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Maximum-entropy model for quantum-mechanical interference effects in metallic conductors

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We show that all the known quantum-mechanical interference effects characteristic of disordered conductors in the metallic limit can be obtained from a "maximum-entropy model," based on a transfer-matrix formulation, which is *independent* of any particular form of the disordered microscopic Hamiltonian. In particular, we have derived the weak-localization effect and the associated backscattering peak, as well as the universal conductance fluctuations and the associated long-range correlations in transmission probabilities. We find precise quantitative agreement with microscopic Green-function calculations evaluated in the quasi-one-dimensional limit. We define two random-matrix ensembles characterizing systems with and without time-reversal symmetry (the analogs of the well-known orthogonal and unitary ensembles), and show that, within the model, breaking of time-reversal symmetry has the expected effect on these phenomena. The model has not been shown to yield behavior characteristic of the two- or three-dimensional limits, which appear to be outside its current range of validity.

I. INTRODUCTION

The basic physical assumption underlying the scaling approach to disordered conductors[1,2] is that the quantum-transport properties of systems probed at length scales much longer than the elastic mean free path l should be insensitive to the microscopic origin of the disorder that causes the elastic scattering. It was therefore natural to seek a formulation of the theory that is relatively independent of the particular model of the disordered Hamiltonian, as was done by Anderson *et al.*,[3] who introduced a transfer-matrix formulation based on the Landauer approach[4] and then obtained its statistical and scaling properties by means of a *random-phase hypothesis.* This random-phase model was very successful in describing the purely one-dimensional (1D) limit; but this limit does not allow the study of the metallic regime in which the sample length L is much longer than the elastic mean free path l, but shorter than the localization length ξ (the two lengths are equal[5] in 1D). Quantitative extensions of the approach to the experimentally relevant metallic regime have not been made, and no such model has been found to be analytically tractable. Quantitative predictions for the behavior of disordered conductors in the metallic regime have been made instead using the impurity-averaged Green-function technique,[6-15] a diagrammatic expansion in the small parameter $(k_F l)^{-1}$ (where k_F is the Fermi wave vector), or from numerical simulations.[16,17,9,10,14,15]

A noteworthy effect predicted by these techniques was the weak-localization effect on the average conductance, which appears both as a temperature-dependent decrease in the Boltzmann conductivity and as a low-field negative magnetoresistance.[6,7] More recently, fluctuation phenomena have been widely observed in the conductance of small metal samples at low temperature. In particular, time-independent "reproducible noise" in the resistance as a function of magnetic field or Fermi energy[18,19] and a type of low-frequency noise[13,14] have been observed. The theory of these effects[8-12] made the striking prediction that the variance of the conductance fluctuations is always of order $(e^2/h)^2$ when the sample is measured on the scale of the phase-coherence length, $L_{\rm in}=\sqrt{D}\,\tau_{\rm in}$, i.e.,

$$\mathrm{Var}(g)\sim 1 \ , \tag{1.1}$$

where g is the dimensionless conductance (measured in units of e^2/h), D is the elastic diffusion constant, and $\tau_{\rm in}^{-1}$ is the elastic scattering rate. The exact value of $\mathrm{Var}(g)$ depends on the measuring geometry and can be calculated in simple cases.[9,11,12] Both these "universal conductance fluctuations" (UCF) and the weak-localization correction to the conductance can be interpreted as interference effects arising from a particular subset of electronic paths which form closed loops while propagating through the sample.[7,20,14]

In the case of UCF, it was explicitly shown that diagrammatic and numerical calculations based on different disordered microscopic Hamiltonians gave exactly the same values for the variance of the conductance,[9] again strongly suggesting the possibility of formulating a theory of these phenomena which is independent of a particular choice for the disordered Hamiltonian. Recently two such theories have been developed on the basis of "maximum-entropy models." One approach, which we have termed *global,* treats the conductor as a whole; the other, which we develop further below is termed *local* be-

cause it is based on an ansatz for the behavior of small segments of the conductor. The local approach presented below builds on the original random-phase model of Anderson *et al.*,[3] and was developed by Mello and coworkers;[21–24] the global approach[25–29] built on initial work of Imry[25] and Muttalib, Pichard, and Stone,[26] and emphasizes the relationship between UCF and the theory of random matrices developed by Wigner, Dyson, and others.[30–35] Very recently, Mello and Pichard[36] have proved the equivalence of the two approaches in the large-N limit (where N is the number of channels), an important limit studied in detail below.

Our starting point is very similar to that of Anderson *et al.*[3] In Sec. II we consider a disordered conductor of fixed width W and variable length L, described by a *transfer matrix* which is the product of many (random) transfer matrices determining the wave propagation through each cross section of the conductor of microscopic length. Fixing the width defines a discrete number N of propagating "channels" at the Fermi energy, and we will be interested in the multichannel case ($N \gg 1$) as opposed to the purely 1D case ($N=1$). We note that even the narrowest metal wires studied have $N \sim 10^4 - 10^5$ (although semiconducting microstructures are often in the few-channel limit). The conductor in our model can be scaled in the quasi-one-dimensional sense simply by increasing the length of the random product of transfer matrices. The conductance g of the system (in the two-probe limit) is a simple function of the parameters specifying the full transfer matrix, allowing a natural definition of $g(L)$ for each impurity configuration. The essence of the approach is to factorize the transfer matrix in a manner which separates out the "phase" factors and the "intensity" factors, where the conductance can be shown to depend only on the latter. As we shall see below, in the multichannel theory, the phase factors correspond to unitary matrices, whereas the intensity factors correspond roughly to the eigenvalues of $\underline{M}\underline{M}^\dagger$, where $\underline{M}$ is the transfer matrix. In Sec. III we define two random-matrix ensembles, characterizing systems with and without time-reversal symmetry. The statistical distribution associated with systems of very small length is then selected on the basis of a *maximum-entropy* criterion; the "evolution" with length of that distribution is governed by a Fokker-Planck or diffusion equation in N dimensions. We shall see that the maximum-entropy "ansatz" implies that the above-mentioned phases are uniformly distributed (we shall refer to such a distribution as "isotropic"), so that they can be averaged over with the appropriate invariant measure, leaving a tractable theory for the statistical behavior of the conductance and related quantities.

It is well understood from the theory of random matrices that if one chooses the distribution of a set of matrices to be "as random as possible" (in the sense of maximum-information entropy), then its eigenvalues will be highly correlated statistically.[37,30] The crucial new feature of our approach as compared to the original work of Ref. 3 is that we treat correctly the statistical correlations of the eigenvalues determining g, which are essential for producing the correct statistical behavior of the conductance. A number of authors have recently stressed that the underlying origin of UCF is the eigenvalue correlations ("spectral rigidity"[30]) characteristic of random-matrix ensembles;[11,25,26] here we combine that insight with the random-phase model to arrive at a computationally tractable theory of this and other effects independent of a particular microscopic Hamiltonian.

In Sec. IV we shall see that just from the isotropy property of the statistical distribution we can obtain the structure of the averages and correlations of transmission and reflection coefficients, as a function of the channel indices. Specific values for the various coefficients require the solution of the diffusion equation, which is found in the metallic regime in Sec. V. In that section we also calculate the average conductance, and show that the weak-localization correction comes out properly. The variance is also computed, and found to be a pure number, independent of size and degree of disorder as expected from Eq. (1.1).

The original calculations of UCF which arrived at Eq. (1.1) were based on the Kubo formulation of linear-response theory applied to a finite disordered conductor connected to two semi-infinite ordered leads maintained at a fixed voltage difference. It can be shown[38] that the conductance defined in this manner is exactly equal to the sum of all the transmission coefficients relating the incident flux in channel a on one side to the outgoing flux in channel b,

$$g = \mathrm{Tr}[\underline{t}\underline{t}^\dagger] = \sum_{a,b} T_{ab} \equiv T \tag{1.2}$$

where $\underline{t}$ is the transmission matrix. We refer to this expression as the *two-probe Landauer formula* because it relates the conductance of the system to the total scattering matrix of the system, an approach pioneered by Landauer;[4] this formula was first proposed by Fisher and Lee[38] and differs from the original Landauer formula.[4] It is now understood[39,40] that such a model corresponds to an ideal two-probe measurement, in which the electrochemical potential difference is measured between reservoirs serving as current source and sink. However, the majority of experiments on these systems are not performed in this way, but are instead done in a four-probe configuration in which a current is injected from a source into a sink, and the voltage induced across the sample is measured by separate voltage probes. It is now clear that the behavior in this situation can differ significantly from that predicted by Eq. (1.1), when the voltage probes are spaced much less than an inelastic scattering length apart, and an adequate extension of the theory,[40] based upon Büttiker's multiprobe generalization of Landauer's formula[41,48] has been developed. However, for the general questions addressed here relating to the minimal physical assumptions necessary to generate the phenomena of weak localization and UCF, we do not need to consider the additional complications introduced by the multiprobe theory. We nonetheless emphasize the necessity of using the multiprobe theory if one wishes to obtain quantitative agreement (or even in some cases qualitative agreement) with many experiments.

II. THE SCATTERING APPROACH TO DISORDERED CONDUCTORS: THE TRANSFER MATRIX

Employing the two-probe model described above, we imagine the disordered system of interest to be placed between two semi-infinite perfectly conducting leads of finite width. We assume the existence of boundary conditions at the transverse surfaces which quantize the energy of the transverse part of the wave function; the theory is not sensitive to the detailed nature of the boundary conditions and so we take them to be infinite hard walls for definiteness. Then, in the perfect conductors, the scattering states at the Fermi energy satisfy the relation $k_F^2=k_n^2+k^2$, where k_F is the Fermi momentum, k the longitudinal momentum, and k_n the quantized transverse momentum. The various k_n $(n=1,2,\ldots,N)$, which satisfy this relation such that $k^2>0$, define the N channels. Since each channel can carry two waves traveling in opposite directions, the wave function on either side of the disordered region is specified by a $2N$-component vector; the first N components are the amplitudes of the waves propagating to the right, and the remaining N components are the amplitudes of the waves traveling to the left. The scattering matrix $\underline{S}$ relates the incoming flux to the outgoing flux

$$\underline{S}\begin{bmatrix} I \\ I' \end{bmatrix}=\begin{bmatrix} O \\ O' \end{bmatrix} \tag{2.1}$$

where I,O,I',O' are the N-component vectors describing the wave amplitudes on the left and right, respectively. In this quasi-one-dimensional geometry, the S matrix is a $2N\times 2N$ matrix of the form

$$\underline{S}=\begin{bmatrix} \underline{r} & \underline{t} \\ \underline{t}' & \underline{r}' \end{bmatrix}, \tag{2.2}$$

where $\underline{t}$ is the transmission matrix which yields the conductance in Eq. (1.1) and $\underline{r}$ is the reflection matrix. Henceforth 2×2 matrices with underlined entries denote $2N\times 2N$ matrices divided into $N\times N$ blocks. Current conservation implies that

$$|I|^2+|I'|^2=|O|^2+|O'|^2 , \tag{2.3}$$

which is equivalent to the unitarity of the $\underline{S}$ matrix.

Although the $\underline{S}$ matrix determines the conductance through Eq. (1.1), it does not satisfy a simple composition rule suitable for introducing a scaling approach. Therefore we instead consider the transfer matrix which contains the same information in a different form.

By definition, the $2N\times 2N$ transfer matrix relates the flux amplitudes on the left-hand side of the disordered region to those on the right,

$$\underline{M}\begin{bmatrix} I \\ O \end{bmatrix}=\begin{bmatrix} O' \\ I' \end{bmatrix} . \tag{2.4}$$

(See Fig. 1.) Just as with $\underline{S}$, we can write $\underline{M}$ in terms of four $N\times N$ blocks α,β,γ, and δ, so that

$$\underline{M}=\begin{bmatrix} \underline{\alpha} & \underline{\beta} \\ \underline{\gamma} & \underline{\delta} \end{bmatrix} \tag{2.5}$$

and from the definitions (2.2) and (2.5) one finds the relations

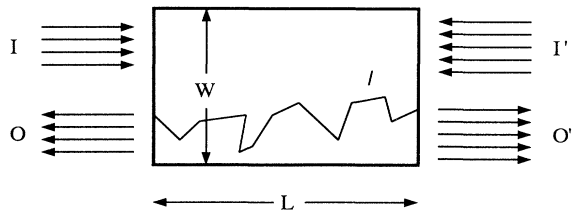

FIG. 1. Schematic diagram indicating the action of the transfer matrix $\underline{M}$ which connects incident flux I and outgoing flux O on the left-hand side of the sample to the same quantities O',I' on the right-hand side. The sample has width W and length L, and the number of channels $N\sim(k_F W)^{d-1}$; this is the number of components to the vectors I,O,I',O'. Below we characterize the transfer matrix of a small segment solely by the value of the electron elastic mean free path l, indicated by the schematic of an electron traversing the sample in random steps of this length.

$$\underline{\alpha}=(\underline{t}^{\dagger})^{-1},\quad \underline{\beta}=\underline{r}'(\underline{t}')^{-1} ,$$
$$\underline{\gamma}=-(\underline{t}')^{-1}\underline{r},\quad \underline{\delta}=(\underline{t}')^{-1},\quad \underline{t}=(\underline{\alpha}^{\dagger})^{-1} , \tag{2.6}$$

$$\underline{r}=-\underline{\delta}^{-1}\underline{\gamma} . \tag{2.7}$$

The flux conservation constraint on the transfer matrix is easily found by reexpressing Eq. (2.3) in the form

$$|I|^2-|O|^2=|O'|^2-|I'|^2 , \tag{2.8}$$

which means from Eq. (2.4) that $\underline{M}$ preserves the *hyperbolic* norm of the vector $\begin{pmatrix} I \\ O \end{pmatrix}$. This defines a $U(N,N)$ (pseudounitary) matrix. An alternative way of expressing the flux conservation constraint on $\underline{M}$ is

$$\underline{M}^{\dagger}\underline{\Sigma}_z\underline{M}=\underline{\Sigma}_z . \tag{2.9}$$

Here Σ_z denotes the matrix

$$\underline{\Sigma}_z=\begin{bmatrix} \underline{1} & \underline{0} \\ \underline{0} & -\underline{1} \end{bmatrix} , \tag{2.10}$$

$\underline{1}$ and $\underline{0}$ designating the $N\times N$ unit and zero matrices. Thus the requirement of flux conservation implies that the transfer matrices $\underline{M}$ form a pseudounitary group $U(N,N)$; $\nu=(2N)^2$ is the number of independent parameters specifying such a matrix, just as for the more familiar unitary group $U(2N)$.

In Ref. 26 it was shown that by performing a unitary transformation on each $\underline{M}$ and imposing time-reversal symmetry it was possible to map the set of transfer matrices onto the real symplectic group $Sp(2N,\mathbb{R})$. This mapping allowed a minor simplification in imposing time-reversal symmetry, but is not particularly useful in the general case considered here, and will not be used below.

If our transfer matrices $\underline{M}$ do obey time-reversal symmetry then (neglecting spin) they also satisfy the requirement

$$\underline{M}^{*}=\underline{\Sigma}_x\underline{M}\underline{\Sigma}_x , \tag{2.11}$$

where

$$\underline{\Sigma}_x = \begin{bmatrix} \underline{0} & \underline{1} \\ \underline{1} & \underline{0} \end{bmatrix} . \tag{2.12}$$

If a magnetic field is present, time-reversal symmetry is broken and our transfer matrices satisfy the requirement (2.9) alone.

Any matrix $\underline{M}$ satisfying the flux-conservation requirement (2.9) can be parametrized as

$$\underline{M} = \begin{bmatrix} \underline{u}^{(1)} & \underline{0} \\ \underline{0} & \underline{u}^{(3)} \end{bmatrix} \begin{bmatrix} (\underline{1}+\underline{\lambda})^{1/2} & \underline{\lambda}^{1/2} \\ \underline{\lambda}^{1/2} & (\underline{1}+\underline{\lambda})^{1/2} \end{bmatrix} \times \begin{bmatrix} \underline{u}^{(2)} & \underline{0} \\ \underline{0} & \underline{u}^{(4)} \end{bmatrix} \equiv \underline{U}\,\underline{\Gamma}\,\underline{V} , \tag{2.13}$$

where $\underline{u}^{(i)}$ ($i=1,2,3$, and 4) are arbitrary $N\times N$ unitary matrices and $\underline{\lambda}$ is a real, diagonal matrix with nonnegative elements $\lambda_1,\lambda_2,\ldots,\lambda_N$.

Note that the N real parameters $\{\lambda_n\}$ are *not* the eigenvalues of the transfer matrix; however, it can be shown that they play an analogous role to the N real eigenvalues of Hermitian matrices (the more frequently studied case) in that they are the natural parameters appearing in the invariant measure (see Appendix A and Ref. 21). Alternatively, it can be shown from Eq. (2.13) that $\{\lambda_n\}$ are the N nondegenerate eigenvalues of the $2N\times 2N$ matrix $\underline{Q}=\frac{1}{4}[\underline{M}^{\dagger}\underline{M}+(\underline{M}^{\dagger}\underline{M})^{-1}-\underline{2}]$. This connection is used in the global approach, where the relationship of this ensemble to the Wigner-Dyson random-matrix ensembles was emphasized.

The time-reversal symmetry requirement (2.11) imposes the additional constraints

$$\underline{u}^{(3)}=(\underline{u}^{(1)})^{*},\quad \underline{u}^{(4)}=(\underline{u}^{(2)})^{*} . \tag{2.14}$$

In this latter case, $\underline{u}^{(1)}$ and $\underline{u}^{(3)}$ give rise to N^2 parameters each, and $\underline{\lambda}$ to N additional ones, in agreement with the expected number of parameters, $\nu'=N(2N+1)$. If all the λ_n are different, the parametrization (2.13) is unique, except for a set of zero measure. However, in the absence of time-reversal symmetry, (2.13) contains $4N^2+N$ parameters, i.e., N more than is needed to specify an arbitrary matrix with $\mathrm{U}(N,N)$ symmetry. This redundancy of the parametrization arises because a matrix $\underline{M}$ parametrized as in (2.13) is unchanged if the unitary matrices appearing in the parametrization are subject to the transformation

$$\underline{U}\to\underline{U}\,\underline{G},\quad \underline{V}\to\underline{G}^{-1}\underline{V} , \tag{2.15}$$

where $\underline{G}$ is the diagonal "phase matrix"

$$\underline{G} = \begin{bmatrix} \underline{\mathscr{L}}^{(\eta)} & \underline{0} \\ \underline{0} & \underline{\mathscr{L}}^{(\eta)} \end{bmatrix} \tag{2.16}$$

where $(\underline{\mathscr{L}}^{(\eta)})_{ij}=\delta_{ij}\exp[i\eta_j]$. Up to the N phases involved in (2.16), the parametrization (2.13) is unique, except for a set of zero measure (if all the λ_n are distinct). The invariance of $\underline{M}$ under (2.15) could be used to eliminate N parameters in the $\underline{u}^{(i)}$ of (2.13). However, we shall choose the alternative of keeping the generality of (2.13) and adding N trivial phase parameters to those needed to specify an arbitrary matrix in $\mathrm{U}(N,N)$. This is a convenient procedure because *any* sensible probability distribution for the $\underline{M}$ matrices must be invariant under the transformation (2.15) and therefore the N additional phases in (2.16) will drop out of the calculation of any statistical quantity just by normalization of the distribution. Thus the parametrization (2.13) will be used systematically throughout the present paper.

Having established our canonical parametrizations, we can express the transmission and reflection matrices and other related quantities in terms of these parameters using Eqs. (2.6) and (2.7). One then finds for the reflection and transmission matrices

$$\underline{r}=-\underline{\delta}^{-1}\underline{\gamma}=-(\underline{u}^{(4)})^{\dagger}[\underline{\lambda}(\underline{1}+\underline{\lambda})^{-1}]^{1/2}\underline{u}^{(2)} , \tag{2.17}$$

$$\underline{t}=(\underline{\alpha}^{\dagger})^{-1}=\underline{u}^{(1)}(\underline{1}+\underline{\lambda})^{-1/2}\underline{u}^{(2)} . \tag{2.18}$$

The reflection and transmission coefficients R_a, and T_a into channel a, when the channels are fed from the left with N incoherent unit fluxes, are then given by

$$R_a=(\underline{r}\underline{r}^{\dagger})_{aa}=[\underline{u}^{(4)\dagger}\underline{\lambda}(\underline{1}+\underline{\lambda})^{-1}\underline{u}^{(4)}]_{aa} = \sum_b |u^{(4)}_{ba}|^2\frac{\lambda_b}{1+\lambda_b} , \tag{2.19}$$

$$T_a=(\underline{t}\underline{t}^{\dagger})_{aa}=[\underline{u}^{(1)}(\underline{1}+\underline{\lambda})^{-1}\underline{u}^{(1)\dagger}]_{aa} = \sum_b \frac{|u^{(1)}_{ab}|^2}{1+\lambda_b} . \tag{2.20}$$

The total reflection and transmission coefficients (for *one* direction of spin) are then given by

$$R=\sum_a R_a=\sum_a \frac{\lambda_a}{1+\lambda_a} , \tag{2.21}$$

$$T=\sum_a T_a=\sum_a \frac{1}{1+\lambda_a} . \tag{2.22}$$

The two-probe conductance g, given by Eq. (1.2), is exactly equal to T, the total transmission coefficient for *both* spin directions; as we will not be considering spin-dependent scattering below, spin degeneracy only contributes a trivial factor of two to Eq. (2.22); so for convenience we will henceforth consider the case of spinless particles. Spin-dependent scattering, leading to the analog of the symplectic ensemble,[34] has been treated recently by Zanon and Pichard.[27] Equations (2.19)–(2.22) represent the connection between the parametrization of the transfer matrices introduced in Eq. (2.13) and physically measurable quantities. Note the crucial fact that the conductance depends only on the real parameters $\{\lambda_n\}$, and is independent of the matrix elements of the unitary matrices $\underline{u}^{(i)}$. Thus all $\underline{M}$ matrices with the same values of $\{\lambda_n\}$ have the same value of g, naturally suggesting that we consider probability distributions for the $\underline{M}$ matrices in which all $u^{(i)}$ are equally probable, i.e., which depend on the $\underline{u}^{(i)}$ only through their invariant

measure. We introduce such a statistical ensemble in the next section.

III. THE ENSEMBLE OF TRANSFER MATRICES

We first define the invariant measure (or Haar's measure) $d\mu(\underline{M})$ on our group of $\underline{M}$ matrices: this concept will be essential in the analysis that follows. By definition, the measure $d\mu(\underline{M})$ remains invariant when all the $\underline{M}$'s are multiplied by a fixed one $\underline{M}_0$. We shall express $d\mu(\underline{M})$ in terms of the parameters of Eq. (2.13); as in the classic random matrix ensembles,[34] the cases with and without time-reversal symmetry can be treated at the same time, by introducing a parameter β that takes on the values 1 and 2 in the two cases, respectively. The case $\beta=1$ is studied in Ref. 21 and case $\beta=2$ is outlined in Appendix A. The result is

$$d\tilde{\mu}(\underline{M})\equiv d\mu(\underline{M})[d\mu(\underline{G})]^{\beta-1} = J_\beta(\underline{\lambda})\prod_a d\lambda_a \prod_{i=1}^{2\beta} d\mu(\underline{u}^{i}) , \tag{3.1a}$$

$$J_\beta(\underline{\lambda})=\prod_{a<b}|\lambda_a-\lambda_b|^\beta . \tag{3.1b}$$

Here, $d\mu(\underline{G})$ is relevant for $\beta=2$ and, in terms of the phases of (2.16), is given by

$$d\mu(\underline{G})=(2\pi)^{-N}\prod_{a=1}^{N} d\eta_a . \tag{3.2}$$

We now consider a collection or ensemble of random conductors of *macroscopic* length L, and describe it in terms of an ensemble of $\underline{M}$ matrices defined by means of a differential probability $dP_L(\underline{M})$. For $\beta=2$, as discussed above, in order to use the parametrization (2.13) it is useful to add the redundant phase variables $\eta_1,\ldots,\eta_N$ of (2.16) in the analysis, even though $dP_L(\underline{M})$ never involves them. We thus define

$$d\tilde{P}_L(\underline{M})\equiv dP_L(\underline{M})[d\mu(\underline{G})]^{\beta-1} \tag{3.3}$$

which we split as

$$d\tilde{P}_L(\underline{M})=p_L(\underline{M})d\tilde{\mu}(\underline{M}) , \tag{3.4}$$

where the probability density $p_L(\underline{M})$ is independent of η_a and $d\tilde{\mu}(\underline{M})$ is given by (3.1).

The probability density $p_L(\underline{M})$ must satisfy the following *combination requirement.* Suppose that we put together two wires of lengths L' and L'' and transfer matrices $\underline{M}',\underline{M}''$: the resulting length and transfer matrix are $L=L''+L'$ and $\underline{M}=\underline{M}''+\underline{M}'$, respectively. If we designate by $p_{L'}(\underline{M}')$ and $p_{L''}(\underline{M}'')$ the two respective probability densities and assume $\underline{M}',\underline{M}''$ to be statistically independent, the resulting probability density is given by the "convolution"

$$p_{L''+L'}(\underline{M})=\int p_{L''}(\underline{M}\,\underline{M}'^{-1})p_{L'}(\underline{M}')d\tilde{\mu}(\underline{M}') . \tag{3.5}$$

In other words, $p_L(\underline{M})$ must be *reproducible under convolutions:* the convolution of two p_L's is another p_L, whose index L is the sum of the two indices. We shall see that his requirement imposes severe restrictions on our probability density.

In Eq. (3.5) let us set $L''=L$, $L'=\delta L$, considering δL as a small, but still macroscopic, length. We have

$$p_{L+\delta L}(\underline{M})=\int p_L(\underline{M}\underline{M}'^{-1})p_{\delta L}(\underline{M}')d\tilde{\mu}(\underline{M}') . \tag{3.6}$$

If we knew $p_{\delta L}(\underline{M}')$, we could construct $p_L(\underline{M})$ for arbitrary lengths by successive convolutions: $p_{\delta L}(\underline{M}')$ is thus our *building block.*

In a complete theory, we should be able to impose the appropriate physical requirements that would allow a unique determination of the building block. Since at the present stage we have not reached that goal yet, we propose an "ansatz" for $p_{\delta L}(\underline{M}')$. We choose the statistical distribution that maximizes Shannon's *information entropy*

$$S[p_{\delta L}]=-\int p_{\delta L}(\underline{M}')\ln[p_{\delta L}(\underline{M}')]d\tilde{\mu}(\underline{M}') , \tag{3.7}$$

constrained by the condition that $p_{\delta L}$ be normalized and that the average

$$\frac{N^{-1}\langle \mathrm{tr}\underline{\lambda}'\rangle_{\delta L}}{\delta L}\equiv\frac{1}{l} \tag{3.8}$$

be fixed. It follows from (2.21) that for small $\{\lambda_n'\}$; $N^{-1}\mathrm{tr}\underline{\lambda}'$ is the total reflection coefficient (summed over the exit channels and averaged over the incident ones); thus (3.8) represents the reflection probability per unit length, which is the inverse of the mean free path l for backward scattering.

The resulting distribution can be written as

$$p_{\delta L}(\underline{M}')=\exp(\mu-\nu\mathrm{tr}\underline{\lambda}') , \tag{3.9}$$

where μ,ν are Lagrange multipliers. It represents, for small lengths, an ensemble of $\underline{M}$ matrices that is as *random as possible* given the constraint (3.8) and the normalization condition.

The building block (3.9) is *isotropic,* i.e., independent of the unitary matrices $\underline{u}^{(i)}$ of (2.13). The proof given in Ref. 21 that the convolution of two isotropic functions is again isotropic is independent of whether we have time-reversal symmetry or not. It implies that the building block (3.9) generates, by successive convolutions, only functions of $\underline{\lambda}$. The distribution $p_L(\underline{M})$ for an arbitrary length L is thus isotropic and will be designated by $p_L(\underline{\lambda})$.

An isotropic distribution implies, by (2.18), that flux incident on one channel is transmitted with the same probability into any channel. Intuitively, this seems reasonable if the system is long compared with the mean free path l and the width W. We shall indeed see that the present model yields the same results as detailed microscopic calculations performed in the diffusive regime ($L\gg l$) and for quasi-one-dimensional systems ($L\gg W$).

Additional constraints would have to be imposed in order to describe a more selective feeding of the channels. The resulting $p_L(\underline{M})$ would no longer be isotropic and the whole analysis would certainly be more complicated. We have not undertaken such a generalization yet, and we leave its study for the future.

We now go back to the "evolution" equation (3.6), with

the ansatz (3.9) for the building block. If we expand (3.6) in powers of δL and take the limit $\delta L \to 0$, we find a Fokker-Planck or diffusion equation for our probability density. The same procedure used in Ref. 21 for $\beta=1$ can be used for $\beta=2$ and we shall not repeat the details here. The essence of the derivation is the following: Eq. (3.6) relates the probability density of transfer matrices $\underline{M}(L+\delta L)$ to that of $\underline{M}''(L)$ and $\underline{M}'(\delta L)$, where the latter is given by Eq. (3.9). $p_{L+\delta L}(\underline{M})$ depends only on the parameters $\{\lambda_n\}$, whereas $P_L(\underline{M}'')$ depends only on the parameters $\{\lambda_n''\}$, with $\lambda_n''\equiv\lambda_n+\delta\lambda_n$, which, as noted above, are related to the eigenvalues of the Hermitian matrix $\underline{H}(L)=\underline{M}^{\dagger''}\underline{M}''$. One can express $\underline{H}(L+\delta L)=\underline{H}(L)+\Delta\underline{H}(\underline{M}'',\underline{M}')$, and then use perturbation theory to express $\underline{\lambda}+\delta\underline{\lambda}$ in terms of the $\underline{\lambda}$ and the matrix elements of $\Delta\underline{H}$. The average (3.6) is then performed, over the sets $\underline{u}^{(i)'},\underline{u}^{(i)''}$ with the invariant measure of U(N), and over the parameters $\{\lambda_n'\}$ using Eq. (3.9). Just as in Ref. 21, it is convenient to introduce the probability density $w_L(\underline{\lambda})$ for the variable $\underline{\lambda}$, which, from Eq. (3.1) for the invariant measure, is related to $p_L(\underline{\lambda})$ by

$$w_L(\underline{\lambda})=p_L(\underline{\lambda})J_\beta(\underline{\lambda}) . \tag{3.10}$$

The resulting diffusion equation for $w_L(\underline{\lambda})$ can be written in a unified way for $\beta=1,2$ as

$$\frac{\partial w_s^{(\beta)}(\underline{\lambda})}{\partial s}=\frac{2}{\beta N+2-\beta}\times\sum_{a=1}^{N}\frac{\partial}{\partial\lambda_a}\left[\lambda_a(1+\lambda_a)J_\beta(\underline{\lambda})\frac{\partial}{\partial\lambda_a}\frac{w_s^{(\beta)}(\underline{\lambda})}{J_\beta(\underline{\lambda})}\right], \tag{3.11}$$

where

$$s=L/l \tag{3.12}$$

is the length of the conductor measured in units of the mean free path. From (3.9) and (3.10) we see that the initial condition must be

$$w_0(\underline{\lambda})=\delta(\underline{\lambda}) . \tag{3.11'}$$

IV. SOME CONSEQUENCES OF THE ISOTROPY OF THE STATISTICAL DISTRIBUTION

The fact that our ansatz (3.9) for the building block is isotropic and, as a result, the statistical distribution $p_L(\underline{M})$ for arbitrary lengths—obtained from $p_{\delta L}(\underline{M})$ by successive convolutions—is also isotropic, has important consequences. We shall see that just from the isotropy assumption we can obtain the structure of both the average and covariance of the transmission and reflection coefficients as a function of channel indices.[23] It is only the specific value of the various coefficients which depends on the solution of the diffusion equation (3.11).

The transmission coefficient T_{ab} is defined as $|t_{ab}|^2$, t_{ab} being the ab matrix element of Eq. (2.18). From Eq. (3.1) for the invariant measure, $\underline{u}^{(1)}$ and $\underline{u}^{(2)}$ are statistically independent (in both cases $\beta=1$ and 2), each being distributed according to the invariant measure of the unitary group U(N).

The average of T_{ab} is then given by

$$\langle T_{ab}\rangle_s^{(\beta)}=\sum_{\alpha,\alpha'}Q_{a\alpha'}^{a\alpha}Q_{\alpha' b}^{\alpha b}\langle(\tau_\alpha\tau_\beta)^{1/2}\rangle_s^{(\beta)} , \tag{4.1}$$

where

$$\tau_\alpha=\frac{1}{1+\lambda_\alpha} . \tag{4.2}$$

The last factor in (4.1) is an average evaluated with the probability distribution $w_s^{(\beta)}(\underline{\lambda})$, i.e., the solution of the diffusion equation (3.10), for which we need not be more specific for the time being. The factors $\underline{M}$ occurring in Eq. (4.1) are a particular case of the general average[43–45]

$$Q_{a_1'\alpha_1',\ldots,a_m'\alpha_m'}^{a_1\alpha_1,\ldots,a_m\alpha_m}=\langle(u_{a_1'\alpha_1'}\cdots u_{a_m'\alpha_m'})(u_{a_1\alpha_1}\cdots u_{a_m\alpha_m})^*\rangle_0 \tag{4.3}$$

performed with the invariant measure of the unitary group (indicated by the index 0).

In Ref. 44 it is shown that

$$M_{a'\alpha'}^{a\alpha}=\frac{\delta_{a'a}\delta_{\alpha'\alpha}}{N} \tag{4.4}$$

so that Eq. (4.1) becomes

$$\langle T_{ab}\rangle_s^{(\beta)}=\frac{\langle T\rangle}{N^2} . \tag{4.5}$$

Here

$$T=\sum_{a,b}T_{ab}=\sum_a\frac{1}{1+\lambda_a} \tag{4.6}$$

[see Eq. (2.22)] is the total transmission factor into all channels, when the incident ones are fed with N incoherent unit fluxes.

Next we calculate, from Eq. (2.18), the crossed second moment

$$\langle T_{ab}T_{a'b'}\rangle_s^{(\beta)}=\sum_{\alpha,\gamma,\alpha',\gamma'}Q_{a\alpha',a'\gamma'}^{a\alpha,a'\gamma}Q_{\alpha' b,\gamma' b'}^{\alpha b,\gamma b'}\times\langle(\tau_\alpha\tau_\gamma\tau_{\alpha'}\tau_{\gamma'})^{1/2}\rangle_s^{(\beta)} . \tag{4.7}$$

In Refs. 44 and 45 the Q coefficients of Eq. (4.7) are shown to be

$$Q_{a'\alpha',b'\beta'}^{a\alpha,b\beta}=\frac{1}{N^2-1}(\delta_{a'}^a\delta_{b'}^b\delta_{\alpha'}^\alpha\delta_{\beta'}^\beta+\delta_{a'}^a\delta_{b'}^b\delta_{\alpha'}^\beta\delta_{\beta'}^\alpha)-\frac{1}{N(N^2-1)}(\delta_{a'}^a\delta_{b'}^b\delta_{\alpha'}^\beta\delta_{\beta'}^\alpha+\delta_{a'}^a\delta_{b'}^b\delta_{\alpha'}^\alpha\delta_{\beta'}^\beta) . \tag{4.8}$$

The covariance

$$C_{ab,a'b'}^T=\langle T_{ab}T_{a'b'}\rangle-\langle T_{ab}\rangle\langle T_{a'b'}\rangle \tag{4.9}$$

can now be calculated, with the result

$$C_{ab,a'b'}^T=[A_N\langle T^2\rangle-B_N\langle T_2\rangle]\delta_{aa'}\delta_{bb'}+[A_N\langle T_2\rangle-B_N\langle T^2\rangle](\delta_{aa'}+\delta_{bb'})+[A_N\langle T^2\rangle-N^2B_N\langle T_2\rangle-C_N\langle T\rangle] . \tag{4.10}$$

Here we have defined

$$T_k=\sum_a \frac{1}{(1+\lambda_a)^k} ,$$
$$A_N=\frac{(N^2+1)}{N^2(N^2-1)^2} ,$$
$$B_N=\frac{2}{N^3(N^2-1)^2} , \tag{4.11}$$
$$C_N=\frac{1}{N^4} .$$

Equation (4.10) is exact. As a check, we can easily verify that the sum of (4.10) over a,b,a',b' gives precisely Var$T$. The structure of (4.10) is the same for $\beta=1,2$ although the explicit value of the coefficients of the δ functions does depend on the specific value of β.

In Ref. 14, Eq. (3), three types of terms are also obtained: setting $W \ll L$ (quasi-1D systems), they are seen to have essentially the structure provided by the δ functions of our Eq. (4.10). The difference is that our Kronecker δ's (that we can write as $\delta_{aa'}=\delta_{\Delta\underline{q}_a=0}$, with $\Delta\underline{q}_a=|\underline{q}_a-\underline{q}_{a'}|$, $\underline{q}_a$ being the transverse wave vector labeling the channel (the eigenmode a), are replaced by functions which are peaked at the wave vectors which satisfy the appropriate Kronecker δ's in our calculation, but decay over some distance in momentum space.

We now turn to the study of the reflection coefficient $R_{ab}=|r_{ab}|^2$, r_{ab} being the ab matrix element of Eq. (2.17). Because of the relations (2.14) for $\beta=1$ the structure of $\langle R_{ab}\rangle$ now depends on the specific value of β.

We start with $\beta=1$ (time-reversal symmetry), when $\underline{u}^{(4)}=\underline{u}^{(2)*}$. We have

$$\langle R_{ab}\rangle_s^{(\beta=1)}=\sum_{\alpha,\gamma} Q_{\alpha a,\alpha b}^{\gamma a,\gamma b}\langle(\rho_\alpha\rho_\beta)^{1/2}\rangle_s^{(\beta=1)} , \tag{4.12}$$

where

$$\rho_\alpha=\frac{\lambda_\alpha}{1+\lambda_\alpha} . \tag{4.13}$$

Making use of Eq. (4.8) we obtain

$$\langle R_{ab}\rangle_s^{(\beta=1)}=(1+\delta_{ab})\frac{\langle R\rangle_s^{(\beta=1)}}{N(N+1)} , \tag{4.14}$$

where

$$R=\sum_{a,b} R_{ab}=\sum_a \frac{\lambda_a}{1+\lambda_a} \tag{4.15}$$

is the total reflection coefficient.

Equation (4.14) means that backward scattering to the same channel is enhanced by a factor 2, as compared with the scattering to any other channel. This is precisely the prediction of weak-localization theory, where the argument is that the various paths contribute with random phases, except for a path and its time-reversed one, which contribute coherently and give rise to a factor of 2 in the backward direction. The same argument predicts that when a magnetic field is present and hence time-reversal symmetry is destroyed, the above-mentioned enhancement is absent.

We can easily check the prediction of our model for the case $\beta=2$. In this case $\underline{u}^{(2)}$ and $\underline{u}^{(4)}$ are no longer related, but, according to (3.1), they are statistically independent, each being distributed according to the invariant measure of the unitary group U(N). We then find

$$\langle R_{ab}\rangle_s^{(\beta=2)}=\sum_{\alpha,\gamma} Q_{\gamma a}^{\alpha a} Q_{\alpha b}^{\gamma b}\langle(\rho_\alpha\rho_\gamma)^{1/2}\rangle_s^{(\beta=2)} \tag{4.16}$$

and, using (4.4)

$$\langle R_{ab}\rangle_s^{(\beta=2)}=\frac{\langle R\rangle_s^{(\beta=2)}}{N^2} , \tag{4.17}$$

showing, indeed, the absence of the backscattering enhancement.

We can similarly calculate higher moments of R_{ab}. We start with $\beta=1$. We have the crossed second moment

$$\langle R_{ab}R_{a'b'}\rangle_s^{(\beta=1)}=\sum_{\alpha,\gamma,\alpha',\gamma'} Q_{\alpha a,\alpha b,\alpha' a',\gamma' b'}^{\gamma a,\gamma b,\gamma' a',\alpha' b'} \times\langle(\rho_\alpha\rho_\gamma\rho_{\alpha'}\rho_{\gamma'})^{1/2}\rangle_s^{(\beta=1)} . \tag{4.18}$$

The Q factor needed in Eq. (4.18) was calculated in Ref. 45; using that result one finds, for the covariance

$$C_{ab,a'b'}^R=\langle R_{ab}R_{a'b'}\rangle-\langle R_{ab}\rangle\langle R_{a'b'}\rangle , \tag{4.19}$$

the expression

$$\begin{aligned}C_{ab,a'b'}^{R(\beta=1)}=&\frac{1}{N^2(N^2-1)(N^2-4)(N^2-9)(N+1)^2}\\
&\times\{2N(N+1)^2(N^3-6N^2+11N-6)[\langle R^2\rangle^{\beta=1}+2\langle R_2\rangle^{(\beta=1)}]\delta_{aba'b'}\\
&+(N+1)^2[(N^4-4N^3+3N^2-4N+12)\langle R_2\rangle^{(\beta=1)}-2(N^3-4N^2+N+6)\langle R^2\rangle^{(\beta=1)}]\\
&\times[2(\delta_{aba'}+\delta_{abb'}+\delta_{aa'b}+\delta_{ba'b'})+(\delta_{aa'}+\delta_{bb'}+\delta_{ab'}+\delta_{a'b})]\\
&+(N+1)^2[(N^4-2N^3-5N^2-2N+24)\langle R_2\rangle^{(\beta=1)}-4(N^3-4N^2+N+6)\langle R^2\rangle^{(\beta=1)}]\\
&\times[\delta_{aa'}\delta_{bb'}+\delta_{ab'}\delta_{a'b'}+(1+\delta_{ab})(1+\delta_{a'b'})]\\
&-(N^2-1)(N^2-4)(N^2-9)(\langle R\rangle^{(\beta=1)})^2(1+\delta_{ab})(1+\delta_{a'b'})\} .\end{aligned} \tag{4.20}$$

In (4.20), the generalized δ functions are defined to vanish unless all its indices coincide, in which case it takes the value 1.

In (4.20) we have defined

$$R_2=\sum_a\left[\frac{\lambda_a}{1+\lambda_a}\right]^2 . \tag{4.21}$$

Equation (4.20) is exact. As a check, we can verify that the sum of (4.20) over a,b,a',b' gives precisely Var$R$.

On the other hand, for $\beta=2$ we obtain an expression with exactly the same structure as that of Eq. (4.10) for the covariance of transmission coefficients, i.e.,

$$C_{ab,a'b'}^{R(\beta=2)}=(A_N\langle R^2\rangle^{(\beta=2)}-B_N\langle R_2\rangle^{(\beta=2)})\delta_{aa'}\delta_{bb'}+(A_N\langle R_2\rangle^{(\beta=2)}-B_N\langle R^2\rangle^{(\beta=2)})(\delta_{aa'}+\delta_{bb'})$$
$$+[A_N\langle R^2\rangle^{(\beta=2)}-B_N\langle R_2\rangle^{(\beta=2)}-C_N(\langle R\rangle^{(\beta=2)})^2] . \tag{4.22}$$

Equations (4.20) and (4.22) for the covariance of reflection coefficients constitute a new prediction of the present model.

The expectation values of T,T_2,T^2,R,R_2,R^2 [evaluated with the probability density $w_s^{(\beta)}(\underline{\lambda})$ of Eq. (3.11)] which appear in all the above expressions will be calculated in the next section.

V. CALCULATION OF EXPECTATION VALUES

We now go back to our basic diffusion equation (3.11). This equation governs the "evolution" with length of the probability density $w_s^{(\beta)}(\underline{\lambda})$ and, as a consequence, the evolution of the expectation value of any function $F(\underline{\lambda})$ of interest.

Multiplying both sides of (3.11) by $F(\underline{\lambda})$ and integrating over $\{\lambda_n\}$, we obtain, for the expectation value

$$\langle F\rangle_s^{(\beta)}=\int F(\underline{\lambda})w_s^{(\beta)}(\underline{\lambda})\prod_{a=1}^{N}d\lambda_a , \tag{5.1}$$

the evolution equation

$$\tfrac{1}{2}(\beta N+2-\beta)\partial_s\langle F\rangle_s^{(\beta)}=\left\langle\sum_a\left[\lambda_a(1+\lambda_a)\frac{\partial^2F}{\partial\lambda_a^2}+(1+2\lambda_a)\frac{\partial F}{\partial\lambda_a}\right]+\frac{\beta}{2}\sum_{a\neq b}\frac{\lambda_a(1+\lambda_a)\frac{\partial F}{\partial\lambda_a}-\lambda_b(1+\lambda_b)\frac{\partial F}{\partial\lambda_b}}{\lambda_a-\lambda_b}\right\rangle_s^{(\beta)} . \tag{5.2}$$

Two expectation values considered in Ref. 21 which can be computed by an exact solution of the diffusion equation are the centroid of the $\{\lambda_a\}$, $\mu=\langle N^{-1}\sum_{a=1}^{N}\lambda_a\rangle$, and the quantity $\zeta=\langle N^{-1}\sum_a\ln(1+\lambda_a)\rangle$. In the general case one finds the same solutions hold, *independent of* β,

$$\mu=\tfrac{1}{2}(e^{2s}-1) , \tag{5.3}$$

$$\zeta=s . \tag{5.4}$$

One expects on very general grounds that for a system of fixed N (fixed width) as $L\to\infty$ the states will become localized and the conductance will become exponentially small with L. It is worth noting that neither Eq. (5.3) for μ nor Eq. (5.4) for ζ rigorously imply strong localization in this sense, since the conductance in the strongly localized limit would be dominated by the behavior of the smallest λ, unlike μ or ζ. For example, μ could grow exponentially even if one of the λ_a were identically unity, whereas the average conductance would always be larger than unity. Nonetheless, in 1D ($N=1$), the diffusion equation is known to predict strong localization[46] and recently it was shown by LeDoussal[47] that (3.11) does give exponential localization as $L\to\infty$ for fixed $N>1$. In the rest of this section we will focus on the study of the moments of the conductance (total transmission coefficient) in the weakly localized (or metallic) limit of $\langle g\rangle\gg1$, which we find can be obtained by studying the limit $N\to\infty$ for fixed s.

Setting $F=T^p$, Eq. (5.2) gives the evolution of the pth moment of T as

$$(2-\beta+\beta N)\partial_s\langle T^p\rangle_s^{(\beta)}=\langle -\beta pT^{p+1}-(2-\beta)pT^{p-1}T_2+2p(p-1)T^{p-2}(T_2-T_3)\rangle_s^{(\beta)} , \tag{5.5}$$

where T_k was defined in (4.11)

We see that on the right-hand side of (5.5) there appear quantities other than $\langle T^{\underline{q}}\rangle_s^{(\beta)}$, so that their evolution equations are needed as well. In Appendix B we obtain the evolution equation for the quantity $\langle T^pT_{\underline{q}}^r\rangle_s^{(\beta)}$, from which we can find the various expressions of interest as particular cases. Aside from (5.5) we obtain

$$(2-\beta+\beta N)\partial_s\langle T^pT_2\rangle_s^{(\beta)}=\langle[2\beta T^{p+2}-\beta(p+4)T^{p+1}T_2+2(2-\beta)T^pT_2-4(2-\beta)T^pT_3-(2-\beta)pT^{p-1}T_2^2]$$
$$+8pT^{p-1}(T_3-T_4)+2p(p-1)T^{p-2}(T_2^2-T_2T_3)\rangle_s^{(\beta)} , \tag{5.6}$$

$$(2-\beta+\beta N)\partial_s\langle T^pT_3\rangle_s^{(\beta)}=\langle[-\beta(p+6)T^{p+1}T_3+6\beta T^{p+1}T_2-3\beta T^pT_2^2]+6(2-\beta)T^pT_3-9(2-\beta)T^pT_4$$
$$-(2-\beta)pT^{p-1}T_2T_3+12pT^{p-1}(T_4-T_5)+2p(p-1)T^{p-2}(T_2T_3-T_3^2)\rangle_s^{(\beta)}\ , \tag{5.7}$$

$$(2-\beta+\beta N)\partial_s\langle T^pT_2^2\rangle_s^{(\beta)}=\langle[-\beta(p+8)T^{p+1}T_2^2+4\beta T^{p+2}T_2]-(2-\beta)pT^{p-1}T_2^3+4(2-\beta)T^pT_2^2-8(2-\beta)T^pT_2T_3$$
$$+2p(p-1)T^{p-2}(T_2-T_3)T_2^2+16T^p(T_4-T_5)+16pT^{p-1}T_2(T_3-T_4)\rangle_s^{(\beta)}\ . \tag{5.8}$$

We observe that every time we write down an evolution equation, we find, on the right-hand side, new quantities that had not appeared before. This being the case, looking for an exact solution is out of the question. An approximate solution can be found, though, if we realize that in the cases of physical interest the number N of channels is very large, so that N^{-1} can be thought of as playing the role of an expansion parameter. In the limit $N\gg1$ we shall thus seek the solution to the foregoing coupled equations as a series in decreasing powers of N. This procedure is similar to that devised by van Kampen in statistical mechanics, where the inverse volume V^{-1} is the expansion parameter. We note that because the statistical quantities of interest, the moments of the conductance, can have a very strong dependence on the sample length L, this procedure is not guaranteed to give the dominant behavior as the system size is scaled to infinity at fixed shape. Nonetheless, since it is in some sense an expansion for large conductance, we may expect it to give a reasonable description of the metallic regime, which we can compare with calculations based on microscopic models; it turns out to work very well in certain limits to be discussed below.

We thus propose the following expansions:

$$\langle T^p\rangle_s=N^pf_{p0}(s)+N^{p-1}f_{p1}(s)+N^{p-2}f_{p2}(s)+\cdots\ , \tag{5.9}$$

$$\langle T^pT_2\rangle_s=N^{p+1}g_{p+1,0}(s)+N^pg_{p+1,1}(s)+N^{p-1}g_{p+1,2}(s)+\cdots\ , \tag{5.10}$$

$$\langle T^pT_3\rangle_s=N^{p+1}h_{p+1,0}(s)+N^ph_{p+1,1}(s)+N^{p-1}h_{p+1,2}(s)+\cdots\ , \tag{5.11}$$

$$\langle T^pT_2^2\rangle_s=N^{p+2}l_{p+2,0}(s)+N^{p+1}l_{p+2,1}(s)+N^pl_{p+2,2}(s)+\cdots \tag{5.12}$$

with the initial conditions

$$f_{pm}(0)=g_{pm}(0)=\cdots=\delta_{m0}\ . \tag{5.13}$$

We introduce the above expansions in Eqs. (5.5)–(5.8) and equate the coefficients of the various powers of N. This is done in Appendix C, where it is shown that at every order we have a closed set of coupled equations to solve.

One of the basic results of the analysis is the pth moment of the total transmission coefficient. From Eqs. (C23) we find

$$\langle T^p\rangle_s^{(\beta)}=\frac{N^p}{(1+s)^p}-\delta_{\beta1}\frac{ps^3}{3(1+s)^{p+2}}N^{p-1}$$
$$+\frac{p}{90(1+s)^{p+6}}\{[(11p-9)s^8+\cdots]\delta_{\beta1}+[(3p-5)s^8+\cdots]\delta_{\beta2}\}N^{p-2}+\cdots\ . \tag{5.14}$$

The two square brackets in Eq. (5.14) are polynomials in s, explicitly given in Appendix C.

We are interested in the *diffusive regime*, in which the length L of the system is much larger than the mean free path l, but much smaller than the 1D localization lengths $\xi_1\sim Nl$, i.e.,

$$l\ll L\ll Nl \tag{5.15a}$$

or

$$1\ll s\ll N\ . \tag{5.15b}$$

We analyze separately the first and second moments of T.

A. The first moment of T

We study independently the two cases $\beta=1,2$.

1. *Time-reversal invariance* ($\beta=1$)

If we use the left-hand side (LHS) of (5.15) to expand the denominators of (5.14) in powers of s we get

$$\langle T\rangle_s^{(1)}=\left[\frac{N}{s}-\frac{N}{s^2}+\cdots\right]-\frac{1}{3}\left[1-\frac{3}{s}+\cdots\right]+\left[\frac{s}{45N}+\cdots\right]\ . \tag{5.16}$$

From the right-hand side (RHS) of (5.15), the leading term in each bracket of (5.16) is smaller than the leading term in the preceding bracket, i.e.,

$$\frac{N}{s}\gg\frac{1}{3}\gg\frac{s}{45N}\gg\cdots\ . \tag{5.17}$$

In what follows we shall find particularly interesting the

situation in which the second and higher terms of the first bracket are negligible compared with the leading term $-\frac{1}{3}$ of the second bracket. For this to occur, it is enough to request that $N/s^2 \ll 1/3$, which implies the additional condition

$$\sqrt{N} \ll s \ . \tag{5.18}$$

In this case we have

$$\langle T \rangle_s^{(1)} = \frac{Nl}{L} - \frac{1}{3} + \cdots \ . \tag{5.19}$$

Conditions (5.15) and (5.18) can be combined as

$$\sqrt{N} \ll s \ll N \ . \tag{5.20}$$

Since

$$N = (k_F W)^{d-1} \ , \tag{5.21}$$

W being the transverse dimension of the system and d the dimensionality, (5.20) can be written as

$$(k_F l)(k_F W)^{(d-3)/2} \ll \frac{L}{W} \ll (k_F l)(k_F W)^{d-2} \ . \tag{5.22}$$

For $d=2$

$$\frac{k_F l}{\sqrt{k_F W}} \ll \frac{L}{W} \ll k_F l \tag{5.23}$$

and for $d=3$

$$k_F l \ll \frac{L}{W} \ll (k_F l)(k_F W) \ . \tag{5.24}$$

Since $k_F l \gg 1$ in the metallic regime, the LHS of (5.22) implies that, for $d \geq 3$, $L/W \gg 1$, which corresponds to a quasi-1D system. This situation is thus very relevant to us, since it is the case where, in Sec. IV, we were able to obtain a good description of the correlation between transmission coefficients. For $d=2$, the LHS of (5.23) hardly implies any restriction of L/W, since it could be satisfied, for instance, for $L/W=1$ [which would imply $W \gg l(k_F l)$].

Now we go back to Eq. (5.19), which was obtained under the above restrictions. We recognize the first term as *Ohm's law,* while the second term is a negative correction, just as occurs in *weak-localization* theory. It is shown in Appendix D that the weak-localization correction to $\langle T \rangle$, calculated by diagrammatic techniques, gives precisely the value $-\frac{1}{3}$ found in Eq. (5.19), in the limit $L/W \gg 1$ and for arbitrary d. The fact that for $d=2$ Eq. (5.19) gives the same correction even for a square is a weakness of the present model, since it is well known that the weak-localization correction in 2D depends logarithmically on the system size.[1] We noted this shortcoming of the model earlier, and we shall return to it later. If we were to request that the second and higher terms of the first and second bracket in (5.16) be negligible compared with the leading term $s/(45N)$ of the third bracket, it would be enough to request that $N/s^2, 3/s \ll s/(45N)$, which, in addition to (5.18), would imply the condition

$$N^{2/3} \ll s \ . \tag{5.25}$$

$\langle T \rangle$ could then be expressed as

$$\langle T \rangle_s^{(1)} = \frac{Nl}{L} - \frac{1}{3} + \frac{1}{45Nl} + \cdots \ . \tag{5.26}$$

We could thus write an expansion of $\langle T \rangle_s^{(1)}$, with the first m terms consisting of descending powers of Nl/L (which then plays the role of the *expansion parameter*), provided that

$$N^{(m-1)/m} \ll s \ll N \ . \tag{5.27}$$

For a given m, we need N large enough, in order to leave "enough room" between the LHS and RHS of (5.27) to fit s in between. Substituting $N=(k_F W)^{d-1}$, (5.27) gives

$$(k_F l)(k_F W)^{[(m-1)(d-1)-m]/m} \ll \frac{L}{W} \ll (k_F l)(k_F W)^{d-2} \ . \tag{5.28}$$

For instance, for $d=3$ we have

$$(k_F l)(k_F W)^{(m-2)/2} \ll \frac{L}{W} \ll (k_F l)(k_F W) \ , \tag{5.29}$$

which implies a quasi-1D system ($L/W \gg 1$) for any fixed $m \geq 2$.

If instead of a quasi-1D system we were to consider a cube ($W=L$), Eq. (5.16) shows that the leading correction to Ohm's law would be given by the second term, $-N/s^2 = -(k_F L)^{d-3}(k_F l)^2$, whereas the result from weak-localization theory is proportional to $-[(L/l)^{d-2}-1]/(d-2)$. We shall thus restrict our analysis to a quasi-1D geometry.

2. *No time-reversal invariance* ($\beta=2$)

Now the second term in (5.14) is absent, so that using $s \gg 1$ we obtain

$$\langle T \rangle_s^{(2)} = \left[\frac{N}{s} - \frac{N}{s^2} + \cdots \right] - \left[\frac{s}{45N} + \cdots \right] + \cdots \ . \tag{5.30}$$

In the quasi-1D approximation discussed above we have

$$\langle T \rangle_s^{(2)} = \frac{Nl}{L} - \frac{L}{45Nl} + \cdots \ . \tag{5.31}$$

The fact that now there is no contribution from weak localization to the order $(Nl/L)^0$ is a well-known effect of the absence of time-reversal symmetry.

B. The variance of T

Even before expanding the first two terms of (5.14) in inverse powers of s, they *cancel exactly* in the calculation of Var$T$, giving

$$\mathrm{Var}T = \frac{1}{45(1+s)^8} [\delta_{\beta 1}(6s^8 + \cdots) + \tfrac{1}{2}\delta_{\beta 2}(6s^8 + \cdots)] \ . \tag{5.32}$$

In the limit $s \gg 1$ we thus have

$$\mathrm{Var}T=\begin{cases}\frac{2}{15}+\cdots, & \beta=1\\ \frac{1}{15}+\cdots, & \beta=2\end{cases}. \tag{5.33}$$

The leading term occurring in $\mathrm{Var}(T)$ is thus *independent* of the number N of channels (determined by the cross section of the wire), the length L of the conductor, and the mean free path l. The rms conductance fluctuation (now including the factor of 2 for spin degeneracy) is thus a *pure* number, i.e.,

$$\sqrt{\mathrm{Varg}}=\begin{cases}\sqrt{8/15}=0.730\\ \sqrt{4/15}=0.516\end{cases}. \tag{5.34}$$

These are precisely the values found in Ref. 9 in the quasi-1D case, with the use of microscopic Green-function techniques. The reader is reminded of our caveat in the Introduction: the experimentally measured value of $\sqrt{\mathrm{Varg}}$ depends on the inelastic scattering length, and typically on the geometry of the measuring configuration. Although it is always found to be of order unity when the voltage probes are spaced approximately one elastic length apart, the conductance fluctuation experiments cannot be quantitatively compared with the $T=0$ two-probe theory. However, in an optical experiment in which one directly measures transmission fluctuations, there appears to be no reason why one could not measure the "two-probe" values to high accuracy. In this case it is crucial to remember that the conductance is the total transmission probability if all incident modes are equally and incoherently occupied. Since this is not the usual boundary condition in optical experiments (there is usually one incident mode occupied, or many coherently occupied), one would have to sum the transmission coefficients for N different measurements in order to study the analog of the conductance.

So far we have analyzed the consequence of Eq. (5.14) for the pth moment of the transmission coefficient, for which we proposed the series expansion (5.9). In the process of solving the coupled equations in Appendix C, the other quantities appearing in (5.10)–(5.12) are also obtained. For instance, for $\langle T^pT_2\rangle_s^{(\beta)}$ one obtains

$$\langle T^pT^2\rangle_s^{(\beta)}=\frac{2s^3+6s^2+6s+3}{3(1+s)^{p+4}}N^{p+1}-\frac{10p+14)s^7+\cdots}{45(1+s)^{p+7}}N^p+\cdots. \tag{5.35}$$

In the quasi-1D approximation discussed above, we have

$$\langle T^pT^2\rangle_s^{(\beta)}=\frac{2}{3}\left[\frac{N}{s}\right]^{p+1}-\delta_{\beta 1}\frac{10p+14}{45}\left[\frac{N}{s}\right]^p+\cdots. \tag{5.36}$$

We now realize that with the results obtained in this section we can complete the calculation of the covariance of transmission and reflection coefficients initiated in Sec. IV. This we carry out as our last task in this section.

We denote the three terms occurring in $C^T_{ab,a'b'}$, Eq. (4.10), as $C^{(1)},C^{(2)},C^{(3)}$, and realize that, with regards to their N dependence, they have the structure

$$C^{T,\beta}_{ab,a'b'}=\left[\frac{A}{N^2}+\frac{B}{N^3}+\frac{C}{N^4}+\cdots\right]\delta_{aa'}\delta_{bb'}+\left[\frac{D}{N^3}+\frac{E}{N^4}+\cdots\right](\delta_{aa'}+\delta_{bb'})+\left[\frac{F}{N^4}+\cdots\right]. \tag{5.37}$$

Suppose a,b,a',b' are such that $C^{(1)}$, $C^{(2)}$, and $C^{(3)}$ contribute. Then A gives the leading term of C^T. If the δ functions eliminate some of the $C^{(i)}$'s, the leading term of C^T is given by the leading term of the first nonzero row that occurs in (5.37). Suppose that we calculate only the leading term of each $C^{(i)}$, i.e., A, D, and F. We are clearly neglecting higher-order terms in one row, while keeping terms of that same order in the following ones. But by doing so, we get precisely the leading term of the resulting C^T, no matter if some of the $C^{(i)}$ are identically zero or not; what we lose, of course, is the next-order term of the resulting C^T. For instance, for $C^T_{12,12}$, $C^{(1)}$, $C^{(2)}$, and $C^{(3)}$ contribute and A/N^2 is the leading term; for $C^T_{12,13}$, $C^{(2)}$ and $C^{(3)}$ contribute and D/N^3 is the leading term; for $C^T_{13,24}$, only $C^{(3)}$ contributes and F/N^4 is the leading term: only A,D,F are thus needed if all we want is the leading term in each case. Using the results of the present section, appearing in Eqs. (5.14), (5.26), (5.31), (5.33), and (5.36), we can write the covariance of the transmission coefficients as

$$C^{T,\beta}_{ab,a'b'}=\langle T_{ab}\rangle_s\langle T_{a'b'}\rangle_s\left[\delta_{aa'}\delta_{bb'}+\frac{2}{3\langle T\rangle_s}(\delta_{aa'}+\delta_{bb'})+\frac{1+\delta_{\beta 1}}{15\langle T_s\rangle^2}\right]. \tag{5.38}$$

Here we have dropped the index β in $\langle T_{ab}\rangle_s$ and $\langle T\rangle_s$ because the leading term is independent β [see Eqs. (5.26) and (5.31)]. We have also assumed $s=L/l\gg 1$, as usual.

Equation (5.38) has precisely the structure of Eq. (3) of Ref. 13 (where $\beta=1$ was considered). Thus, as already noted in the previous section, while our approach may miss some subtle correlations between nearby channels, it does correctly describe the "global" fluctuations in transmission and conductance. Assume, for instance, that a single mode b is excited on the left and we are interested in fluctuations of T_{ab} and $T_b=\sum_a T_{ab}$, which represent, respectively, the transmission coefficient to a single channel a and to all channels on the right. From (5.38) we find

$$\mathrm{Var}(T_{ab})=\langle T_{ab}\rangle^2 \tag{5.39}$$

just as in Eq. (5) of Ref. 48, and

$$\mathrm{Var}(T_b)=\frac{2l}{3NL} \tag{5.40}$$

in agreement with Refs. 13 and 49.

Finally, we turn to the covariance of the reflection coefficients, Eqs. (4.20) and (4.22). The various averages of R which are needed are related to the corresponding averages of T by

$$\langle R\rangle=N-\langle T\rangle\ , \tag{5.41a}$$

$$\langle R^2\rangle=N^2-2N\langle T\rangle+\langle T^2\rangle\ , \tag{5.41b}$$

$$\mathrm{Var}R=\mathrm{Var}T\ , \tag{5.41c}$$

$$\langle R_2\rangle=N-2\langle T\rangle+\langle T_2\rangle\ . \tag{5.41d}$$

The same discussion following Eq. (5.37) applies here, so that only the leading term in the factor of each δ function is needed. In the limit $s=L/l\gg 1$ we then get

$$C^{R,\beta=1}_{ab,a'b'}=\langle R_{ab}\rangle\langle R_{a'b'}\rangle\left[\frac{\delta_{aa'}\delta_{bb'}+\delta_{ab}\delta_{a'b}}{1+\delta_{ab}}+\frac{1}{\langle R\rangle}(\delta_{ab}\delta_{a'b'}\delta_{aa'}-\delta_{aa'}-\delta_{bb'}-\delta_{ab'}-\delta_{ba'})+\frac{32}{15\langle R\rangle^2}\right]\ . \tag{5.42}$$

In the same approximation we find for $\beta=2$, from Eq. (4.22),

$$C^{R,\beta=2}_{ab,a'b'}=\langle R_{ab}\rangle\langle R_{a'b'}\rangle\left[\delta_{aa'}\delta_{bb'}-\frac{1}{\langle R\rangle}(\delta_{aa'}+\delta_{bb'})+\frac{16}{15\langle R\rangle^2}\right]\ . \tag{5.43}$$

From Eqs. (5.42) and (5.43) we find

$$\mathrm{Var}(R_{ab})=\langle R_{ab}\rangle^2\ , \tag{5.44}$$

just as in Eq. (10) of Ref. 48.

We see from Eqs. (5.42) and (5.43) that we are not entitled to neglect correlations between reflection coefficients (as was suggested in Ref. 48), any more than we are in the case of transmission coefficients. In particular, both have a "long-range" term (the one with no Kronecker δ's) of the same order of magnitude. Such correlations between reflection coefficients can be measured in an optical experiment; however, a careful analysis of the data is required to compare with the above results.

VI. SUMMARY AND CONCLUSIONS

In this and earlier work we have defined two additional random-matrix ensembles consisting of random-transfer matrices whose statistical properties are determined by the diffusion equation (3.11), for the case $\beta=1,2$. In terms of their symmetry properties these ensembles bear a close relationship to the orthogonal and unitary ensembles of the classical theory of random matrices.[34,30] However, their statistical properties are much richer since they depend in general on two parameters, N, the numbers of channels, and $s=L/l$, the system length in units of the elastic mean free path. We have shown that in the large-N limit these ensembles give statistical behavior of the conductance consistent with all the known quantum-interference effects in the metallic regime for samples much longer than they are wide. Since the ensembles are specified only by symmetry considerations, the isotropy assumption, and the maximum-entropy hypothesis for the transfer matrix describing an infinitesimal segment, we have demonstrated a minimal set of assumptions needed to generate these quantum-interference effects without any specifications relating to the microscopic Hamiltonian. The generality of these properties, made clear in our approach, is in our view the essence of the scaling theory of disordered conductors.

The weakness of the theory we have presented up to this point is that it has not been shown to generate the crucial dependence of the scaling properties of the conductance on the spatial dimensionality. As noted above, the large-N approximation is not really a controlled expansion, since the dependence of the various statistical quantities on length is not known in advance. Only after performing the expansion is one able to identify the limit to which it corresponds, which turns out to be the quasi-one-dimensional metallic limit. An important question is whether the theory in its present form contains all of the important behavior, and is it only our inability to calculate statistical quantities exactly which prevents us from seeing this or whether the basic assumptions of the model need to be changed.

It appears that in fact the model is essentially incomplete, since it only depends on the total number of channels $N\propto(k_FL)^{d-1}$, and not on their connectivity. Thus the model necessarily predicts, e.g., the same average conductance for a 2D strip or a 3D bar of the same length, as long as the total number of transverse channels is the same. This property can lead to a contradiction with the behavior expected from the scaling theory of localization.[24] Imagine choosing a value of $k_Fl\gg 1$ such that in 3D the system has extended states, and start with a small square of side L_0 and bar with the same value of $N=k_FL_0/\pi\gg 1$; initially both systems are Ohmic and they have the same value of $\langle g\rangle=Nl/L_0$. If we increase the size of both systems, keeping $N=k_FL/\pi$ equal in each conductor (so the square remains square, the bar becomes more and more elongated), then eventually the square will cross over to the 2D strongly localized regime, and $\langle\ln g\rangle\sim-2L/\xi$. For the bar it is easy to see that it will never have exponentially small conductance, since the conditions of $N=k_FL/\pi$ and $k_Fl\gg 1$ imply for the bar that $(k_Fl)L\ll Nl$, i.e., the sample length is always much less than the quasi-1D localization length, Nl.[3] Hence the two systems must eventually have different values of $\langle g\rangle$, a difference that cannot be generated within the present model. We believe that at

minimum the isotropy assumption, which is only intuitively plausible for quasi-1D systems, must be dropped in order to describe correctly the 2D and 3D behavior. In conclusion, we have presented a new model for disordered conductors which naturally incorporates the notion of universality implicit in the scaling theory of localization, and which is analytically tractable in a physically relevant limit; quasi-one-dimensional metallic conduction.

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APPENDIX A: THE INVARIANT MEASURE

We shall find the invariant measure for the case $\beta=2$ (no time-reversal symmetry), following very closely the derivation given in Appendix B of Ref. 21 for $\beta=1$. Therefore, we merely indicate the main steps of the proof, presenting with a little more detail only those points that involve a conceptual difference.

The method is based on the following observation. If we have ν independent variables x_i $(i=1,2,\ldots,\nu)$ and the arc element (not necessarily positive definite)

$$ds^2=\sum_{i,j=1}^{\nu} g_{ij}(x)dx_i dx_j \tag{A1}$$

remains invariant under the transformation $x_i=x_i(x'_1,\ldots,x'_\nu)$, then the volume element

$$dV=|\det \underline{g}(x)|^{1/2}\prod_{i=1}^{\nu} dx_i \tag{A2}$$

remains invariant under the same transformation.

Using (2.2) we can prove that

$$ds^2=\mathrm{Tr}(\underline{\Sigma}_z\, d\underline{M}^{\dagger}\,\underline{\Sigma}_z\, d\underline{M}) \tag{A3}$$

(where Tr indicates the trace of $2N\times 2N$ matrix) remains invariant under the transformation $\underline{M}\to\underline{M}_0\underline{M}\,\underline{M}_1$.

In terms of the parameters of (2.13) we can write (A3) as

$$ds^2=\mathrm{tr}\left[\sum_{i=1}^{4}(\delta\underline{u}^{1})^{\dagger}\delta\underline{u}^{(i)}-[2\underline{\lambda}(\underline{1}_N+\underline{\lambda})]^{-1}d\underline{\lambda}\,d\underline{\lambda}+2\,\mathrm{Re}\sqrt{\underline{1}_N+\underline{\lambda}}(\delta\underline{u}^{1})^{\dagger}\sqrt{\underline{1}_N+\underline{\lambda}}(\delta\underline{u}^{2})\right.$$
$$\left.-2\,\mathrm{Re}\sqrt{\underline{\lambda}}(\delta\underline{u}^{3})^{\dagger}\sqrt{\underline{\lambda}}(\delta\underline{u}^{3})-2\,\mathrm{Re}\sqrt{\underline{\lambda}}(\delta\underline{u}^{1})^{\dagger}\sqrt{\underline{\lambda}}(\delta\underline{u}^{4})+2\,\mathrm{Re}\sqrt{\underline{1}_N+\underline{\lambda}}(\delta\underline{u}^{3})^{\dagger}\sqrt{\underline{1}_N+\underline{\lambda}}(\delta\underline{u}^{4})\right]. \tag{A4}$$

Here Tr indicates the trace of $N\times N$ matrices. For $i=1,3$ on the one hand, and $i=2,4$ on the other, we have defined the $N\times N$ anti-Hermitian matrices

$$\delta\underline{u}^{i}=(\underline{u}^{i})^{\dagger}d\underline{u}^{i},\quad i=1,3 \tag{A5a}$$

$$\delta\underline{u}^{i}=d\underline{u}^{i}(\underline{u}^{i})^{\dagger},\quad i=2,4 \tag{A5b}$$

which, in turn, can be expressed in terms of real antisymmetric and symmetric matrices as

$$\delta\underline{u}^{i}=\delta\underline{a}^{i}+i\delta\underline{s}^{i}. \tag{A6}$$

We can now write the ds^2 of (A4) as

$$ds^2=\sum_{a=1}^{N}\left[(\delta x_{aa})^2+(\delta y_{aa})^2+2\lambda_a(\delta z_{aa})(\delta x_{aa}-\delta y_{aa}-\delta z_{aa})-\frac{(d\lambda_a)^2}{2\lambda_a(1+\lambda_a)}\right]$$
$$+2\sum_{a<b}\left[\sum_{i=1}^{4}[(\delta s^i_{ab})^2+(\delta a^i_{ab})^2]+2\sqrt{(1+\lambda_a)(1+\lambda_b)}(\delta a^1_{ba}\delta a^2_{ba}+\delta s^1_{ba}\delta s^2_{ba}+\delta a^3_{ba}\delta a^4_{ba}+\delta s^3_{ba}\delta s^4_{ba})\right.$$
$$\left.-2\sqrt{\lambda_a\lambda_b}(\delta a^2_{ba}\delta a^4_{ba}+\delta s^1_{ba}\delta s^4_{ba}+\delta a^3_{ba}\delta a^2_{ba}+\delta s^3_{ba}\delta s^2_{ba})\right], \tag{A7}$$

where we have defined

$$\delta x_{aa}=(\delta\underline{s}^{1})_{aa}+(\delta\underline{s}^{2})_{aa}, \tag{A8a}$$

$$\delta y_{aa}=(\delta\underline{s}^{3})_{aa}+(\delta\underline{s}^{4})_{aa}, \tag{A8b}$$

$$\delta z_{aa}=(\delta\underline{s}^{1})_{aa}-(\delta\underline{s}^{3})_{aa}. \tag{A8c}$$

Let us first count the independent variations that appear on the RHS of (A7): $(\delta x)_{aa},(\delta y)_{aa},(\delta z)_{aa}$ $(a=1,2,\ldots,N)$ contribute $3N$ variables; $(\delta a)^i_{ab}$ $(i=1,2,3,4;a<b)$ contribute $2N(N-1)$; (δs^i_{ab}) $(i=1,2,3,4;a<b)$ contribute $2N(N-1)$; $\delta\lambda_a$ contribute N variables; we thus have $4N^2$ variables altogether, which is the correct number of independent parameters for $U(N,N)$, as indicated after (2.10).

Equation (A2) now gives, for the invariant measure

$$d\mu(\underline{M}) \sim \prod_{a<b} (\lambda_a-\lambda_b)^2 \prod_c d\lambda_c \times \prod_{a=1}^{N} (\delta x_{aa}\delta y_{aa}\delta z_{aa}) \prod_{a<b}\prod_{i=1}^{4} (\delta s^i_{ab}\delta a^i_{ab}) \ . \tag{A9}$$

We recall that in the parametrization (2.13) we have the freedom expressed by the transformation (2.15), which leaves $\underline{M}$ invariant. Under that transformation, the δs^i_{ab} $(i=1,2,3,4)$ become

$$(\delta \underline{s}^{\,1})_{aa} \to (\delta \underline{s}^{\,1})_{aa} + \eta_a \ , \tag{A10a}$$

$$(\delta \underline{s}^{\,3})_{aa} \to (\delta \underline{s}^{\,3})_{aa} + \eta_a \ , \tag{A10b}$$

$$(\delta \underline{s}^{\,2})_{aa} \to (\delta \underline{s}^{\,2})_{aa} - \eta_a \ , \tag{A10c}$$

$$(\delta \underline{s}^{\,4})_{aa} \to (\delta \underline{s}^{\,4})_{aa} - \eta_a \ . \tag{A10d}$$

Of the four independent variables $(\delta \underline{s}^{\,i})_{aa}$ $(i=1,2,3,4)$, only the three combinations (A8) appear in the expression (A7) for ds^2, and hence in the variant measure (A9). We notice that (A8) remains invariant under the transformation (A10), while any other linear combination of $(\delta \underline{s}^{\,i})_{aa}$ can either be written in terms of $\delta x_{aa}, \delta y_{aa}, \delta z_{aa}$ or else it does not stay invariant under (A10). An example of the latter possibility is $(\delta \underline{s}^{\,1})_{aa}$, which transforms according to (A10a): but we can take advantage of the freedom mentioned above and get rid of $(\delta \underline{s}^{\,1})_{aa}$ by a proper choice of η_a in (A10a). On the other hand, the combinations (A8) are insensitive to the transformation (A10) and therefore they cannot be gotten rid of: this is why they appear explicitly in (A7) and (A9).

The same freedom mentioned in the above paragraph can be used to make $(\delta \underline{s}^{\,1})_{aa}$ arbitrary and equal, say, to $(2\pi)^{-1} d\eta_a$. We multiply both sides of (A9) by δs^1_{aa} and notice that

$$\delta x_{aa}\, \delta y_{aa}\, \delta z_{aa} (\delta \underline{s}^{\,1})_{aa} = \prod_{i=1}^{4} (\delta \underline{s}^{\,i})_{aa} \tag{A11}$$

and that

$$\prod_a (\delta \underline{s}^{\,i})_{aa} \prod_{a<b} [(\delta \underline{s}^{\,i})_{a<b} (\delta \underline{a}^{\,i})_{a<b}] = d\mu(\underline{u}^{\,i}) \tag{A12}$$

is the invariant measure of the unitary group $U(N)$ associated with the matrices $\underline{u}^{\,(i)}$. Using (3.2) we then get

$$d\mu(\underline{M}) d\mu(\underline{G}) = \prod_{a<b} (\lambda_a-\lambda_b)^2 \prod_c d\lambda_c \prod_{i=1}^{4} d\mu(\underline{u}^{\,i}) \ , \tag{A13}$$

which is (3.1) for $\beta=2$. Notice in (A13) we have $4N^2+N$ independent parameters on both sides.

APPENDIX B: THE EVOLUTION EQUATION FOR $\langle T^p T^r_{\underline{q}} \rangle^{(\beta)}_s$

We apply Eq. (5.2) to the quantity

$$F = T^p T^r_{\underline{q}} \ . \tag{B1}$$

T and $T_{\underline{q}}$ are defined in Eqs. (4.6) and (4.11), respectively. By direct differentiation one obtains

$$\left\langle 2\sum_a \lambda_a(1+\lambda_a)\frac{\partial^2 F}{\partial \lambda_a^2} \right\rangle = \langle 2p(p-1)T^{p-2}(T_2-T_3)T^r_{\underline{q}} + 4pT^{p-1}(T-T_2)T^r_{\underline{q}} + 4\underline{q}rpT^{p-1}T^{r-1}_{\underline{q}}(T_{\underline{q}+1}-T_{\underline{q}+2}) + 2\underline{q}^{\,2}r(r-1)T^pT^{r-2}_{\underline{q}}(T_{2\underline{q}}-T_{2\underline{q}+1}) + 2r\underline{q}(\underline{q}+1)T^pT^{r-1}_{\underline{q}}(T_{\underline{q}}-T_{\underline{q}+1})\rangle \tag{B2}$$

and

$$\left\langle 2\sum_a (1+2\lambda_a)\frac{\partial F}{\partial \lambda_a} \right\rangle = \langle -2pT^{p-1}(2T-T_2)T^r_{\underline{q}} - 2\underline{q}rT^pT^{r-1}_{\underline{q}}(2T_{\underline{q}}-T_{\underline{q}+1})\rangle \ . \tag{B3}$$

The last term occurring in (5.2) is proportional to

$$K = \left\langle \sum_{a\neq b} \frac{1}{\lambda_a-\lambda_a} \left[\lambda_a(1+\lambda_a)\frac{\partial F}{\partial \lambda_a} - \lambda_b(1+\lambda_b)\frac{\partial F}{\partial \lambda_b} \right] \right\rangle \tag{B4}$$

which can be found to be

$$K = -p\langle T^{p-1}T^r_{\underline{q}}(T^2-T_2)\rangle + \underline{q}r\left\langle T^pT^{r-1}_{\underline{q}} \sum_{a\neq b} \frac{\lambda_b(1+\lambda_a)^{\underline{q}} - \lambda_a(1+\lambda_b)^{\underline{q}}}{(\lambda_a-\lambda_b)(1+\lambda_a)^{\underline{q}}(1+\lambda_b)^{\underline{q}}} \right\rangle \ . \tag{B5}$$

We can easily show that

$$\frac{\lambda_b(1+\lambda_a)^{\underline{q}} - \lambda_a(1+\lambda_b)^{\underline{q}}}{\lambda_a-\lambda_a} = -\delta_{\underline{q}0} + \sum_{n=0}^{\underline{q}-2} (1+\lambda_a)^{\underline{q}-1-n}(1+\lambda_b)^{n+1} - \sum_{n=0}^{\underline{q}-1} (1+\lambda_a)^{\underline{q}-1-n}(1+\lambda_b)^n \tag{B6}$$

with the understanding that a summation is taken as identically zero when the upper index is negative. Substituting (B6) in (B5) we find

$$K=\left\langle -pT^{p-1}T_{\underline{q}}^{r}(T^{2}-T_{2})+\underline{q}rT^{p}T_{\underline{q}}^{r-1}\left[-(T_{0}^{2}-T_{0})\delta_{\underline{q}0}+\sum_{n=0}^{\underline{q}-2}(T_{1+n}T_{\underline{q}-n-1}-T_{\underline{q}})-\sum_{n=0}^{\underline{q}-1}(T_{1+n}T_{\underline{q}-n}-T_{1+\underline{q}})\right]\right\rangle . \quad \text{(B7)}$$

We now substitute the partial results (B2), (B3), and (B7) in (5.2) to obtain

$$\begin{aligned}(\beta N+2-\beta)\partial_{s}\langle T^{p}T_{\underline{q}}^{r}\rangle_{s}^{(\beta)}&=\Big\langle 2p(p-1)T^{p-2}(T_{2}-T_{3})T_{\underline{q}}^{r}-p(2-\beta)T^{p-1}T_{2}T_{\underline{q}}^{r}+2\underline{q}^{2}r(r-1)T^{p}T_{\underline{q}}^{r-2}(T_{2\underline{q}}-T_{2\underline{q}+1})\\&\quad-\beta pT^{p+1}T_{\underline{q}}^{r}+2r\underline{q}(\underline{q}-1)T^{p}T_{\underline{q}}^{r}-2r\underline{q}^{2}T^{p}T_{\underline{q}}^{r-1}T_{\underline{q}+1}+4\underline{q}rpT^{p-1}T_{\underline{q}}^{r-1}(T_{\underline{q}+1}-T_{\underline{q}+2})\\&\quad+\beta\underline{q}rT^{p}T_{\underline{q}}^{r-1}\left[\sum_{n=0}^{\underline{q}-2}(T_{1+n}T_{\underline{q}-n-1}-T_{\underline{q}})-\sum_{n=0}^{\underline{q}-1}(T_{1+n}T_{\underline{q}-n}-T_{1+\underline{q}})\right]\Big\rangle_{s}^{(\beta)}\end{aligned} \quad \text{(B8)}$$

with the convention indicated right after (B6). Equations (5.5)–(5.8) are particular cases of Eq. (B8).

APPENDIX C: SOLUTION OF THE COUPLED EQUATIONS (5.5)–(5.8)

If we introduce the series expansions (5.9)–(5.8) we obtain, for the functions $f_{pm}(s),g_{pm}(s),h_{pm}(s),l_{pm}(s)$, the four coupled equations

$$\begin{aligned}&\beta N^{p+1}f_{p0}'(s)+\beta N^{p}f_{p1}'(s)+\beta N^{p-1}f_{p2}'(s)+\cdots+(2-\beta)N^{p}f_{p0}'(s)+(2-\beta)N^{p-1}f_{p1}'(s)+\cdots\\&\quad=-\beta pN^{p+1}f_{p+1,0}(s)-\beta pN^{p}f_{p+1,1}(s)-\beta pN^{p-1}f_{p+1,2}(s)+\cdots\\&\qquad-(2-\beta)pN^{p}g_{p0}(s)-(2-\beta)pN^{p-1}g_{p1}(s)+\cdots+2p(p-1)N^{p-1}g_{p-1,0}(s)+\cdots\\&\qquad-2p(p-1)N^{p-1}h_{p-1,0}(s)+O(N^{p-2}) ,\end{aligned} \quad \text{(C1)}$$

$$\begin{aligned}&\beta N^{p+2}g_{p+1,0}'(s)+\beta N^{p+1}g_{p+1,1}'(s)+\cdots+(2-\beta)N^{p+1}g_{p+1,0}'(s)+\cdots\\&\quad=-2\beta N^{p+2}f_{p+2,0}(s)+2\beta N^{p+1}f_{p+2,1}(s)+\cdots\\&\qquad-\beta(p+4)N^{p+2}g_{p+2,0}(s)-\beta(p+4)N^{p+1}g_{p+2,1}(s)+\cdots+2(2-\beta)N^{p+1}g_{p+1,0}(s)+\cdots\\&\qquad-4(2-\beta)N^{p+1}h_{p+1,0}(s)+\cdots-p(2-\beta)N^{p+1}l_{p+1,0}(s)+O(N^{p}) ,\end{aligned} \quad \text{(C2)}$$

$$\beta N^{p+2}h_{p+1,0}'(s)+\cdots=-\beta(p+6)N^{p+2}h_{p+2,0}(s)+\cdots+6\beta N^{p+2}g_{p+2,0}(s)+\cdots-3\beta N^{p+2}l_{p+2,0}(s)+O(N^{p+1}) , \quad \text{(C3)}$$

$$\beta N^{p+3}l_{p+2,0}'(s)+\cdots=-\beta(p+8)N^{p+3}l_{p+3,0}(s)+\cdots+4\beta N^{p+3}g_{p+3,0}(s)+O(N^{p+2}) . \quad \text{(C4)}$$

In Eqs. (5.6)–(5.8) we have enclosed in square brackets those terms that were used to obtain the expansions (C2)–(C4).

Equating the coefficients of N^{p+1} in Eq. (C1) we obtain the differential equations

$$f_{p0}'(s)+pf_{p+1,0}(s)=0 , \quad \text{(C5)}$$

subject to the initial conditions (5.13)

$$f_{p0}(0)=1 . \quad \text{(C6)}$$

(C5) constitute a *closed,* albeit infinite, set of coupled equations.

Evaluating (C5) at $s=0$ and combining it with (C6), we obtain

$$f_{p0}'(0)=-p . \quad \text{(C7)}$$

Similarly, successive differentiations of (C5) give

$$f_{p0}''(0)=(-p)(-p-1) , \quad \text{(C8)}$$

$$f_{p0}'''(0)=(-p)(-p-1)(-p-2) , \quad \text{(C9)}$$

We can thus write, for $f_{p0}(s)$, the series expansion

$$f_{p0}(s)=1-ps+(-p)(-p-1)\frac{s^{2}}{2!}+(-p)(-p-1)(-p-2)\frac{s^{3}}{3!}+\cdots , \quad \text{(C10)}$$

which, for $s<1$, converges to

$$f_{p0}(s)=\frac{1}{(1+s)^{p}} . \quad \text{(C11)}$$

For $s>1$, (C10) diverges, but we see that (C11) satisfies (C5) and (C6) for arbitrary s, so we take (C11) as our solution for both $s<1$, $s\geq 1$.

We now equate the coefficients of N^{p} in (C1) and of N^{p+2} in (C2). Since $f_{p0}(s)$ is already known, we obtain for $f_{pl}(s)$ and $g_{p0}(s)$ the closed set of coupled equations

$$f_{p1}'(s)+pf_{p+1,1}(s)=-\frac{2-\beta}{\beta}[f_{p0}'(s)+pg_{p0}(s)] , \quad \text{(C12a)}$$

$$g_{p0}'(s)+(p+3)g_{p+1,0}(s)=2f_{p+1,0}(s) , \quad \text{(C12b)}$$

subject to the initial conditions (5.13)

$$f_{p1}(0)=0\ , \tag{C13a}$$

$$g_{p0}(0)=1\ . \tag{C13b}$$

The inhomogeneous term on the RHS of (C12b) is the solution (C11) of our previous step. Having solved (C12b), we thus know the inhomogeneous term of the RHS of (C12a).

Just as before, successive differentiations of (C12b) give the various derivatives of $g_{p0}(s)$ at $s=0$. Since the resulting power series for $g_{p0}(s)$ is not easily identifiable, we substitute it in (C12a) and find for the nth derivative of $f_{p1}(s)$ at $s=0$

$$f_{p1}^{(n)}(0)=(-)^n\frac{2-\beta}{\beta}\frac{1}{3}n(n-1)(n-2)p(p+2)\times(p+3)\cdots(p+n-2)\ . \tag{C14}$$

The resulting power series for $f_{p1}(s)$ can be easily summed to give

$$f_{p1}(s)=-\frac{2-\beta}{\beta}\frac{ps^3}{3(1+s)^{p+2}}\ . \tag{C15}$$

Substituting (C15) in (C12a) we find, for $g_{p0}(s)$

$$g_{p0}(s)=\frac{2s^3+6s^2+6s+3}{3(1+s)^{p+3}}\ . \tag{C16}$$

The comment made right after (C11) applies here, too.

Next, we equate the coefficients of N^{p-1} in (C1), of N^{p+1} in (C2), of N^{p+2} in (C3), and of N^{p+3} in (C4). Since $f_{p0}(s)$, $f_{p1}(s)$, and $g_{p0}(s)$ are known from the previous steps, we now obtain, for $f_{p2}(s)$, $g_{p1}(s)$, $h_{p0}(s)$, and $l_{p0}(s)$, the closed coupled equations

$$f'_{p2}(s)+pf_{p+1,2}(s)=-\frac{2-\beta}{\beta}[f'_{p1}(s)+pg_{p1}(s)]+\frac{2}{\beta}p(p-1)[g_{p-1,0}(s)-h_{p-1,0}(s)]\ , \tag{C17a}$$

$$g'_{p1}(s)+(p+3)g_{p+1,1}(s)=\frac{2-\beta}{\beta}[-g'_{p0}(s)+2g_{p0}(s)-4h_{p0}(s)-(p-1)l_{p0}(s)]+2f_{p+1,1}(s)\ , \tag{C17b}$$

$$h'_{p0}(s)+(p+5)h_{p+1,0}(s)=6g_{p+1,0}(s)-3l_{p+1,0}(s)\ , \tag{C17c}$$

$$l'_{p0}(s)+(p+6)l_{p+1,0}(s)=4g_{p+1,0}(s)\ , \tag{C17d}$$

with the initial conditions (5.13)

$$f_{p2}(0)=g_{p1}(0)=0,\quad h_{p0}(0)=l_{p0}(0)=1\ . \tag{C18}$$

Just as in the previous step, it is convenient to solve the above four equations starting from the last one and working our way out to the top.

The procedure followed above, of finding the derivatives at $s=0$ of each function and then summing up the series, rapidly becomes too cumbersome. Therefore, guided by our previous results, (C11), (C15), and (C16), we propose a solution with the structure of a polynomial in s, divided by a power of $(1+s)$. In all cases we have been able to find a fully consistent solution. We just quote here the results:

$$l_{p0}(s)=\frac{1}{9(1+s)^{p+6}}(4s^6+24s^5+60s^4+84s^3+72s^2+36s+9)\ , \tag{C19}$$

$$h_{p0}(s)=\frac{1}{15(1+s)^{p+6}}(8s^6+48s^5+120s^4+165s^3+135s^2+60s+15)\ , \tag{C20}$$

$$g_{p1}(s)=-\frac{2-\beta}{\beta}\frac{1}{45(1+s)^{p+6}}[(10p+4)s^7+(40p+28)s^6+(60p+84)s^5+(45p+135)s^4+(15p+75)s]\ , \tag{C21}$$

$$f_{p2}(s)=\left[\frac{2-\beta}{\beta}\right]^2\frac{p}{90(1+s)^{p+6}}[(11p-9)s^8+(58-42)s^7+(173p-117)s^6+(336p-294)s^5+(420p-450)s^4+(300p-330)s^3+90(p-1)s^2]+\frac{2}{\beta^2}(\beta-1)(4-\beta)\frac{p}{90(1+s)^{p+4}}[(3p-5)s^6+(18p-30)s^5+(45-75)s^4+(60p-90)s^3+45(p-1)s^2]\ . \tag{C22}$$

Substituting (C11), (C15), and (C22) in the series expansion (5.9), we obtain

$$\langle T^p\rangle_s^{(\beta)}=\frac{N^p}{(1+s)^p}-\left[\frac{2-\beta}{\beta}\right]N^{p-1}\frac{ps^3}{3(1+s)^{p+2}}$$
$$+N^{p-2}\left[\left[\frac{2-\beta}{\beta}\right]^2\frac{p}{90(1+s)^{p+6}}[(11p-9)s^8+(58p-42)s^7$$
$$+(173p-117)s^6+(336p-294)s^5+(420p-450)s^4$$
$$+(300p-330)s^3+90(p-1)s^2]$$
$$+\frac{2}{\beta^2}(\beta-1)(4-\beta)\frac{p}{90(1+s)^{p+4}}[(3p-5)s^6+(18p-30)s^5+(45p-75)s^4$$
$$+(60p-90)s^3+45(p-1)s^2]\Bigg]+O(N^{p-3})\ . \tag{C23}$$

Similarly, substituting (C16) and (C21) in the series expansion (5.10) we obtain

$$\langle T^pT_2\rangle_s^{(\beta)}=N^{p+1}\frac{2s^3+6s^2+6s+3}{3(1+s)^{p+4}}$$
$$-N^p\frac{2-\beta}{\beta}\frac{1}{45(1+s)^{p+7}}$$
$$\times[(10p+4)s^7+(40p+28)s^6+(60p+84)s^5+(45p+135)s^4+(15p+75)s^3]+O(N^{p-1})\ . \tag{C24}$$

APPENDIX D: PERTURBATION CALCULATION OF THE WEAK-LOCALIZATION CORRECTION TO THE CONDUCTANCE FOR A QUASI-1D SYSTEM

Consider a finite sample of dimensions L_x,L_y,L_z with z the direction of the current flow. For spinless particles, the weak-localization correction to the conductivity is given by[6,7]

$$\delta\sigma=-\frac{e^2}{\pi\hbar}\frac{1}{L_xL_yL_z}\sum_k\frac{1}{k^2}\ . \tag{D1}$$

With the boundary conditions appropriate to a finite sample,[12] the three components of the wave vector k are given by

$$k_i=\frac{m_i\pi}{L_i},\quad i=x,y,z \tag{D2a}$$

with

$$m_x,m_y=0,1,2,\ldots\ ,$$
$$m_z=1,2,\ldots\ . \tag{D2b}$$

We write the conductance as

$$G=\frac{\sigma L_xL_y}{L_z}\equiv\frac{e^2}{h}g\ , \tag{D3}$$

then

$$\delta g=-\frac{2}{\pi^2}\sum_{m_x,m_y=0}^{\infty}\sum_{m_z=1}^{\infty}\frac{1}{m_x^2(L_z/L_x)^2+m_y^2(L_z/L_y)^2+m_z^2}. \tag{D4}$$

For a quasi-1D system, $L_x,L_y\ll L_z$, so that the most important contribution to the sum in (D4) comes from $m_x=m_y=0$, i.e.,

$$\delta g_{1D}\approx-\frac{2}{\pi^2}\sum_{m_z=1}^{\infty}\frac{1}{m_z^2}=-\frac{2}{\pi^2}\zeta(2)\ , \tag{D5}$$

where $\zeta(n)$ is Riemann's ζ function. In particular

$$\zeta(2)=\pi^2/6\ , \tag{D6}$$

so that

$$\delta g=-\tfrac{1}{3} \tag{D7}$$

as quoted in Sec. V.

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