_F/Δ to be fixed and equal to 329, which is reasonable for a material with such a low electron concentration. This assumption is not based on any theoretical considerations, but is motivated to simplify a disorder analysis of the experimental values. While this should not affect the results obtained about the temperature dependence of τ_{in}^{-1}, it is not possible with the information at hand to reliably test the disorder dependence of Eqs. (5.1) and (5.4) against these data. Finally, we need the dimensionless residual resistivity $\hat{\rho}$. Reference 17 gives $\rho_{4.2}$, the resistivity at 4.2 K. We write the T-dependent conductivity as $\sigma(T) = \sigma_0 + \delta\sigma(T)$, where $\sigma_0 = 1/\rho_0$ is the residual conductivity and $\delta\sigma(T) \to 0$ for $T \to 0$. The metal-insulator transition for this material occurs at $\rho_{4.2} = 9\ \mathrm{m}\Omega\,\mathrm{cm}$.[17] This gives the following relation between $\hat{\rho}$ and $\rho_{4.2}$:

$$\hat{\rho} = (\rho_{4.2}/\rho_M)[1 - \rho_{4.2}/9\ (\mathrm{m}\Omega\,\mathrm{cm})]^{-1} \ . \tag{6.1}$$

For ρ_M we choose 1.85 mΩ cm, which is again a reasonable value.

With these values the theory yields the solid lines in Figs. 1 and 2. Since InO_x is strongly coupled ($\Delta/T_c \simeq 2.2$), the correlation-gap contribution to Γ is greater than the Coulomb contribution and thus the power-law temperature dependence should be dominant. Therefore, we first analyze the temperature dependence of $1/\tau$ in Fig. 1 by normalizing $1/\tau$ by the disorder dependence F of the contribution due to interference effects [Eq. (5.4)]. Setting $\mu = \frac{1}{2}$, the disorder parameter F is given by

$$F = \left[1 + 3\frac{8\lambda - 3}{\pi}\hat{\rho} \right] \sqrt{\hat{\rho}} \ . \tag{6.2}$$

If the films were all of the same dimensionality, the theory lines in Fig. 1 would be indistinguishable from each other regardless of disorder. The family of lines indicates that these films are indeed between asymptotically 2D behavior (which is given by the upper dashed line) and asymptotically 3D behavior (the lower dashed line). The agreement with the data is quite favorable. Figure 1 shows that the power-law temperature dependence predicted by the theory compares well with experiment. Further, the magnitude and disorder dependence of the rates also compare quite favorable with theory, within the assumption made to fix the ratio of the Fermi energy to the gap, Δ. This is shown in Fig. 2, which plots experimental data and our theoretical result for the six data samples of Ref. 17 separately.

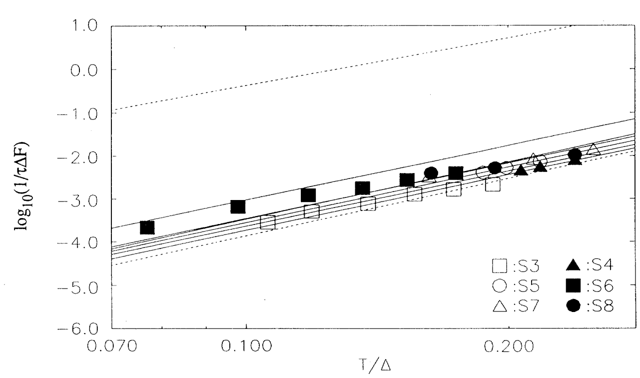

FIG. 1. Comparison of experimental data and the sum of Eqs. (5.1) and (5.4), normalized by the disorder dependence F as defined in text. The upper dashed line corresponds to the theoretical τ_{in}^{-1} for an asymptotically 2D sample, while the lower dashed line corresponds to a bulk sample. Data points are taken from Ref. 17.

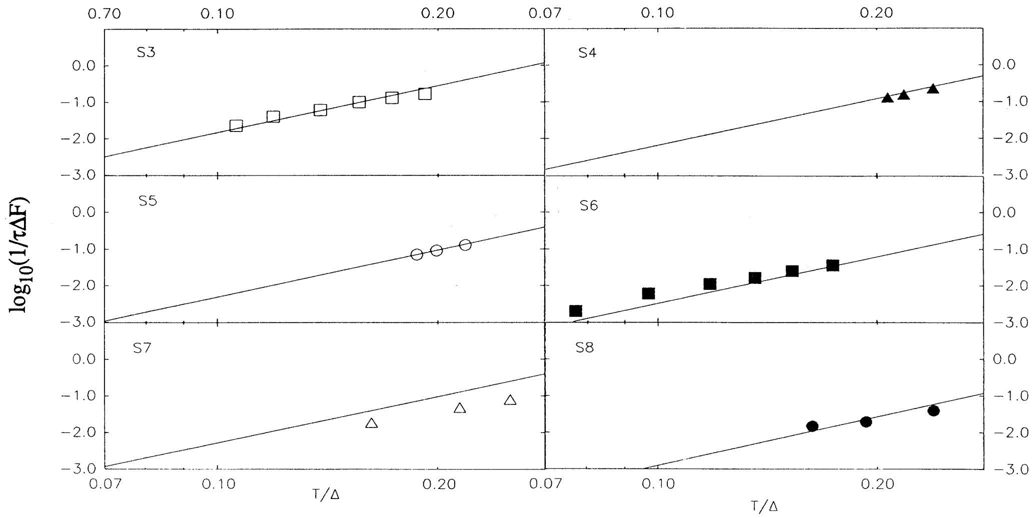

FIG. 2. Separate comparison of our theory with experimental data points taken from Ref. 17.

The disorder dependence of $\tau_{\rm in}^{-1}$ can be better checked against the purely 2D data on Sn films by White, Dynes, and Garno[16] and the purely 3D data on aluminum by Dynes *et al.*[15] These experiments were performed on granular materials, and the amount of disorder was controlled by varying the thickness of their films. Therefore, one might expect that the parameters entering Eqs. (5.1) and (5.4) remain relatively unchanged. The theoretical rate has a disorder dependence that is consistent with both sets of data, namely, $R_\square$ in 2D and $\rho^{3/2}$ in 3D. However, the theory cannot account for the large magnitude of the measured rates, which are larger than the sum of Eqs. (5.1) and (5.4) by a factor of 10 in Sn and larger still in Al. We note, however, that a deviation of, e.g., c_L/c_T from its clean value by a factor less than 2 would account for this discrepancy in Sn. The temperature dependence of $\tau_{\rm in}^{-1}$ in these materials has not been measured.

The experiments of Ref. 15 were performed on aluminum samples that were very close to a metal-insulator transition. The mechanism for the breakdown of superconductivity in this region is currently not well understood, and our theory is expected not to apply there. From a theory for $\Gamma_{\rm in}$ in normal metals,[36] we expect the theory to be valid for $\hat{\rho} \lesssim 10$ and $\Gamma_{\rm in}$ to saturate for larger $\hat{\rho}$. The latter behavior was indeed observed in Ref. 15. We believe that nonequilibrium effects might be important in this case. This is beyond the scope of the present paper. Our treatment only considers the case when the number of injected quasiparticles is small compared to the number of thermal-equilibrium quasiparticles. This is not valid at very low temperatures and/or high injection current densities. If the number of excess quasiparticles is not small, then the lifetime will be reduced and is expected to have quite a different behavior than the one obtained in perturbation theory. The temperature dependence of the rate is expected to be most affected by these possible nonequilibrium effects. This will be most pronounced in 2D because of the higher quasiparticle injection densities[37] and also the possibility of Joule heating.[38] One would have to use nonequilibrium distribution functions in order to treat this case properly.[39] This remains to be explored in the future.

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APPENDIX: A SELF-CONSISTENT TREATMENT OF THE COULOMB CONTRIBUTION

In this appendix we describe the self-consistent calculation mentioned at the end of Sec. III. The basic idea is that the scattering rate Γ should be included in the Coulomb propagator and in the Green's functions in Eq. (3.8). (Γ must not be included in the Kubo function which by definition is a correlation function for the noninteracting electron system.) To keep things simple, we approximate the effect of $\Gamma \neq 0$ on the Green's functions by the substitution $\omega \rightarrow \omega - i\Gamma$. For the Coulomb propagator, we focus on the recombination B_r'' [Eq. (3.6a)]. B_r'' yields the leading contribution to the scattering rate, and B_s'' [Eq. (3.6b)] shows less structure near the gap and will be less affected by Γ. We make the above replacement in Eq. (2.21) of Ref. 19 and find, for $Dq^2 \gg \sqrt{\omega(\omega-2\Delta)}$,

$$C_r''(q,\omega) = -(\pi\Delta N_f/2Dq^2) \times \mathrm{Im}\left[P_{-1/2}\left[1-\frac{(\omega-i\Gamma)^2}{2\Delta^2}\right] - P_{1/2}\left[1-\frac{(\omega-i\Gamma)^2}{2\Delta^2}\right]\right], \quad \text{(A1)}$$

where the $P_{\pm 1/2}$ denote Legendre functions. For $\Gamma \to 0$, Eq. (A1) reduces to Eq. (3.6a).

We use these expressions in Eq. (3.8) and ignore the frequency dependence of Γ on the right-hand side. We find that for weak inelastic scattering $\gamma \equiv \Gamma_C/2\Delta \ll 1$, the rate for a quasiparticle at the gap edge is given by the solution of the transcendental equation

$$\gamma = \gamma_0 - \frac{2\gamma_0\gamma}{\pi}\ln\gamma \ , \quad \text{(A2)}$$

where γ_0 is given by Eq. (3.12b) normalized by 2Δ. For larger values of γ, we have solved Eq. (3.8) numerically by iteration. We find that the self-consistency leads to a somewhat weaker temperature dependence of the rate, but for $\gamma_0 \ll 1$, the rate is well described by the perturbative result.

For larger values of γ, self-consistency raises the value of the rate for low temperatures and reduces it for higher temperatures until one reaches the region where $\gamma \simeq \frac{1}{2}$. In this region both the interaction and superconducting density of states approach their normal-state behavior as the gap becomes more and more smeared. The rates then rise sharply and approach their normal-state limits, and the iteration procedure takes longer and longer to converge. However, this occurs only for values of the disorder parameters which physically correspond to a superconductor well within the region of a localization transition to an insulator, a region which the present theory is inadequate to describe. We therefore conclude that the perturbation-theory expression for the scattering rate is sufficient unless the disorder is so large that one approaches the metal-insulator transition.

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[25]An easy way to estimate the error in this approximation is as follows. We break up the wave-number integral in Eq. (3.4) into the regions $q\xi < 1$ and $q\xi > 1$. In the former region, we use the gauge-invariant potential for $q\xi \ll 1$ from Ref. 19, and in the latter we use the pair approximation. We then find for the lifetime the result expressed in Eq. (3.10), modified by a factor of $\frac{1}{2}\{1+[(4/\pi)\arctan g(\pi/2)-1]/(\pi-1) -(A/10)[1/\ln(\Delta/T)+\frac{1}{2}\mathrm{li}\sqrt{T/\Delta}]\}$, where li is the logarithm integral and A is a number of order unity. The gauge-invariant potential thus yields only logarithmic corrections to an exponential temperature dependence.

[26]Since our approximation for the Coulomb potential [Eq. (3.7)] differs slightly from the one used in Ref. 18, the screening factor F given in Eq. (13) of Ref. 18 must be replaced by $(\pi^{3/2}-1)/(\pi^2-1)$ to obtain the present result.

[27]Note that Eqs. (3.9) and (3.10) do not recover the normal-metal result in the limit $(\Delta/\omega) \to 0$, since the integrand in Eq. (3.8) was expanded for frequencies near 3Δ. In order to recover the normal-metal limit, one must perform the integrals in Eq. (3.8) exactly. This can be done, but little information is gained thereby, and we therefore do not give the result.

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$$L(x,y)=\Theta(1-x)(1-(1/x^2)\sin^{-1}\{x^2[1-(4/\pi)y^2]\})+(x^2/4)[1-(2/\pi)\sin^{-1}(1-1/2y^2)]\ .$$

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