Umklapp process and resistivity in one-dimensional fermion systems

T. Giamarchi*
AT&T Bell Laboratories, 600 Mountain Avenue, Murray Hill, New Jersey 07974-2070
(Received 7 December 1990)

The influence of umklapp scattering on the resistivity of an interacting-one-dimensional-fermion system (Luttinger liquid) is studied. By using a renormalization-group calculation and a memory-function approximation for the conductivity, it is possible to obtain its frequency and temperature dependence at arbitrary filling. At high temperature the conductivity behaves as a power law of the temperature with an exponent depending on the interactions. Away from half filling there is a crossover between this behavior and an exponential increase of the conductivity. At half filling, the low-temperature conductivity behaves as $e^{-\Delta/T}$, where Δ is the gap in the charge spectrum. It is argued that to get such behavior other scattering processes or phase-breaking processes are needed since in the presence of only electron-electron scattering the conductivity should, strictly speaking, be infinite at every finite temperature, even at half filling. Finally some results on the exponents of correlation functions and on the weight of the Drude peak obtained previously for the Hubbard model are shown to be generic features of any Luttinger liquid.

I. INTRODUCTION

Due to the discovery of high-T_c superconductivity[1] there has recently been a considerable interest in strongly correlated electronic systems and in particular in the two-dimensional Hubbard model.[2] A possible approach to this very difficult problem is to understand the physics of the one-dimensional Hubbard model and to try to apply it in higher dimensions.[3] The one-dimensional models are usually much easier to handle than their counterpart in higher dimensions and can even prove to be exactly solvable, as is the case for the 1D Hubbard model.[4] Even for more complicated models, very efficient techniques, such as bosonization or renormalization calculations,[5] are still applicable and are expected to give the correct physics. In addition to the physical insight that such one-dimensional interacting electron models can offer to understand higher dimensions, they also have proven to be of fundamental importance for purely one- or quasi-one-dimensional specific problems, e.g., quasi-one-dimensional organic conductors[6] or conducting polymers,[7] where interactions are known to play a major role.

As is well known, in one dimension an interacting-electron gas will be, for repulsive interactions, an insulator at half filling due to the existence of the umklapp process,[5] whereas away from half filling the umklapp process is expected to be irrelevant and the system is expected to be a conductor. It is therefore of considerable interest to understand how this Mott metal-insulator transition takes place. Such a question has received much attention in the recent past for the 1D Hubbard model. Using the Bethe-ansatz solution or related methods a number of authors[8-12] have focused on the zero-temperature behavior of the conductivity. It has been shown that away from half filling the conductivity presents a drude peak $\delta(\omega)$. The weight of the Drude peak has been computed and it has been shown that such weight vanishes at half filling, a signature of the metal-insulator transition.[10,12]

Despite this important breakthrough, such Bethe-ansatz methods are mainly limited to zero-temperature and zero-frequency properties, since it would otherwise entail a calculation of correlation functions, a very difficult task in the Bethe-ansatz framework. For the same reasons the conductivity of more general 1D models than the pure Hubbard one have been considered only a little although some attempts have been made.[10,11]

In this paper we will try to address the question of the conductivity of a general Luttinger liquid in one dimension, using bosonization techniques. These techniques have the disadvantage of being tractable only if the interactions are small compared with the bandwidth, but on the other hand they allow for many more physical quantities to be computed. Nevertheless, since there is no expected phase transition as the interactions are increased, as is well known from exact solutions[4] and renormalization-group arguments,[5] these techniques are expected to give the correct physical behavior for the whole interaction range. They have already been used with success for other scattering potentials such as impurities.[13,14]

The plan of the paper is as follows. In Sec. II, I review the Hamiltonian of a general Luttinger liquid that I will use in the following as well as the general formula for the conductivity. In Sec. III the conductivity is computed for arbitrary filling, temperature, and frequency, by using a memory-function approximation.[15] Such an approximation is valid if the umklapp term is small. In Sec. IV we show how the use of renormalization equations allows one to obtain the conductivity even in the case of a general umklapp process at, and away from, half filling, provided that the temperature is not too low or the doping too small. Finally, in Sec. V, using the Luther-Emery method,[16] I study the low-temperature behavior of the conductivity and argue that the 1D Hubbard model

shows a peculiar behavior as a function of the temperature. Using a method introduced by Schulz[17] for the commensurate-incommensurate transition, I also show that some results obtained for the Hubbard model can be generalized to an arbitrary Luttinger liquid.

II. HAMILTONIAN

Only a short derivation will be given here in order to fix the notation; more details can be found in Ref. 5.

Let us consider, for example, the discrete 1D Hubbard model:

$$H=-t\sum_{\langle i,j,\sigma\rangle}c^{\dagger}_{i,\sigma}c_{j,\sigma}+U\sum_{i}n_{i,\uparrow}n_{i,\downarrow}\ , \tag{2.1}$$

where $\langle\ \rangle$ stands for nearest neighbors. If the interaction term is small enough, then it is legitimate to linearize the spectrum close to the Fermi surface. One introduces left- and right-going fermions with momentum close to $\pm k_F$, denoted by $c_{\pm,k}$. The kinetic-energy term becomes

$$H_{\rm kin}=\sum_{k,\sigma}v_F k(c^{\dagger}_{+,k,\sigma}c_{+,k,\sigma}-c^{\dagger}_{-,k,\sigma}c_{-,k,\sigma})\ , \tag{2.2}$$

where $v_F=2t\sin(k_F\alpha)$, where α is the lattice spacing. As is well known, in one dimension the fermion operators can be represented in terms of the boson ones.[5] To do so, one defines the ρ and σ density operators by

$$\rho_{\pm}=\frac{1}{\sqrt{2}}(\rho_{\pm,\uparrow}+\rho_{\pm,\downarrow}),\quad \sigma_{\pm}=\frac{1}{\sqrt{22}}(\rho_{\pm,\uparrow}-\rho_{\pm,\downarrow})\ , \tag{2.3}$$

and the phase fields

$$\phi_{\nu}(x),\theta_{\nu}(x)=\mp\frac{i\pi}{L}\sum_{p\ (\neq 0)}\frac{1}{p}e^{-\alpha|p|/2-ipx}\times[\nu_{+}(p)\pm\nu_{-}(p)]\ , \tag{2.4}$$

where $\nu=\rho$ or σ and in A,B the upper sign refers to A. Using the fact that the fermion operators are expressed by

$$\psi_{r,\sigma}=\frac{1}{\sqrt{2\pi\alpha}}e^{irk_F x}e^{i/\sqrt{2}[r\phi_{\rho}-\theta_{\rho}+\sigma(r\phi_{\sigma}-\theta_{\sigma})]}\ , \tag{2.5}$$

the complete Hamiltonian (2.1) becomes[5]

$$H=H_{\rho}+H_{\sigma}+\frac{2g_3}{(2\pi\alpha)^2}\int dx\cos[\sqrt{8}\phi_{\rho}(x)+\delta x]+\frac{2g_{1\perp}}{(2\pi\alpha)^2}\int dx\cos(\sqrt{8}\phi_{\sigma}(x))\ , \tag{2.6}$$

where H_{ρ} and H_{σ} are defined by

$$H_{\nu}=\frac{1}{2\pi}\int dx\left[(u_{\nu}K_{\nu})(\pi\Pi_{\nu})^2+\left[\frac{u_{\nu}}{K_{\nu}}\right](\partial_x\phi_{\nu})^2\right]\ . \tag{2.7}$$

Π and ϕ are canonically conjugate variables and $\pi\Pi=\partial_x\theta$. The ρ and σ parts of the Hamiltonian (2.6) describe the charge and spin degrees of freedom of the system, respectively. The $g_{1\perp}$ term is the scattering between electrons of opposite spins with an exchange of momentum of $2k_F$. The g_3 term is the umklapp process and is the only process that does not conserve momentum. $\delta=4k_F-2\pi/\alpha$ measures the distance to half filling. $\pi/(2\alpha)$ would be the Fermi wave vector for a half-filled band, therefore if d is the doping ($d=0$ at half filling and $d=1$ for a filled band) one has $d=(\alpha\delta)/(2\pi)$. Note that here we assume that we work at a fixed number of particles, since k_F is directly related to the filling. Another way to incorporate deviations from half filling would have been to start from the half-filled Hamiltonian [$\delta=0$ in (2.6)] and to add a chemical potential μ.[18] Although the two representations are equivalent, to compute the conductivity it is inconvenient to work at fixed chemical potential since μ has a discontinuity when going away from half filling (see Sec. V). In the following (except Sec. V) we will work with a fixed number of particles.

For the Hamiltonian (2.1) the various coefficients in (2.6) and (2.7) are given by

$$\begin{aligned}&u_{\rho}K_{\rho}=u_{\sigma}K_{\sigma}=v_F\ ,\\ &u_{\rho}/K_{\rho}=v_F+U/\pi\ ,\\ &u_{\sigma}/K_{\sigma}=v_F-U/\pi\ ,\\ &g_{1\perp}=g_3=U\ .\end{aligned} \tag{2.8}$$

In fact, (2.6) describes the most general 1D Hamiltonian for a fermion system (with spin-conserving interactions) and the parameters u,K,g are the only ones necessary to describe the long-range properties of the system. In the following I will consider the more general Hamiltonian (2.6) without necessarily referring to the Hubbard model. The various u, K, and g will be taken as the parameters. In particular, in contrast to what happens in the 1D Hubbard model the umklapp strength g_3 and the other (momentum-conserving) interactions that enter K and u need not be the same.

Given the Hamiltonian (2.6), the easiest way to obtain the current operator j is to use the continuity equation $\partial_t\rho_t+\partial_x j=0$. The total density of charge is given by

$$\rho_t(x)=\sum_{r=\pm,\sigma}\rho_{r,\sigma}(x)=-\frac{\sqrt{2}}{\pi}\partial_x\phi_{\rho}(x)\ . \tag{2.9}$$

One therefore gets for the current

$$j=\frac{\sqrt{2}}{\pi}\partial_t\phi_{\rho}(x)=\sqrt{2}(u_{\rho}K_{\rho})\Pi_{\rho}\ . \tag{2.10}$$

The current operator depends only on the charge degrees of freedom and can be computed using only the charge part of the Hamiltonian (2.6) since in one dimension there is charge and spin separation as can be seen from Hamiltonian (2.6). Expression (2.10) differs from the naive value of the current of a system of velocity v_F, which would be $j=v_F(\psi^{\dagger}_{+}\psi_{+}-\psi^{\dagger}_{-}\psi_{-})=\sqrt{2}v_F\Pi$. This comes from the fact that for the continuum Hamiltonian (2.6) the density does not necessarily commute with the interactions giving rise to a renormalization of the current operator. If the interactions come from a well-defined lattice Hamiltonian, where of course $[\rho,H]=0$, one has indeed $u_{\rho}K_{\rho}=v_F$ as expected [see (2.8)]. It is also easy to show that the conductivity is given by[11]

$$\sigma(\omega)=\frac{i}{\omega}\left[\frac{2uK}{\pi}+\chi(\omega)\right], \tag{2.11}$$

where $\chi(\omega)$ is the retarded current-current correlation function. In (2.11) and in the following the ρ indices are dropped since only charge variable will now be considered:

$$\chi(\omega)=\langle j;j\rangle_\omega=-\frac{i}{L}\int dx\int_0^\infty dt\langle[j(x,t),j(0,0)]\rangle e^{i\omega t}. \tag{2.12}$$

Note that here what plays the role of the plasma frequency in the usual formulas for the conductivity[15] is uK. In the absence of umklapp scattering, (2.11) is easily evaluated to give

$$\sigma(\omega)=2uK\left[\delta(\omega)+\frac{i}{\pi}\mathrm{P}\left[\frac{1}{\omega}\right]\right], \tag{2.13}$$

P being the principal part thereof. Therefore the strength of the Drude peak is simply given by $2uK$.[10,11]

III. PERTURBATIVE CALCULATION

In this section I will assume that the umklapp process g_3 is sufficiently weak that some perturbative calculation of the conductivity as a function of g_3 can be performed, the other interactions contained in K being treated exactly. Of course the conductivity itself has a singular development in power of the scattering potential g_3, but the perturbative expansion can be performed by using a memory-function formalism.[15] If one assumes that the system is a normal conductor (σ finite) at zero frequency, then from (2.11) one gets $\chi(0)=-2uK/\pi$, and one can express the conductivity in terms of the meromorphic memory function $M(\omega)$ by

$$\sigma(\omega)=\frac{i2uK}{\pi}\frac{1}{\omega+M(\omega)}, \tag{3.1}$$

where[15]

$$M(\omega)=\frac{\omega\chi(\omega)}{\chi(0)-\chi(\omega)}. \tag{3.2}$$

The calculation of the memory function can be carried out perturbatively to give at the lowest order

$$M(\omega)=\frac{(\langle F;F\rangle_\omega^0-\langle F;F\rangle_{\omega=0}^0)/\omega}{-\chi(0)}. \tag{3.3}$$

The F operators take into account that the current is not a conserved quantity $F=[j,H]$ and $\langle F;F\rangle_\omega^0$ stands for the retarded correlation function of the operator F at frequency ω computed in the *absence* of the scattering potential ($g_3=0$). Expression (3.3) is correct at high frequency for arbitrary temperatures, but it does not necessarily remain valid at low frequencies even for finite temperatures. Its validity at low frequency implicitly assumes in a self-consistent way that the true conductivity behaves as (3.1) with $M(\omega\to0)=i\tau$ as some relaxation time. In particular, since expression (3.3) is computed for $g_3=0$, it neglects all effects of self-adjustments of the ground state to the scattering potential, and in that sense is equivalent to a Boltzmann approximation, which assumes phase-breaking processes between each scattering. Such phase-breaking processes would be inelastic processes other than the electron-electron interaction, e.g., the interaction of the electrons with a thermal bath, which, strictly speaking, would be needed for the system to reach thermal equilibrium.[19] The main effect of such a bath on the electrons is to make them lose their phase coherence. The Boltzmann approximation corresponds to the extreme case where such a randomization of the phase of the electron wave functions occurs before each collision process (i.e., here the umklapp interaction). Clearly this approximation breaks down if the effect of g_3 on the ground state is strong and, in particular, as is the case for the Hamiltonian (2.6), if g_3 leads to the opening of a gap[5] Δ in the charge excitation spectrum. We will come back to this point in Sec. V. Nevertheless, we will *assume,* as it is physically sensible at least at high temperatures, that the memory-function approximation gives reasonable results.

Using the expression of the current (2.10) and the umklapp term (2.6), one gets for the F operator

$$F=[j,H]=\frac{8g_3}{(2\pi\alpha)^2}(uK)i\sin[\sqrt{8}\phi(x,\tau)+\delta x]. \tag{3.4}$$

Since the correlation function of F is evaluated in the absence of the umklapp term [$g_3=0$ in (2.6)], one can use known formulas for the correlation functions of the boson operators[20] to get

$$\langle F;F\rangle_\omega=\frac{2g_3^2(uK)^2}{\pi^4\alpha^2u}\left[\sin(2\pi K)\left[\frac{2\pi\alpha T}{u}\right]^{4K-2}\times B(K-iS_+,1-2K)\times B(K-iS_-,1-2K)-\frac{\pi}{1-2K}\right] \tag{3.5}$$

with $S_\pm=(\omega\pm u\delta)/(4\pi T)$ and $B(x,y)=\Gamma(x)\Gamma(y)/\Gamma(x+y)$ is the β function. Using (3.3) one gets

$$M(\omega)=\frac{g_3^2K}{\pi^3\alpha^2}\left[\frac{2\pi\alpha T}{u}\right]^{4K-2}\frac{1}{\omega}\times[B(K-iS_+,1-2K)B(K-iS_-,1-2K)-B(K-iS_+^0,1-2K)B(K-iS_-^0,1-2K)], \tag{3.6}$$

where $S_\pm^0=S_\pm(\omega=0)$.

Equation (3.6) gives the full frequency and temperature dependence of the conductivity at every filling provided that a perturbative expansion in g_3 is possible. We will see in Sec. IV how this limitation can be circumvented by using the renormalization method.

Nevertheless, the limit of small g_3 already exhibits the

generic features of the complete solution in the various physically interesting limits.

A. Half filling

At half filling $\delta=0$, and therefore $S_+=S_-$. Expression (3.6) simplifies greatly in the two physically interesting regimes $\omega \gg T$ and $\omega \ll T$.

1. $\omega \gg T$

One then gets from (3.6)

$$M(\omega)\simeq \frac{g_3^2 K}{\pi^3\alpha^2}\sin(2\pi K) \times \Gamma^2(1-2K)e^{-i\pi(2K-1)}\frac{1}{\omega}\left[\frac{\alpha\omega}{2u}\right]^{4K-2} . \quad (3.7)$$

One can notice that the correction $M(\omega)/\omega$ to the conductivity in the absence of the umklapp process behaves as ω^{4K-4} from (3.7). Therefore, for $K>1$ the correction becomes negligible when $\omega\to 0$, indicating that the umklapp process does not contribute to the resisitivity of the system at sufficiently small frequency, whereas for $K<1$ the "correction" diverges at low frequency indicating that the conductivity of the system is dominated by the umklapp process. One recovers the usual boundary for the relevance of the umklapp term obtained by renormalization in the limit $g_3\to 0$ (see also Sec. V).

The coefficient of the correction is given by (3.7), but since it is obtained from a correlation function with anomalous dimensions it is expected to be nonuniversal. In contrast, the exponent should be a universal quantity. The conductivity behaves, therefore, as $\sigma(\omega)\sim 1/\omega$ if $K>1$ and $\sigma(\omega)\sim\omega^{3-4K}/g_3^2$ if $K<1$. In this last case, the behavior is similar to the temperature dependence (see Sec. III A 2). Obviously this behavior is not correct at low frequencies if one expects a gap in the charge spectrum, since in that case the conductivity vanishes for frequencies below the gap. This is due to the breakdown of the approximation leading to (3.7), as pointed out before.

2. $\omega \ll T$

One then gets from (3.3)

$$M(\omega)\simeq i\frac{g_3^2 K}{\pi^3\alpha^2}B^2(K,1-2K)\cos^2(\pi K)\frac{1}{T}\left[\frac{2\pi\alpha T}{u}\right]^{4K-2} . \quad (3.8)$$

Using (3.1) one gets

$$\sigma(0)=\frac{i2uK}{\pi M(0)} . \quad (3.9)$$

The temperature dependence of the conductivity is therefore $\sigma\sim T^{3-4K}$. The noninteracting electron gas corresponds to $K=1$ and one recovers $\sigma\sim 1/T$. Such a behavior is the Fermi-liquid behavior, since in one dimension due to phase-space restrictions the usual $1/T^2$ is replaced by a $1/T$ as one can easily see by a simple Boltzmann-equation calculation.[21] For arbitrary interactions the conductivity behaves as a power law of the temperature with an exponent depending on the interactions. This indicates that in one dimension there is a strong renormalization of the scattering process due to the various fluctuations of a one-dimensional electron gas (namely charge-density wave or superconductive fluctuations). A similar effect also happens for other scattering potentials, such as impurities,[13,14] but with a different exponent. One can see that, compared to the Fermi-liquid behavior $1/T$ repulsive interactions ($K<1$) will enhance the scattering potential whereas for attractive interactions ($K>1$) the effect is opposite and the system becomes less and less sensitive to the scattering potential due to the superconducting fluctuations. If the interactions become sufficiently repulsive ($K<\frac{3}{4}$) the conductivity *decreases* with temperature even at temperatures higher than the charge gap Δ.

The above formula would indicate that the conductivity behaves as a power law of the temperature. Again this approximation is not valid for temperatures lower than the charge gap Δ, since one would expect then an exponential decay of the conductivity. We will come back to this point in the Sec. V.

B. Away from half filling

Again two interesting regimes occur depending on whether $T\ll(\omega,u\delta)$ or $T\gg(\omega,u\delta)$. We will not consider here the cases $(\omega,T)\gg u\delta$ since in that case we are led back to expressions similar to those obtained for the half-filled case, since at sufficiently large temperatures or frequencies the system is unable to distinguish whether or not it is at half filling.

1. $\omega \gg T$

Here one gets for $\omega < u\delta$

$$M(\omega)=\frac{g_3^2 K}{\pi^3\alpha^2}\sin(2\pi K)\Gamma^2(1-2K)\left[\frac{\alpha}{2u}\right]^{4K-2} \times\frac{1}{\omega}\{[(u\delta)^2-\omega^2]^{2K-1}-(u\delta)^{4K-2}\} . \quad (3.10)$$

By expanding in ω one sees that the first term is proportional to ω:

$$M(\omega)\simeq\frac{g_3^2 K}{\pi^3\alpha^2}\sin(2\pi K)\Gamma^2(1-2K)\left[\frac{\delta\alpha}{2}\right]^{4K-2} \times(1-2K)\frac{\omega}{(u\delta)^2} . \quad (3.11)$$

Again, if $K>1$ the memory function vanishes when $\delta\to 0$, whereas it is strongly relevant if $K<1$, indicating again that the umklapp process plays or does not play an important role depending on K.

The next-order correction in $M(\omega)$ would be of order ω^3 giving a correction of order ω to the conductivity. One can interpret this result by noting that away from half filling one can forget about the umklapp term at sufficiently low frequencies or large scales. The physical properties of the system can thus be computed with a

long-range Hamiltonian similar to (2.6) but with $g_3=0$ and a renormalized exponent K^* and velocity u^*. The conductivity therefore behaves as [see (2.13)]

$$\sigma(\omega)=2u^*K^*\left[\delta(\omega)+\frac{i}{\pi}P\left[\frac{1}{\omega}\right]\right] + \text{regular corrections} , \quad (3.12)$$

which is the result suggested by (3.10).

If $\omega > u\delta$ it is easy to see from (3.6) that $M(\omega)$ acquires an imaginary part, whereas it is purely real for $\omega<\delta$ (in the limit where $T\to 0$). In order to explain this behavior one has to note that the memory-function approximation computes the conductivity with the bare Hamiltonian H_0 and all the effects of g_3 are contained in the F terms. It is equivalent to say that the scattering potential g_3 will open an infinitesimally small (compared with ω and T) gap in the charge spectrum at half filling around $k=\pm\pi/2$ instead of a finite one (see Sec. V). Up to frequencies $u\delta$, only intraband transitions among the carriers above half filling (and therefore above the "gap") are allowed. As is well known, such transitions give no contribution to the real part of the conductivity except at $\omega=0$. For higher frequencies interband transitions are allowed across the "gap" giving rise to absorption. Of course when a finite gap is considered (see Sec. V), the value at which absorption appears is not $u\delta$ but $\omega=[(u\delta)^2+\Delta^2]^{1/2}$, where Δ would be the gap in the charge excitation spectrum.

2. $T \gg \omega$

Then one obtains for the memory function

$$M(\omega)=i\frac{g_3^2K}{2\pi^3\alpha^2}\left[\frac{\alpha\delta}{2}\right]^{4K-2}\sin^2(2\pi K) \times\frac{1}{2T}e^{-(u\delta)/2T}\Gamma^2(1-2K) . \quad (3.13)$$

From (3.13) and (3.1) one thus sees that at low temperature the conductivity increases exponentially away from half filling. This corresponds to the fact that in one dimension the umklapp process is very rapidly frozen due to phase-space restrictions. The exponential behavior in the limit $K\to 1$ corresponds to the result obtained by a simple Boltzmann approximation.

IV. FINITE g_3

The results of Sec. III are valid if a perturbation expansion in powers of g_3 is possible. However, it is well known that in one dimension the g_3 term leads to a singular perturbative expansion.[5] One way to handle this difficulty is to vary the cutoff α in the Hamiltonian (2.6) and to generate renormalization equations for the various parameters u, K, and g_3. In order to compute the conductivity for arbitrary g_3, such a renormalization of K and g_3 must also be taken into account.[14] The way to do it is to iterate the renormalization equations up to a point where a perturbative expansion in g_3 can be performed, and then to use the formulas of Sec. III with the renormalized parameters to compute the conductivity.

A. Renormalization equations

First one has to know the equations giving the renormalized parameters as a function of the cutoff α. One can notice that the Hamiltonian (2.6) is exactly the same as the one describing the commensurate-incommensurate transition.[17,18,22] For such a Hamiltonian the renormalization equations are well known and I will not reproduce the derivation here. A complete derivation (with nearly the same notations) can be found in Ref. 23. The equations are

$$\begin{aligned}\frac{dK}{dl}&=-\tfrac{1}{2}y_3^2K^2J_0(\delta(l)\alpha) ,\\ \frac{dy_3}{dl}&=(2-2K)y_3 ,\\ \frac{du}{dl}&=-\frac{y_3^2}{2}uKJ_2(\delta(l)\alpha) ,\\ \frac{d\delta}{dl}&=\delta(l)+\frac{y_3^2}{2\pi\alpha}J_1(\delta(l)\alpha) ,\end{aligned} \quad (4.1)$$

where $y_3=g_3/(\pi u)$, and l describes the renormalization of the cutoff by $\alpha(l)=\alpha e^l$. The J are Bessel functions. The existence of the Bessel function is related to the use of a sharp cutoff in real space, whereas a smooth cutoff would have led to nonoscillatory functions.[22] The important point about the Bessel functions in (4.1) is that when $\delta(l)\sim 1/\alpha$ they start to oscillate and the renormalization coming from the g_3 term is stopped. At this point one can perform a safe perturbative expansion in g_3. If one is at finite temperature the renormalization has also to be stopped at length $\alpha(l)$ comparable to the thermal length u/T, which means for $l\simeq\ln(W/T)$, where $W=u/\alpha$ is of the order of the bandwidth. Here again one can perform an expansion in g_3.

If one is at half filling $\delta=0$ one then recovers the usual Kosterlitz-Thouless equations[24] with a separatrix at $K-1=y_3/2$ between a regime where g_3 is irrelevant and a regime where it is relevant.

I will focus in the following on the case $K<1$ since it would correspond for the 1D Hubbard model to repulsive interactions. In that case q_3 increases under renormalization, and flows to strong coupling. The renormalization equations, therefore, become inapplicable at length scales of the order of the gap opened by the umklapp processes at half filling. Note that if one is away from half filling, since $\delta\neq 0$ there is always one length scale at which δ stops the renormalization. Therefore we can expect to get from the renormalization and memory function the correct temperature dependence of the conductivity as long as the temperature is not too small at half filling ($T>\Delta$) or the doping is not too small away from half filling.

B. Half filling

In this case only the temperature stops the renormalization. The simple limit of Sec. III is equivalent to

neglecting the renormalization of K due to g_3 and to consider only the renormalization of g_3 itself. One has from (4.1)

$$g_3(l)=g_3 e^{(2-2K)l} . \tag{4.2}$$

Since one stops the renormalization when $l=\ln(u/T\alpha)$ by using formula (3.8) with the new cutoff $\alpha(l)=u/T$, one gets

$$\sigma(T)\sim\frac{1}{g_3^2}T^{3-4K} \tag{4.3}$$

recovering the result of Sec. III A 2. The prefactor is meaningless since it depends on the precise way of doing the renormalization and the precise value at which the renormalization is stopped. Of course such an approximation is only valid if the initial value of g_3 is sufficiently small *and* the renormalization is stopped sufficiently early (high temperatures) that g_3 remains small compared to K. If one takes, for example, the K and g_3 that would come from a 1D Hubbard model [see (2.8)] such an approximation would seldom be valid.

If now g_3 is finite the renormalization of K has to be taken into account and the simple power law (4.3) is no longer valid. By numerically integrating (4.1) and using (3.8) one can obtain the full temperature dependence of the conductivity for various values of the interactions, at least in the regime where the equations are still valid, which corresponds roughly to $T>\Delta$. The results are shown in Fig. 1, for the special case of the Hubbard model, K and g_3 given as a function of U by (2.8). The renormalization equations have been stopped when $g_3\sim 1$, which corresponds roughly to $T\sim\Delta$. One can notice that the power-law behavior (4.3) that is valid at high temperature is modified by the renormalization of K and the exponent is weakened, indicating that the decay of the conductivity with temperature is sped up. The validity of the renormalization equations breaks when $T\sim\Delta$ and the low-temperature behavior has to be obtained in a different way. The gap Δ can also be obtained by the renormalization equations, using the method of Ref. 25, and is shown in Fig. 1 for the various values of U studied. Of course for small U the value obtained is identical to the Bethe-ansatz solution.[4,26] Some comments on the low-temperature solution will be given in Sec. V.

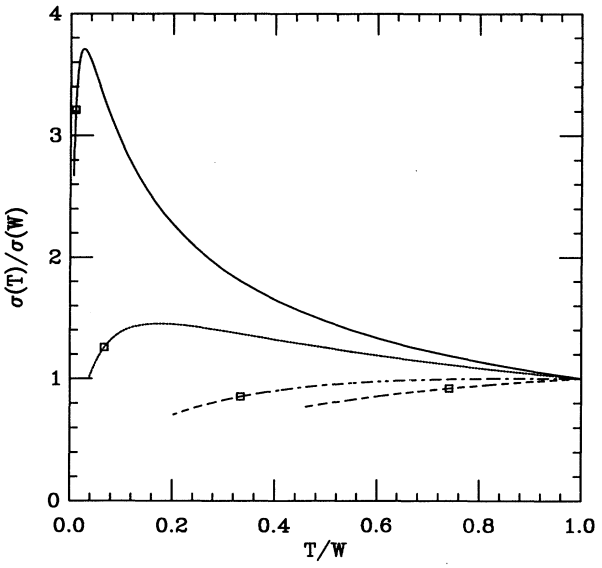

FIG. 1. Temperature dependence of the conductivity at half filling for the Hubbard model. $W\simeq u/\alpha\sim 2t$ is half the bandwidth and all curves have been normalized with $\sigma(W)$. The solid small dots, dash dots, and dashes are for $U/(2\pi t)=0.2$, 0.3, 0.5, and 0.7, respectively. The curves are obtained by iteration of the renormalization equations up to $g_3/(\pi v_F)=1$. The value of the normalized charge gap Δ/W is indicated as a square for each value of U. $T\sim\Delta$ is roughly the scale at which the renormalization equation stops being valid.

C. Away from half filling

In this case there are two possible cutoffs for the renormalization equations, T and δ. If $T\gg u\delta$ the renormalization equations are stopped at a length $l\simeq\ln(W/T)$ and δ introduces little changes in the renormalization flow (4.1) except for a small renormalization of the velocity u. One therefore basically recovers the half-filled behavior described in Sec. IV B. On the other hand, if $T\ll u\delta$ the renormalization will be cut off by the Bessel functions and by using (3.13) with a cutoff $\alpha(l)\sim 1/\delta$ one will recover the exponential increase of the conductivity with the temperature. The full temperature dependence of the conductivity can again be studied by the same method as in Sec. IV B. For the sake of simplicity I will now focus on the determination of the crossover temperature T^* separating the high temperature "half-filled"-like region from the low-temperature exponential increase of

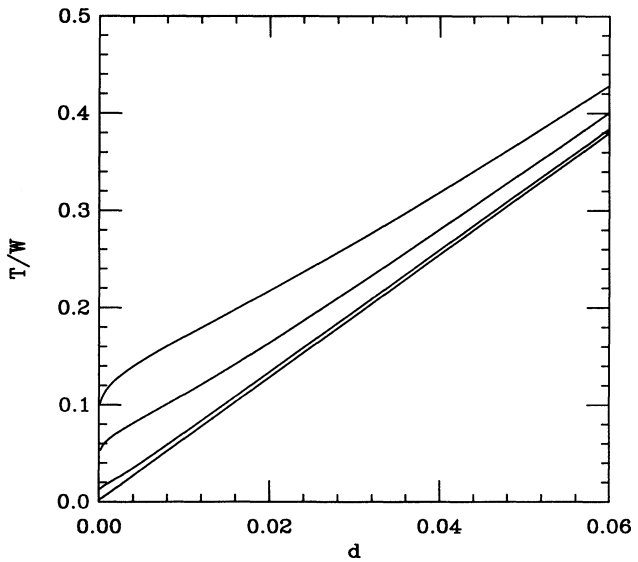

FIG. 2. Crossover temperature between the half-filled-like behavior and an exponential increase of the conductivity away from half filling as a function of the doping $d=(\alpha\delta)/(2\pi)$ for the Hubbard model. The four curves starting from the lower one are for $U/(2\pi t)=0.2$, 0.3, 0.5, and 0.7, respectively.

the conductivity. If there are no interactions the scale at which δ stops the renormalization is, using (4.1), $e^{l}=1/(\alpha\delta)$, and the crossover occurs when this scale is just the thermal scale, i.e., when $T^*/W=\alpha\delta$. When interactions are present, δ is renormalized in a nontrivial way and T^* is changed. The result is shown in Fig. 2. One can see that the effects of the interactions are in some sense to enhance the departure from half filling, in the sense that the umklapp term will be frozen (exponential increase of the conductivity) at a higher temperature than for a noninteracting system. At very small doping the renormalization equations flow to too large coupling before δ cuts the renormalization, for the first-order renormalization equations to be trustable. So T^* for very small doping cannot be obtained this way.

V. LUTHER-EMERY METHOD

Clearly although the renormalization and memory-function approximations provide a useful tool to study the conductivity in the high-temperature or -frequency region at half filling or the full behavior not too close to half filling, they break down in the interesting region of low-temperature behavior at half filling or very close to half filling.

To try to obtain the correct behavior in this region we will rewrite the Hamiltonian (2.6) in a representation with a constant chemical potential μ. The charge part of the Hamiltonian becomes

$$H=H_\rho-\mu\int dx\,\partial_x\phi_\rho+\frac{2g_3}{(2\pi\alpha)^2}\int dx\,\cos(\sqrt{8}\phi_\rho)\ . \tag{5.1}$$

One can use then the procedure introduced by Luther and Emery.[16] If one introduces the new field $\tilde{\phi}=\sqrt{2}\phi_\rho$, the Hamiltonian can be written as an Hamiltonian of spinless fermions:

$$H=H_0+\frac{\pi u\sinh(2\theta)}{L}\sum_p\{2\rho_+(p)\rho_-(-p)-f_1[\rho_+(p)\rho_+(-p)+\rho_-(p)\rho_-(-p)]\}\ , \tag{5.2}$$

where now the ρ operators are the density operators of the new spinless fermions. H_0 is the Hamiltonian

$$H_0=\sum_k vk(c^\dagger_{1,k}c_{1,k}-c^\dagger_{2,k}c_{2,k})+\frac{g_3}{2\pi\alpha}\sum_k c^\dagger_{1,k}c_{2,k}+\text{H.c.} \tag{5.3}$$

with a velocity given by

$$v=u[\cosh(2\theta)+f_1\sinh(2\theta)]\ ,\qquad e^{2\theta}=1/2K_\rho\ . \tag{5.4}$$

The f_1 term has been introduced by Schulz.[10] Since the last term in (5.2) is just the kinetic energy written with boson operators, f_1 is arbitrary. Its utility will be clear below. For $K=\frac{1}{2}$ one is on the so-called Luther-Emery line[16] and the interaction between the two species of fermions disappears and H_0 can be diagonalized by a Bogoliubov transformation. This gives the new energies $E_{1,2}=\pm\{(vk)^2+[g_3/(2\pi\alpha)]^2\}^{1/2}$ and leads to the opening of a gap. If one is away from the Luther-Emery line the interaction term in (5.2) gives a renormalization of the gap but the energies keep the form $E_{1,2}=\pm[(vk)^2+(\Delta/2)^2]^{1/2}$. The true gap Δ can either be obtained from renormalization-group arguments or from an exact solution.[17] If one is at half filling the chemical potential is in the gap so that all 2 states are occupied and all 1 states are empty. At zero temperature the system is obviously an insulator, and the real part of the conductivity is zero up to frequencies equal to the gap. For frequencies higher than the gap, interband transitions can occur and the conductivity is finite.

One can also compute the temperature dependence of the conductivity on the Luther-Emery line, since now one ends basically with a two-band free-fermion system with the lower band filled at $T=0$. The "fermions" here correspond to solitons of charge in the original Hamiltonian and are therefore current-carrying objects. The current of the original Hamiltonian can be expressed in terms of the spinless fermions of (5.3) by

$$j=\frac{\sqrt{2}}{\pi}\partial_t\phi_\rho=\frac{1}{\pi}uK\tilde{\phi}=\sum_k v(c^\dagger_{1,k}c_{1,k}-c^\dagger_{2,k}c_{2,k}) \tag{5.5}$$

and up to a numerical coefficient is the current of the spinless fermion problem. It is easy to see that at finite temperature, although the number of thermally excited carriers at a given temperature is exponentially small, $n\sim e^{-\Delta/T}$, these carriers do not scatter since the only source of scattering comes from the umklapp term already absorbed in the diagonalization of the Hamiltonian. So, strictly speaking, if the umklapp term is the only source of scattering, the conductivity would turn out to be infinite at every finite temperature, a rather unphysical result. One can ask whether or not such a result is an artifact of our continuum Hamiltonian and/or the special Luther-Emery line. In fact, assuming that all the eigenstates of the Hamiltonian are known, the real part of the conductivity is given by[27]

$$\sigma(\omega)=\frac{\pi\beta}{\nu}\sum_{n,m}|\langle n|j|m\rangle|^2e^{-\beta E_n}\delta(\omega+E_n-E_m)\ , \tag{5.6}$$

where n,m are the exact eigenstates and E_n the exact energies. Therefore, if a thermodynamic number of excited states such that $\langle n|j|n\rangle\neq 0$ exists, then from (5.6) there will be a $\delta(\omega)$ part in the conductivity and therefore an infinite static conductivity. Such states exist in the Hubbard model,[28] and reduce on the Luther-Emery line to the simple solitons. One can therefore expect the result of an infinite conductivity not to be an artifact of the continuum Hamiltonian but due to absence of phase-breaking processes that would give a finite lifetime to such states at nonzero temperature. Of course if other *finite* scattering is present the conductivity will be regular with a characteristic lifetime depending on such processes. But in order to have a well-defined electron-electron driven conductivity one would need the conductivity to become finite as soon as the *smallest amount* of the

phase-breaking process is included in the Hamiltonian. Whether or not this supposition is correct is not clear at the moment, although it seems reasonable from physical grounds. Note that the memory-function approximation corresponds to the other extreme limit where one assumes that the phase coherence is lost before each collision, not allowing the eigenstates to adapt to the scattering potential.

Although the temperature dependence is somehow pathological, one can expect the frequency dependence to be given correctly. One can therefore use (5.2) to address the question of the weight in the Drude peak when the system is close to half filling. Away from half filling there is still a gap in the charge spectrum, but now the chemical potential is above the gap to accommodate the extra carriers, and as far as only very-low-frequency behavior is concerned, the gap will not be of practical importance, one being able to consider only the dispersion relation very close to the Fermi level. Of course if the frequency is sufficiently high to allow transitions across the gap, then there will be interband absorption, as was already discussed in Sec. IV B 1.

In fact, a similar problem was studied a long time ago by Schulz[17] in the context of the commensurate-incommensurate transition, and I will follow his method here. Away from half filling the chemical potential is in the upper band and in addition to the 2 states the 1 states up to a value k_c are occupied. If δn is the filling, $\delta n = k_c/\pi$. Then assuming that we are interested only in very-low-frequency behavior, the spectrum can again be linearized around k_c. Only the interaction processes scattering electrons at the (new) Fermi level need to be kept from (5.2). As pointed out by Schulz, if f_1 is chosen to cancel all the $\rho_r(p)\rho_r(-p)$ (so-called g_4 terms) that appear after the linearization, then one has[17]

$$H=\sum_k v_c k(\alpha_k^\dagger\alpha_k-\beta_k^\dagger\beta_k) +2\pi u \sinh(2\theta) f(k_c)\sum_p \rho_1(p)\rho_2(-p) \tag{5.7}$$

with

$$v_c=\frac{\partial E}{\partial k}=\frac{v^2 k_c}{\sqrt{(vk_c)^2+(\Delta/2)^2}}\ , \qquad f_1=\frac{1}{1+4(vk_c)^2/\Delta}\ , \qquad f(k_c)=v_c^2/v^2\ . \tag{5.8}$$

The important point is that now the interaction term is $\sim k_c^2$ so sufficiently close to half filling the interaction term becomes negligible compared to the Fermi energy so that the linearization of the spectrum is well justified. The exponents of the various correlation functions can be determined similarly:[17]

$$K=\frac{1}{2}\left[1-\frac{4uk_c}{\Delta}\sinh(2\theta)\right]\ . \tag{5.9}$$

Therefore, $K\to\frac{1}{2}$ at half filling, as already obtained for the Hubbard model. (5.9) shows that it is a much more general result holding for any Luttinger liquid. One can see that this value is approached as k_c, which again should be a universal behavior although the prefactor clearly depends on the details of the model. Such a result agrees with the numerical results of Schulz.[10] The velocity of the excitations is given by v_c and from (5.8) behaves as $v_c\sim k_c/\Delta$. Since uK gives the weight of the Drude peak [see (2.11)], one recovers that close to half filling the weight of the $\delta(\omega)$ part of the conductivity vanishes *linearly with doping and with a slope inversely proportional the umklapp gap.* Again such a result is consistent with the numerical[10] and analytical[12] results obtained for the Hubbard model. The present analysis shows the results should hold even for more-complicated models.

VI. CONCLUSION

In this paper I have studied the influence of umklapp scattering on the conductivity of a one-dimensional Luttinger liquid. By using a renormalization-group method and a memory-function approximation, it is possible to obtain the temperature and frequency dependence of the conductivity for arbitrary fillings, at least not too close to half filling or at not too low temperatures or frequencies. One finds that for a very small umklapp term the conductivity behaves at half filling as $\sigma\sim T^{3-4K}$, where K is an exponent characteristic of the Luttinger liquid and entering into the decay of various correlation functions. K depends on the interactions and $K<1$ for repulsive ones. If, as is the case in the Hubbard model, the umklapp term is of the same order as the other interactions, the renormalization of K by the umklapp has to be taken into account and this simple power law is no longer valid, but similar features remain. The renormalization of K tends to decrease the conductivity.

Clearly this is to be contrasted with what one can expect from electron-electron interactions in higher dimensions, where the resisitivity behaves as T^2. In the one-dimensional case for repulsive interactions the resisitivity increases always *faster* than T. Such a measure of the resisitivity could prove to be a tool to determine if one is in the one- or three-dimensional regime for quasi-one-dimensional conductors.

Away from half filling there is a crossover temperature above which the temperature behavior is similar to the one at half filling and below which the conductivity starts to increase exponentially with the temperature. Again the full dependence can be obtained from the method. Interactions make the system more sensitive to doping in the sense that it crosses over to a non-half-filled behavior (exponential increase of the conductivity) for a higher temperature than in the absence of interactions.

I also argue that, strictly speaking, if one does not take into account phase-breaking mechanisms the conductivity should be infinite at every finite temperature if only electron-electron interactions are considered. This pathology occurs because of the lack of phase-breaking processes.

Some results that have been proven only for the Hubbard model on the behavior of the weight of the Drude peak D and exponents, namely $K\to\frac{1}{2}$ and $D\sim\delta/\Delta$,

where K is the exponent controlling the decay of density-density correlation functions, δ the doping, and Δ the gap, are shown to be valid for any one-dimensional Luttinger liquid.

Another question of interest would be the competition between the disorder and the umklapp process. As pointed out by Shankar, the smallest amount of disorder will destroy the umklapp gap,[11] and the system should behave as an Anderson insulator at least for sufficiently large length scales. But if the disorder is very weak, the umklapp gap will dominate the short-range properties of the system, so that one could expect to see a crossover between a Mott and an Anderson insulator when varying the temperature. Note that disorder also gives a power-law dependence of the conductivity with temperature but with a different exponent $\sigma(T) \sim T^{1-K_\rho}$.[13,14] How the conductivity will behave as a function of the temperature when both processes are included is unclear.

ACKNOWLEDGMENTS

It is a pleasure to thank H. J. Schulz, A. J. Millis, and B. S. Shastry for many invaluable discussions.

*On leave from Laboratoire de Physique des Solides, Université de Paris–Sud, Bâtiment 510, 91405 Orsay CEDEX, France.

[1]J. G. Bednorz and K. A. Miller, Z. Phys. B **64**, 189 (1986).
[2]P. W. Anderson, Science **235**, 1196 (1987).
[3]Y. Ren and P. W. Anderson (unpublished).
[4]E. H. Lieb and F. Y. Wu, Phys. Rev. Lett. **20**, 1445 (1968).
[5]V. J. Emery, *Highly Conducting One-Dimensional Solids,* edited by J. T. Devreese *et al.* (Plenum, New York, 1979), p. 247; J. Solyom, Adv. Phys. **28**, 209 (1979), and references therein. The bosonization has been formulated as an operator identity by R. Heindenreich, R. Seiler, and D. A Uhlenbrock, J. Stat. Phys. **22**, 27 (1980); F. D. M. Haldane, J. Phys. C **12**, 4791 (1979); **14**, 2585 (1981).
[6]D. Jérome and H. J. Schulz, Adv. Phys. **31**, 299 (1982).
[7]A. J. Heeger, S. Kivelson, J. R. Schrieffer, and W. P. Su, Rev. Mod. Phys. **60**, 781 (1988).
[8]B. S. Shastry and B. Sutherland, Phys. Rev. Lett. **65**, 243 (1990).
[9]C. J. Hamer, G. R. W. Quispel, and M. T. Batchelor, J. Phys. A **20**, 5677 (1987).
[10]H. J. Schulz, Phys. Rev. Lett. **64**, 2831 (1990).
[11]R. Shankar, (unpublished).
[12]N. Kawakami and S. K. Yang, Phys. Rev. Lett. **63**, 3063 (1990).
[13]A. Luther and I. Peschel, Phys. Rev. Lett. **32**, 992 (1974).
[14]T. Giamarchi and H. J. Schulz, Europhys. Lett. **3**, 1287 (1987); Phys. Rev. B **37**, 325 (1988).
[15]W. Götze and P. Wölfle, Phys. Rev. B **6**, 1226 (1972).
[16]A. Luther and V. J. Emery, Phys. Rev. Lett. **33**, 589 (1974).
[17]H. J. Schulz, Phys. Rev. B **22**, 5274 (1980).
[18]G. Montambaux, M. Heritier, and P. Lederer, Phys. Rev. B **33**, 7777 (1986).
[19]See, for example, H. U. Baranger and A. D. Stone, Phys. Rev. B **40**, 8169 (1989); N. Trivedi and D. A. Browne, *ibid.* **38**, 9581 (1988); K. S. Chow, D. A. Browne, and V. Ambegaokar, *ibid.* **37**, 1624 (1988); M. Büttiker, Y. Imry, R. Landauer, and S. Pinhas, *ibid.* **31**, 6207 (1985); and references cited therein.
[20]H. J. Schulz and C. Bourbonnais, Phys. Rev. B **27**, 5856 (1983); H. J. Schulz, *ibid.* **34**, 6372 (1986).
[21]J. M. Ziman, *Electrons and Phonons* (Clarendon, Oxford, 1962).
[22]B. Horowitz, T. Bohr, J. M. Kosterlitz, and H. J. Schulz, Phys. Rev. B **28**, 6596 (1983).
[23]T. Giamarchi and H. J. Schulz, J. Phys. **49**, 819 (1988).
[24]J. M. Kosterlitz, J. Phys. C **7**, 1046 (1974).
[25]A. I. Larkin and J. Sak, Phys. Rev. Lett. **39**, 1025 (1977).
[26]A. A. Ovchinnikov, Zh. Eksp. Teor. Fiz. **57**, 2137 (1969) [Sov. Phys.—JETP **30**, 1160 (1970)].
[27]G. D. Mahn, *Many Particle Physics* (Plenum, New York, 1981).
[28]H. J. Schulz and B. S. Shastry (private communication).