PHYSICAL REVIEW B VOLUME 44, NUMBER 7 15 AUGUST 1991-I

Pinning energies and phase slips in weakly pinned charge-density waves

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(Received 8 February 1991)

The distribution of pinning energies for a charge-density wave weakly pinned by random impurities is shown to be strongly non-Gaussian, with strongly pinned regions vastly more probable than naive estimates would indicate. Implications for the observability of phase slips in charge-density-wave experiments are discussed.

I. INTRODUCTION

The dynamics of sliding charge-density waves (CDW's) has been a subject of interest for many years now.[1] Although in principle CDW's are described by both amplitude and phase degrees of freedom, it has been customary to describe weakly pinned CDW's using the Fukuyama-Lee-Rice (FLR) model, which has phase degrees of freedom only.[2] The rationale for ignoring amplitude fluctuations is that long-wavelength phase distortions can have arbitrarily low energy, whereas amplitude changes always cost a finite energy. When the pinning is weak, it is plausible to expect only low-energy distortions to occur.

However, recently it was shown that in an infinite-size sample the FLR model cannot be used to describe CDW's in the presence of a uniform electric field because unbounded strains build up, which leads to the generation of phase slips.[3] The large strains arise because within the FLR model, even in the presence of random pinning, the entire CDW has the same time-averaged velocity, which is imposed by elastic forces. Since a region with an anomalously small or large pinning strength needs a force scaling with the volume of the region to keep it moving at the mean velocity, while the amount of the system available to exert this (elastic) force scales as the surface area of the region, the strain per unit area scales as the linear dimension of the region. Therefore, this strain can become arbitrarily large even for bounded fluctuations in the pinning strength. However, although it was shown that in principle phase slips must occur in a sample of infinite size, the question of how many phase slips occur in experimentally relevant situations (i.e., finite-size samples) was not addressed.

Simple estimates of the number of phase slips in three-dimensional samples[3] yield the result that they should be unobservable in real CDW materials. Despite this, comparison of the dynamical predictions of the FLR model[4] to experiment yields indications that phase slips are important in experimental situations. Although the FLR model appears capable of describing many aspects of CDW experiments, including the observation of a reasonably sharp threshold field where CDW conduction sets in,[5] a few experimental observations are not easily explained using this model. In particular, although broadband noise is commonly observed in the sliding state of a CDW,[6] the FLR model with periodic boundary conditions yields a periodic sliding state with no low-frequency noise components.[7] Further evidence for the relation between broadband noise and phase slips is the observed correlation between the amplitude of the broadband noise and macroscopic velocity inhomogeneities in $NbSe_3$ samples.[8] Thus, it is reasonable to suspect that the phase slips that are omitted in the FLR model lead to experimentally relevant phenomena when the CDW slides.

This paper concerns the distribution of pinning energies as well as the density of regions with phase slips in weakly pinned CDW's, with the goal of estimating whether phase slips are observable in real CDW samples. In the weak-pinning limit, the CDW is not pinned by any single impurity but rather by the collective action of many impurities over a substantial length scale. It will be argued that in this regime it is reasonable that broadband noise (which we speculate to occur only in the presence of phase slips) occurs in samples whose threshold fields appear quite sharp.

First, it is shown that the random pinning potential leads to a non-Gaussian distribution of pinning energies. Because strongly pinned regions are smaller than weakly pinned ones, the number of strongly pinned regions falls off much more slowly than a Gaussian distribution does. Thus one finds that the number of strongly pinned regions is much larger than naive estimates would indicate. The problem of calculating the distribution of pinning energies is closely related to that of calculating the density of states in impurity band tails in semiconductors.[9]

It is also shown that the number of regions that have large strains and hence phase slips at their boundaries is enormously larger than naive estimates based on the ratio of the phase-slip energy to the threshold field would indicate.[3] The number of phase slips is not simply related to the number of regions with a given pinning energy; regions with phase slips are predominantly ones that are basically undistorted, so that the pinning energy per unit volume is the bare impurity potential strength. The number of phase slips is described by a Gaussian distribution, but the width of the distribution is governed by the bare impurity potential rather than the threshold field of a typical region. This result follows because the phase slips occur almost entirely at the boundaries of very small regions whose impurities are correlated so that elastic distortions in their interiors are negligible, whereas the threshold field of a typical region is determined by a com-

petition between elastic and impurity energies. Because the bare impurity strength is much greater than the threshold field in materials with sliding CDW's, phase slips are much more likely to occur on experimental scales than the estimates based on Ref. 3 would suggest.

This paper is organized as follows. In Sec. II the standard argument that shows that for $d<4$ the undistorted state is unstable for any nonzero impurity potential strength[2] is reviewed and generalized to yield the distribution of local pinning energies in the system. The arguments in this section are very similar to those used in the context of band tails in semiconductors. The analogy between these two systems is discussed.

In Sec. III a simple generalization of the scaling analysis used in Sec. II is used to show that the number of phase slips is governed by the ratio $V/E_{\rm ps}$, where V describes the bare impurity potential and $E_{\rm ps}$ describes the phase-slip energy. In Sec. IV it is shown that the scaling arguments used in this paper yield results for the one-dimensional case that are consistent with some previous exact work. Section V is a discussion of the experimental situation, and Sec. VI is the conclusion.

II. DISTRIBUTION OF PINNING ENERGIES

In this paper we consider the distribution of pinning energies in the absence of a field and assume that the threshold field of a given region is proportional to the pinning energy of the region. This assumption is standard when making estimates of this type.[10]

The CDW is described by an order parameter of the form $\psi(\mathbf{x})\cos[\mathbf{Q}\cdot\mathbf{x}+\phi(\mathbf{x})]$. In the FLR model, variations of $\psi(\mathbf{x})$ are ignored. The FLR equation describing the energy E of a system with phase degrees of freedom in the absence of a field is

$$E=k\int d^dx\,[\nabla\phi(\mathbf{x})]^2-\int d^dx\,V(\mathbf{x})\cos[\mathbf{Q}\cdot\mathbf{x}+\phi(\mathbf{x})]\ . \tag{2.1}$$

These equations describe a d-dimensional system where ϕ is the CDW phase, $V(\mathbf{x})$ describes the impurity potential, and k describes the CDW stiffness. The amplitude degrees of freedom are not included. Although phase slips occur in the presence of a field, these equations are adequate to estimate the distribution of pinning energies and hence local threshold fields. Scaling arguments can then be used to obtain the density of phase slips.[3]

In this section we discuss the distribution of the pinning energy per unit volume

$$(1/V_\Omega)\int_\Omega d^dx\,V(\mathbf{x})\cos[\mathbf{Q}\cdot\mathbf{x}+\phi(\mathbf{x})]\ ,$$

where Ω is a given region of the d-dimensional system with volume V_Ω. It is shown that when the pinning arises from the collective effects of many impurities, the probability $p(\varepsilon_{\rm pin})$ of a given pinning-energy density $\varepsilon_{\rm pin}$ obeys

$$-\ln[p(\varepsilon_{\rm pin})]\sim\left[\frac{\varepsilon_{\rm pin}}{E_{\rm LR}}\right]^{(2-d/2)}\ , \tag{2.2}$$

where $E_{\rm LR}$ is the typical (Lee-Rice) pinning energy $E_{\rm LR}\sim(V^4n_i^2/k^d)^{1/(4-d)}$. In three dimensions $-\ln[p(\varepsilon_{\rm pin})]\sim\varepsilon_{\rm pin}^{1/2}$; this result means that strongly pinned regions are vastly more probable than in a Gaussian distribution.

To understand the result (2.2), first recall the standard argument showing that the pinning energy is nonzero.[2] Assume $V(\mathbf{x})$ arises from weak uncorrelated impurities that have density $n_{\rm imp}$. It is assumed that the weak-pinning limit applies; this occurs when the elastic constant k is large enough that the CDW configuration does not conform to each impurity, or $kn_{\rm imp}^{2/d}\gg Vn_{\rm imp}$. This condition is arrived at by balancing the energy cost per unit volume of changing the phase ϕ by an amount of order unity over a distance of the impurity separation with the energy gain per unit volume from the impurity potential.

If ϕ is constant inside an infinite sample, then there is no elastic energy cost and the contributions from different impurities cancel, so the total energy is zero. Now consider distorting the uniform state by taking a rigid spherical region of size ξ and deforming a region of size ξ around it in such a way as to minimize the impurity potential in the interior of the region. This process can be viewed as constructing a variational configuration and calculating its energy per unit volume. The elastic energy per unit volume cost of this process scales as k/ξ^2. This energy must be balanced against the impurity energy gained by adjusting the region. A *typical* energy gain scales as the square root of the number of impurities in the region, so the impurity potential leads to a gain in energy per unit volume of $V(n_i\xi^d)^{1/2}/\xi^d$, where n_i is the density of impurities. Minimization of the energy change as a function of ξ leads to a pinning energy density of $k/\xi_{\rm LR}^2$, where the Lee-Rice length $\xi_{\rm LR}\sim(k/V\sqrt{n_i})^{2/(4-d)}$. Note that the energy gain from the impurity potential and the elastic energy cost are the same order of magnitude on the length scale $\xi_{\rm LR}$.

The above argument gives the pinning-energy density of a typical region, but now we would like to consider the distribution of pinning energies. In this paper we will be particularly interested in regions that are more strongly pinned than average. The main point is that the strongly pinned regions are smaller than the weakly pinned ones, so the probability of finding a fluctuation in the impurity potential leading to a strongly pinned region is much greater than one would estimate by looking at a region of size the Lee-Rice length. It will be assumed that even though the relevant length scale may be much less than a Lee-Rice length, it is still large enough that each region contains a large number of impurities, so that the pinning still arises from the collective action of many impurities.

It is reasonable that on a length scale ξ, the distribution of impurity potential energies is described by a Gaussian with width proportional to the square root of the number of impurities in the region $(n_i\xi^d)^{1/2}$, so that the impurity energy per unit volume is described by a Gaussian of width $(n_i/\xi^d)^{1/2}$. Accordingly, fluctuations leading to large values of the impurity pinning-energy density are more likely to be found in small regions. This result can be demonstrated easily for a simpler problem where impurities with strength $\pm V$ are distributed randomly in the region, with the impurity potential energy

of a region with $N=\xi^d$ sites proportional to the excess of impurities of one sign. This model can be motivated physically by considering each site as a volume of CDW that can have an impurity that either raises (+) or lowers (−) the local energy. No more than one impurity of each sign is allowed in a given volume, so we focus on low impurity densities, where this restriction should be unimportant. We argue that the probability $p(n_{\rm exc},N)$ of observing a particular value of the excess of impurities of one sign per unit volume $n_{\rm exc}$ in a region with N sites (where $N=\xi^d$) is described by a Gaussian distribution:

$$p(n_{\rm exc},N)\sim\exp\{-[(n_{\rm exc})^2N/n_i]\}\ , \tag{2.3}$$

with n_i the impurity density.

To see this result, assume the chance of any site having either a + or − impurity on it is q, and that the positions of the + and − impurities are uncorrelated. The impurity density n_i is hence $2q/a^d$, where a^d is the volume of each site. Let $N_{\rm exc}=n_{\rm exc}N$, where once again N is the number of sites in the region. Then $p(N_{\rm exc},N)$ obeys

$$p(N_{\rm exc},N)=\sum_{N_+=0}^{\infty} q^{N_+}(1-q)^{(N-N_+)}q^{(N_++N_{\rm exc})} \times(1-q)^{(N-N_{\rm exc}-N_+)}\frac{N!}{N_+!(N-N_+)!} \times\frac{N!}{(N_++N_{\rm exc})!(N-N_+-N_{\rm exc})!}\ . \tag{2.4}$$

When N is large enough, this sum is dominated by its largest term, which occurs (up to corrections of order $1/N$) when N_+ takes on the value N_+^*, where

$$\frac{(N_+^*+N_{\rm exc})(N_+^*)}{[N-(N_+^*+N_{\rm exc})](N-N_+^*)}=\left[\frac{q}{1-q}\right]^2\ . \tag{2.5}$$

It is clear that $N_+^*(q,N_{\rm exc}=0)=qN$. If $n_{\rm exc}\equiv N_{\rm exc}/N\ll qN$, then one can expand Eq. (2.5) to find

$$n_{\rm exc}=q\left[1-\frac{1}{2}\frac{n_{\rm exc}}{q}+\frac{(1-2q)}{8(1-q)}\left[\frac{n_{\rm exc}}{q}\right]^2\right]\ . \tag{2.6}$$

The variation of the number of configurations with a given $n_{\rm exc}$ near the value $n_{\rm exc}=0$ is determined by the second variation of the largest term in the sum (2.4) with respect to $n_{\rm exc}$ (the linear variation is zero because $n_{\rm exc}=0$ is a maximum); one finds

$$\frac{1}{N}\frac{d^2}{dn_{\rm exc}^2}[\text{largest term in (2.4)}] =-2\left[\frac{1}{q}+\frac{1}{1-q}\right]+O(n_{\rm exc}/q)\ . \tag{2.7}$$

Thus, for small q, the probability of attaining an impurity excess per unit volume of $n_{\rm exc}$ satisfies Eq. (2.3).

In addition to the restriction $q\ll 1$, which arises because only one impurity of each sign is allowed per site, this form is valid only when $n_{\rm exc}\ll q$. A lower bound on $p(N_{\rm exc},N)$ valid for any $n_{\rm exc}$, including the case $n_{\rm exc}\gg q$, can be obtained by considering regions with only (−) impurities, so that $N_+=0$ in Eq. (2.4). One finds that

$$p(N_{\rm exc},N)\gtrsim(1-q)^N\exp\left\{N\left\{n_{\rm exc}\left[\ln\left[\frac{q}{n_{\rm exc}}\right]\right]+(1-n_{\rm exc})\ln\left[\frac{1-q}{1-n_{\rm exc}}\right]\right\}\right\}\ , \tag{2.8}$$

so that the falloff of the probability distribution (2.3) is faster than the actual tail. Thus, for this simplified model, a Gaussian form is a lower bound on the tails when the impurity density of a region is far from its mean value. However, scaling arguments of the type used here are expected to be valid only when the pinning arises because of the collective effects of many impurities, where Gaussian statistics apply.

The simplified model examined here has the unphysical feature that the pinning energy from each impurity is either $+V$ or $-V$, whereas the density modulation of a rigid CDW leads to a term varying as $\cos(\mathbf{Q}\cdot\mathbf{r}_i+\Phi)$, where $\mathbf{Q}$ is the CDW wave vector, $\mathbf{r}_i$ is the position of the ith impurity, and Φ is a phase that one adjusts to minimize the energy. A simple treatment of the CDW case is to write

$$\sum_i\cos(\mathbf{Q}\cdot\mathbf{r}_i+\Phi)={\rm Re}\left[\sum_i e^{i(\mathbf{Q}\cdot\mathbf{r}_i+\Phi)}\right]\ , \tag{2.9}$$

where Re denotes the real part.[11] For uncorrelated impurities, the real and imaginary parts of this sum are each described by random walks, and the calculation outlined above yields the correct asymptotics for each.[12] The total pinning energy minimized with respect to Φ is the absolute value of the sum in Eq. (2.9), which leads to a power-law correction to the Gaussian form (2.3). We expect power-law corrections of this type to be present, but we will focus on the exponent because its variation dominates the estimates. In any case, the scaling arguments used here should probably not be expected to yield logarithmic corrections to the exponent accurately.

Having shown that the distribution function for the impurity contribution to the energy is described by a Gaussian form, we now include the elasticity contribution. The pinning energy of the region is the sum of the impurity energy gain and the elastic energy cost. The impurity energy gain scales as the impurity excess, and the elastic energy cost once again scales as Ck/ξ^2; the unknown prefactor C depends on the fraction of 2π the re-

gion must translate in order to minimize the impurity energy. Since the pinning energy per unit volume $\varepsilon_{\rm pin}$ is $Vn_{\rm exc}-Ck/\xi^2$, Eq. (2.3) can be used to calculate the probability of observing a value of $\varepsilon_{\rm pin}$ in a region of size ξ, $p(\varepsilon_{\rm pin},\xi)$, yielding

$$p(\varepsilon_{\rm pin},\xi)\sim\exp\left[-\left[\varepsilon_{\rm pin}+\frac{Ck}{\xi^2}\right]^2\frac{\xi^d}{n_i V^2}\right] . \tag{2.10}$$

The total number of regions with pinning energy density $\varepsilon_{\rm pin}$, $P(\varepsilon_{\rm pin})$, is $\int d\xi\, p(\varepsilon_{\rm pin},\xi)$.[13] The exponential dependence of the integrand enables one to evaluate the integral using steepest descents, leading to the emergence of a dominant length scale $\bar{\xi}$ that satisfies

$$\varepsilon_{\rm pin}=C\left[\frac{4}{d}-1\right]\frac{k}{\bar{\xi}^2} . \tag{2.11}$$

Equation (2.11) makes it explicit that regions with larger pinning energies have smaller sizes. Finally, evaluating Eq. (2.10) at $\bar{\xi}$ yields Eq. (2.2).

The arguments leading to Eq. (2.2) are similar to those used to estimate the density of states in the band tail of doped semiconductors.[9] In the band-tail problem one is looking at the distribution of eigenvalues λ of the Schrödinger equation

$$\left[-\frac{\hbar^2}{2m}\nabla^2+V(\mathbf{x})\right]\psi(\mathbf{x})=\lambda\psi(\mathbf{x}) , \tag{2.12}$$

where $V(\mathbf{x})$ is a random potential and $\psi(\mathbf{x})$ is a normalized wave function. Clearly, λ must satisfy

$$\lambda=\int d^dx\left[\frac{\hbar^2}{2m}|\nabla\psi|^2+V(\mathbf{x})|\psi(\mathbf{x})|^2\right] . \tag{2.13}$$

Equation (2.13) has the same gradient term as Eq. (2.1). The problems are different because the couplings to the impurity potential are not the same, but in both situations one expects the energy contribution from the pinning potential to be determined by Gaussian fluctuations in the impurity distributions.

Halperin and Lax discuss how estimates analogous to those given above can be obtained for the band-tail problem by viewing it as a construction of a variational wave function for a state in the band tail.[9] One assumes that states for different values of λ have the same shape and differ only in their characteristic size. The optimal shape determines the coefficient of $\lambda^{2-d/2}$ in the exponent. For the CDW the optimal shape is more difficult to calculate because of the nonlinear dependence of the pinning energy on the configuration, but there is no reason to expect that the functional form of the density of states should depend on this complication.

III. DENSITY OF REGIONS WITH PHASE SLIPS

In this section the arguments in the previous section are adapted so that the number of regions with phase slips can be estimated. First we review the argument that shows that phase slips must occur in CDW's.[3] It relies on the facts that the surface-to-volume ratio of a region R of size L tends to zero as L gets large, and that Newton's third law implies that the total elastic force on the region is bounded above by $kS_{\rm max}A_R$, where A_R is the area of the boundary of R, and $S_{\rm max}$ arises because if the CDW distortion is too large a phase slip will occur. Here consideration is restricted to regions whose volumes Ω_R scale like L^d and whose surface areas A_R scale as L^{d-1}. (The constants of proportionality are suppressed here.)

Consider the equation of motion for the phase $\phi(\mathbf{x})$ that follows when the system described by energy functional (2.1) obeys relaxational dynamics in the presence of an external force F:[14]

$$\dot{\phi}=k\nabla^2\phi-V(\mathbf{x})\sin[\mathbf{Q}\cdot\mathbf{x}+\phi(\mathbf{x})]+F . \tag{3.1}$$

If the pinning force per unit volume of a given region R is $F_{\rm pin}(R)$, then averaging the equation of motion (3.1) over the interior of the region R yields

$$\langle v\rangle_R\le F-F_{\rm pin}(R)+kS_{\rm max}A_R/\Omega_R , \tag{3.2}$$

where $\langle v\rangle_R$ is the spatially averaged velocity of the region and $S_{\rm max}=\max|\nabla\phi|$. This equation implies that the region R remains stationary if $F_{\rm pin}(R)-F>kS_{\rm max}/L$.

Now we must estimate the number of regions with pinning forces that exceed this value. We assume the pinning force $F_{\rm pin}(R)$ is of the same order as the pinning energy at $F=0$, $V_{\rm pin}(R)$, and we use the fact that in three dimensions the dominant contribution will be found to come from regions with the pinning energy per unit volume $V_{\rm pin}(R)$ of the order of the bare impurity potential V, which is much greater than the Lee-Rice pinning energy $E_{\rm LR}$. Thus, for F of the order of the threshold force F_T, the energy from the force F is negligible compared to the contributions from the elasticity and from the impurity potential. Therefore, we estimate the density of regions that satisfy $V_{\rm pin}(R)>kS_{\rm max}/L$.

Again we assume that the pinning force is proportional to the pinning energy of the region in the absence of a field. As before, pinning energy is determined by a competition between the impurity excess in the region that can lower the energy and the elastic energy cost of the distortion needed to align the CDW with the impurities so as to lower the energy. However, the pinning force must also exceed the maximum elastic force exerted by the surrounding CDW on the region. Thus, the impurity energy gain per unit volume in a region of size L must exceed $Ck/L^2+kS_{\rm max}/L$, where C describes the amount of distortion needed to line up the region with the impurities and $kS_{\rm max}$ is the maximum elasticity force that the system can exert without creating a phase slip.

One way to make a specific estimate is to consider how many regions of size L have pinning energies per unit volume that exceed $kS_{\rm max}/L$. As above, it is assumed that the pinning arises from random fluctuations in the distribution of uncorrelated impurities. The number of regions of size L that have pinning energies per unit volume that exceed $S_{\rm max}/L$, $n_{\rm PS}(L)$, is

$$n_{\rm PS}(L)\sim\int_{kS_{\max}/L}^{\infty} d\varepsilon_{\rm pin}\exp\left\{-\left[\left[\varepsilon_{\rm pin}+\frac{Ck}{L^2}\right]^2\times\frac{L^d}{n_i V^2}\right]\right\}. \quad (3.3)$$

The argument of the exponential is minimized when $k/L^2\sim\varepsilon_{\rm pin}$, but at this value of L the pinning energy is of order k/L^2, which is much less than $kS_{\max}/L$ in the regime $L\gg 1$ where these arguments apply. Thus, the maximum of the integrand is outside the region of integration, and the dominant contribution to phase slippage comes from different regions than those that dominate the estimates for local pinning energies. In three dimensions the dominant regions are those that have the smallest possible size for a given value of the pinning energy, and whose bulk elasticity contribution to the pinning energy is negligible. In other words, the regions with phase slips are those where the impurities happen to be arranged so that there is substantial pinning energy even when the regions are undistorted.

An alternate way to see that the dominant contribution leading to phase slips arises from very small regions is to redefine the pinning force of a region $\tilde{F}_{\rm pin}$ to include the boundary term, so that the distribution of pinning forces for regions of size L is written

$$p(\tilde{F}_{\rm pin},L)\sim\exp\left[-\left[\tilde{F}_{\rm pin}+\frac{Ck}{L^2}+\frac{kS_{\max}}{L}\right]^2\frac{L^d}{n_i V^2}\right]. \quad (3.4)$$

In three dimensions the quantity in square brackets in Eq. (3.4) is a monotonically increasing function of L, so that the dominant contribution must come from the smallest possible values of L.

The simple model with + and − sites can be used to estimate the phase-slip probability. As argued above, the dominant contribution is from regions with the smallest possible size. One can use Eq. (2.4) to estimate the number of regions with $N=\xi^d$ sites obeying $N_- - N_+ > kS_{\max}\xi^{d-1}/V$. The dominant contribution arises when $N_+=0$ and $N_-=N$; one finds that N must be larger than $(kS_{\max}/V)^d$, so that the number of regions with phase slips $n_{\rm ps}$ obeys

$$\ln(n_{\rm ps})\sim(kS_{\max}/V)^d\ln[q(1-q)]. \quad (3.5)$$

Thus, the number of regions with phase slips is exponentially small in the ratio $kS_{\max}/V$ rather than the much larger quantity $kS_{\max}/E_{\rm LR}$, where $E_{\rm LR}$ is the typical pinning energy.

In this section the method used to estimate the number of strongly pinned regions has been generalized so that the density of phase slips can be estimated. The number of regions with phase slips is not simply related to the number of strongly pinned regions; rather, it is determined by the ratio of the bare impurity potential to the phase-slip energy. In three dimensions the density of phase slips $n_{\rm ps}$ obeys $n_{\rm ps}\sim\exp[-(kS_{\max}/V)^3]$.

IV. ONE DIMENSION

In this section it is shown that the results in this paper are consistent with previous work that calculates the distribution of anomalously strongly pinned regions in a one-dimensional CDW in the weak-pinning limit.[15] In one dimension the arguments in Sec. II yield a distribution of pinning-energy densities $P(\varepsilon_{\rm pin})$ that satisfies $-\ln[P(\varepsilon_{\rm pin})]\sim\varepsilon_{\rm pin}^{3/2}$. In turn, $\varepsilon_{\rm pin}$ of a region R is defined to be the spatial average $\langle V(x)\cos[Qx+\phi(x)]\rangle_R/L_R$, where L_R is the length of the region. Here these results are compared to Ref. 15, where a transfer-matrix method is used to calculate the distribution of the energy changes when the boundary condition is changed. Specifically, let $E[\phi(L)]$ be the minimum value of the total energy of the system of length L for a given $\phi(L)$. If $\phi^*(L)$ is the value of $\phi(L)$ that minimizes $E[\phi(L)]$, then the variation of E as $\phi(L)$ is varied about $\phi^*(L)$ depends on the pinning energy near the end of the chain. In Ref. 15 it is shown that the probability distribution of the second variation $\tilde{\varepsilon}=d^2E[\phi(L)]/d^2\phi(L)|_{\phi(L)=\phi^*(L)}$, $P(\tilde{\varepsilon})$, obeys $-\ln[P(\tilde{\varepsilon})]\sim\tilde{\varepsilon}^3$.

The apparent discrepancy between these two results arises from the different quantities calculated. The quantity $\tilde{\varepsilon}$ is not proportional to the pinning energy $\varepsilon_{\rm pin}$ because when one perturbs the boundary the disturbance propagates into the system a distance that decreases as $\varepsilon_{\rm pin}$ increases. Simple arguments indicate that this propagation distance varies as the inverse square root of the pinning energy, so that the energy change induced by a perturbation at the boundary is proportional to $\sqrt{\varepsilon_{\rm pin}}$ rather than $\varepsilon_{\rm pin}$. The idea can be illustrated using a commensurately pinned system of length L with total energy E given by

$$E=\int_0^L dx\left[\frac{k}{2}\left[\frac{d\phi}{dx}\right]^2-V\cos\phi\right], \quad (4.1)$$

where k is the spring constant, V is the potential strength, and $\phi(x)$ is the phase at position x. For the ground-state configuration $\phi(x)=0$ for all x, clearly the pinning energy per unit length $\varepsilon_{\rm pin}$ is V. Compare this result to that for $\tilde{\varepsilon}$. For a small perturbation of the phase at $x=0$, the deviations from the potential minimum $\phi(x)=0$ are small, so that the cosine potential can be expanded. The configuration is thus described by

$$0=\frac{d^2\phi(x)}{dx^2}-\frac{V}{k}\phi(x). \quad (4.2)$$

The solution to Eq. (4.2) with the boundary conditions $\phi(x=0)=\phi_0$ and $\phi(x=\infty)=0$ is $\phi(x)=\phi_0 e^{-\alpha x}$, where $\alpha=\sqrt{V/k}$. Thus, the disturbance induced by a change at the boundary propagates a distance proportional to $\sqrt{k/V}$. The energy change induced by changing $\phi(x=0)$ from 0 to ϕ_0, $\delta E(\phi_0)$, is found using Eq. (4.1) to be $\delta E(\phi_0)=\phi_0^2(kV)^{1/2}$. Thus in this example the energy change induced by a perturbation at the boundary, $\tilde{\varepsilon}$, is proportional to $\sqrt{\varepsilon_{\rm pin}}$.

Extension of this argument to the more complicated randomly pinned case is nontrivial. However, we argue that the relationship $d^2E/d\phi_0^2\propto\varepsilon_{\rm pin}^{1/2}$ holds in this case

also. Consider a region with pinning-energy density $\varepsilon_{\rm pin}$. As shown in Sec. II, this value of $\varepsilon_{\rm pin}$ occurs for regions with characteristic size ξ where $\varepsilon_{\rm pin} \sim k/\xi^2$. The pinning energy arises because of a statistical fluctuation in the impurity distribution that leads to an impurity contribution also of order $\varepsilon_{\rm pin}$. If one assumes that the impurity contribution can be estimated up to terms of order unity by examining an undistorted region of size ξ (an assumption implicit in Sec. II), then using the fact that one is examining a configuration that minimizes the energy, one expects the region to be at the minimum of an effective potential of depth $\varepsilon_{\rm pin}$. Changing the configuration at the edge of this region thus yields a perturbation that propagates a distance of order $\sqrt{k/\varepsilon_{\rm pin}}$. This distance is the same order as the size of the region itself, so the perturbation has died away before the edge of the region is reached. (If the decay length for the perturbation were much larger than the region, then $\tilde{\varepsilon}$ would be determined by the pinning energies of many regions.) Thus, once again the total energy change induced by perturbing the boundary, $\tilde{\varepsilon}$, scales as $\sqrt{\varepsilon_{\rm pin}}$, where $\varepsilon_{\rm pin}$ is the pinning energy density of the region at the edge of the system.

That $\tilde{\varepsilon}$ cannot be interpreted as a pinning energy per unit volume is indicated by the fact that its distribution falls off faster than a Gaussian does. Even in the total absence of distortions there will be regions of the CDW where fluctuations in the impurity distribution will lead to regions with large (and negative) impurity contributions to the energies. Since by assumption the impurity positions are independent, the number of these regions obeys a Gaussian distribution. It is hard to imagine how allowing the CDW to lower its energy by distorting will cause the number of regions with anomalously low energies to decrease.

The observation that $\tilde{\varepsilon} \propto \varepsilon_{\rm pin}^{1/2}$ means that the probability distribution of $\varepsilon_{\rm pin}$ that obeys $-\ln[p(\varepsilon_{\rm pin})] \sim \varepsilon_{\rm pin}^{3/2}$ corresponds to a probability distribution of $\tilde{\varepsilon}$ satisfying $-\ln[P(\tilde{\varepsilon})] \sim \tilde{\varepsilon}^3$. (In this paper, any power-law prefactors of the exponentials have been ignored.) Thus, the results of this work in one dimension and those of Ref. 15 are consistent.

The arguments in this section make it clear that it is important to consider the type of perturbation when relating the linear response of the system to the pinning-energy distribution. The response of a region to a uniform field is proportional to the inverse of its pinning energy and is thus fundamentally different from the response to a perturbation imposed only at the boundary. Of course, the response to a uniform applied force is dominated by the contribution of the weakly pinned regions, so that the estimates here are not relevant to conductivity experiments. Whether a region has a phase slip at its boundary is determined not only by its pinning energy but also by its size, so separate estimates are needed, as discussed in Sec. III.

V. EXPERIMENTAL OBSERVABILITY

This section concerns the relevance of phase slips in CDW experiments. It will be seen that there is good evidence that phase slips play a role in the dynamics of almost all samples. Although at this stage it is unclear whether the phase slips arise from macroscopic sample inhomogeneities or from the mechanism discussed in this paper, theoretical estimates indicate that impurity-induced phase slips should be experimentally relevant in at least some situations. In any case, there is evidence that no samples are well described by the Fukuyama-Lee-Rice model in the regime where the phase correlation (or Lee-Rice) length is much smaller than the sample size.

The experimental evidence for phase slips consists mainly of width in narrow-band noise (NBN) peaks and the presence of broadband noise. The relevance of these experiments to this question arises because it is known that the FLR model, which does not allow for phase slips, has a strictly periodic response to a dc applied field.[7] Therefore, low-frequency temporal fluctuations in the current response to an applied voltage must arise from effects not present in the FLR model. Even ignoring this theoretical result, it is clear that samples that have different regions with different time-averaged velocities *must* have phase slips present. However, phase slips can also occur in situations where macroscopic velocity inhomogeneities are not present.

The material with the most homogeneous velocity is $NbSe_3$,[16] and even in this material no CDW experiment done so far appears to be in a sample that has no phase slips and also many Lee-Rice domains. Most samples yield broadband noise,[6] an observation that the FLR model cannot explain. In samples with a thickness step that leads to two regions with different velocities, the broadband noise has been shown to be correlated with the presence of the step.[8] This result is consistent with the claim that broadband noise only occurs in the presence of phase slips. Further possible evidence for the presence of phase slips is the small but unexplained rounding of the CDW threshold transition observed by Bhattacharya, Higgins, and Stokes.[5]

The situation in undoped $NbSe_3$ is complicated by the fact that the Lee-Rice length may be of the same order as the system size.[8] Evidence supporting this possibility includes the fact that in these samples the threshold field depends strongly on sample thickness, a result that indicates that surface perturbations make a significant contribution to the total pinning force. Therefore, phase slips can be caused by inhomogeneous velocity induced by a nonuniform sample cross section. A very small fraction (typically $< 10^{-4}$) of samples in a batch may have a uniform cross section and significantly suppressed broadband noise; these samples appear to have a small number of Lee-Rice domains. This latter conclusion is based on the fact that samples with resolution-limited narrow-band noise peaks and no broadband noise have several properties inconsistent with the infinite-volume limit of the FLR model.[17] The current-voltage characteristic in these samples is qualitatively different from typical samples and is similar to that observed in models with a small number of degrees of freedom.[18]

Restricting consideration to samples with broadband noise, the question remains whether macroscopic velocity inhomogeneity is the only source of temporal fluctuations

in the current measured when a dc applied voltage is applied. It has been shown that in many $NbSe_3$ samples the width of the NBN arises predominantly from temporal fluctuations in the CDW velocity.[19] The instantaneous CDW velocity appears substantially more homogeneous than the time-average value, indicating that noise sources other than spatial velocity inhomogeneity may be present. Once again, these noise sources are not included in the FLR model, so it is reasonable to speculate that they may be identified with phase slips. Since the effects of velocity shear are small in these measurements, it is natural to associate these fluctuations with phase slips that do not lead to macroscopic regions with different velocities. In any case, it is of interest to compare the broadband noise amplitude for samples with different impurity concentrations as well as different materials in an attempt to distinguish impurity-induced phase slips from those induced by thickness variations in a sample with few Lee-Rice domains.

Theoretical estimates of the phase-slip density are inconclusive. It has been shown in this paper that the number density of phase slips varies as $\exp[-(kS_{\max}/V)^d]$, where $kS_{\max}$ is a measure of the maximum elastic force the CDW can exert and V is the bare potential strength. We expect $kS_{\max}/V$ to be roughly the ratio of the CDW gap to the impurity potential strength.[20] This estimate arises because $kS_{\max}$ is a measure of the energy cost of the phase slip, and the bare impurity potential enters (rather than the threshold field) because the dominant contribution comes from regions that are undistorted, as discussed in Sec. III. Although it is reasonable that $kS_{\max}/V$ is a quantity of order unity, since small changes in the ratio $kS_{\max}/V$ lead to large changes in the phase-slip density, a definite experimental prediction cannot be made. However, it is clear that this estimate yields a much larger phase-slip probability than the naive one following from Ref. 3. The two estimates are different by a factor *in the exponent* of the Lee-Rice coherence volume, which is $\gtrsim 10^6$.

A quantitative comparison with experiment requires good estimates of the phase-slip energy and bare impurity potential, neither of which is available. The phase-slip energy is expected to be of the order of the CDW gap (about 1000 K). The pinning strength is expected to be stronger for charged impurities (e.g., Ti in $NbSe_3$) than for isoelectronic impurities (e.g., Ta in $NbSe_3$). The pinning energy for charged impurities has been estimated to be of the order of 0.2 eV (about 2000 K).[2] Thus, it is reasonable that phase slips are important when charged impurities are present. The pinning strength for isoelectronic impurities is not well known, though it is expected to be "considerably less" than for charged impurities.[2] Grüner's estimate of 10^{-2} eV (Ref. 1) leads to a negligible density of phase slips, though factor of 2 changes in the phase-slip energy or the pinning strength lead to observable phase-slip densities. However, in any case it is clear that charged impurities are much more likely to induce phase slips than isoelectronic ones. Therefore, it may be possible to correlate broadband noise amplitudes with the presence of charged impurities.

Since definitive theoretical estimates are impractical, it is not possible to associate the experimentally observed broadband noise with phase slips generated by the mechanism discussed in this paper and in Ref. 3 related to the statistical properties of randomly distributed impurities. Inhomogeneous velocity in real CDW materials also arises from sample inhomogeneities (e.g., clumping of impurities) and from thickness variations as well as nonuniform electric fields. In some sense the issue is semantic, because the arguments here serve to show that an amplification of the variations of the pinning potential occurs (based on surface to volume ratios), so that one is merely asking whether the observed strong-pinning behavior is "intrinsic" or an amplification of weak-pinning behavior. In any case, detailed understanding of CDW experiments requires accounting for phase-slip dynamics.

VI. DISCUSSION

This paper addresses the issue of the relevance of phase slips to experiments on sliding charge-density-wave systems. It is shown that estimating the number of phase slips is nontrivial because CDW's are described by the limit of weak pinning, where the pinning arises from the collective action of many impurities.

It is argued that stationary regions are much more likely to coexist with a moving bulk CDW than the converse because strongly pinned regions tend to be much smaller than weakly pinned ones. Scaling arguments similar to those used to calculate densities of states in semiconductor band tails are used to show that the distribution of strongly pinned regions is strongly non-Gaussian. It is shown that the scaling arguments for the distribution of strongly pinned regions yield results consistent with transfer-matrix calculations in the one-dimensional case.

The number of phase slips in physical situations is estimated and it is shown that the phase-slip density is not simply related to the number of regions with a given pinning energy. The number of phase slips is described by a Gaussian distribution, but the width of the Gaussian is determined not by the bulk threshold field but rather the microscopic impurity strength. The evidence for CDW velocity inhomogeneities in experimental samples is examined, and it is argued that phase-slip processes lead to experimental consequences in almost all samples.

ACKNOWLEDGMENTS

Invaluable conversations with S. Kivelson and P.B. Littlewood are gratefully acknowledged, and the author thanks V. M. Vinokur for drawing attention to Ref. 11 and D. Monroe for useful discussions.

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