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Magnetization curves for thin films of layered type-II superconductors, Kolmogorov-Arnold-Moser theory, and the devil's staircase

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Magnetization curves for a thin-layered superconducting film in parallel magnetic field have been shown to become devil's staircases provided the superconducting layers are perpendicular to the film plane. The transition from an incomplete to a complete devil's staircase with decreasing temperature is predicted. A chain of vortices is described by the generalized Frenkel-Kontorova model.

The interest in vortex lattices in layered anisotropic superconductors has been revived by the discovery of high-T_c compounds which appeared to be layered and anisotropic.[1,2] Application of a magnetic field parallel to the Cu-O layers creates a vortex lattice distorted by the anisotropy of the London penetration depths. In addition, the order parameter variation along the c axis creates a periodic potential with period a equal to the interlayer spacing, which is typically about 10 Å.[1] The potential is strong if the coherence length $\xi_\perp$ is less than a.[3] This condition is met in $YBa_2Cu_3O_7$ ($\xi_\perp=4$ Å, $a=8$ Å), $Bi_2Sr_2CaCu_2O_8$, and $Tl_2Sr_2CaCu_2O_8$ ($\xi_\perp=2$ Å, $a=12$ Å) at low temperatures.[1,4] The periodic potential becomes exponentially small for $\xi_\perp \gg a$,[5] which implies that its strength may be changed simply by raising the temperature. This periodic potential is believed to be responsible for the pinning in these materials.[3,5] Here I report that the same periodic potential makes the magnetization curves nonanalytic due to the incommensurability between the interlayer spacing a and the mean distance l between the vortices.

Consider a thin film with the c axis parallel to the plane of the film. The magnetic field is applied parallel to both the film and the ab planes. Introducing the frame of reference as in Fig. 1 and rescaling x and y make the London equation isotropic

$$x \rightarrow (\lambda_\perp/\lambda_\parallel)x, \quad y \rightarrow (\lambda_\parallel/\lambda_\perp)y, \quad \lambda^2=\lambda_\parallel\lambda_\perp . \tag{1}$$

The interlayer spacing a and the film thickness d are also rescaled according to Eq. (1). Since the incommensurability is more prominent in small fields I restrict myself to $H \ll H_{c2}$ and, therefore, the London approximation. The boundary conditions for the London equation can be met by introducing the images.[6] Every vortex has a train of images of alternating signs (Fig. 1), which modify the flux per vortex ϕ, the effective energy per vortex ε, and, consequently, the lower critical field $H_{c1}=4\pi\varepsilon/\phi$:[6-8]

$$\phi=\phi_0[1-1/\cosh(d/2\lambda)] , \tag{2}$$

$$\varepsilon=\varepsilon_0+\frac{\phi_0}{8\pi}\left[2\sum_{n=1}^{\infty}(-1)^n h(nd)+H[1-1/\cosh(d/2\lambda)]\right] , \tag{3}$$

$$H_{c1}^{*}=\left[H_{c1}+\sum_{n=1}^{\infty}(-1)^n h(nd)\right][1-1/\cosh(d/2\lambda)]^{-1} , \tag{4}$$

where $h(x)=\phi_0 K_0(x/\lambda)/2\pi\lambda^2$ is the magnetic field created by one vortex in a bulk superconductor.[9] The vortex-vortex repulsion

$$U(x)=\frac{\phi_0}{4\pi}\sum_{n=1}^{\infty}(-1)^n h(|\mathbf{x}+n\mathbf{d}|) \tag{5}$$

is convex and decreases as $\exp(-x/\lambda)$ at large x.

Equations (2)-(5) have been derived supposing all the vortices positioned in the middle of the film. Such a one-dimensional chain realizes the minimal energy configuration in small fields only: $H<H_2$. For $H_2<H<H_3$ the vortices form two parallel chains, then three chains, etc.[6,8] The threshold field H_2 has been calculated in Ref. 6 and experimentally observed in Refs. 8 and 10. For films with the thickness $d \simeq \lambda$ the field H_2 appears to be about a couple of H_{c1}'s. The remaining part of the paper will be restricted to the field region $H_{c1}<H<H_2$ in which vortices are situated exactly on the x axis (Fig. 1). The Gibbs free energy becomes

$$G=\frac{1}{l}\left[\sum_{i>j}U(x_i-x_j)+\sum_j V(x_j)-\mu\right] , \tag{6}$$

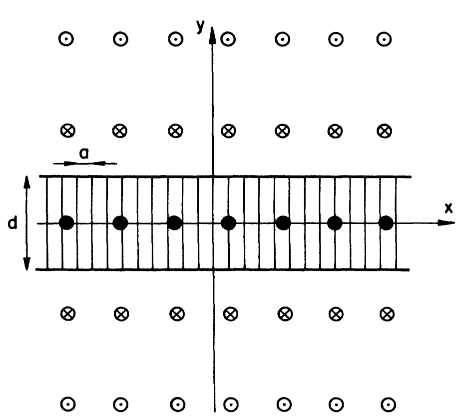

FIG. 1. Chain of vortices (solid circles) and their images (circles with dots, vortices; crossed circles, antivortices). The magnetic field is perpendicular to the xy plane.

where l is the mean spacing between vortices, $V(x+a)=V(x)$ is the periodic potential mentioned in the beginning of the paper, and μ is the chemical potential:

$$\mu=\frac{\phi_0}{4\pi}[1-1/\cosh(d/2\lambda)](H-H_{c1}^{*}) . \quad (7)$$

Equation (6) defines the generalized Frenkel-Kontorova model, which differs from the classical model by assuming an infinite range of interaction. The free energy must be minimized in two steps: first, the vortex configuration $\{x_j\}$ has to be found provided the average spacing l is fixed; then G has to be minimized over l. The first problem has been exactly solved by Aubry[11] for the classical Frenkel-Kontorova model (nearest neighbors only) and then generalized by Zaslavsky[12] for the infinite range of the interaction. The main results can be summarized as follows:

(1) If $l/a=p/q$ then $\{x_n\}$ is periodic: $x_{n+p}=x_n+aq$.

(2) If l/a is irrational then $\{x_n\}$ is quasiperiodic, it is either $x_n=f^+(nl+\alpha)$ for all n or $x_n=f^-(nl+\alpha)$ for all n, where both $f(x)$ are monotonic; $f^+(x)>f^-(x)$ and $f(x+a)=f(x)+a$ [Fig. 2(b)].

(3) If l/a is irrational then for almost all l there exists a critical value of V, such that for $V<V_c$ the $f^+(x)$ and $f^-(x)$ coincide and $f(x)=f^+(x)=f^-(x)$ is an analytic function [the Kolmogorov-Arnold-Moser torus, Fig. 2(a)].

(4) There exists a continuum set of irrational (l/a)'s, albeit of zero measure, such that $V_c=0$,[13] i.e., $\{x_n\}$ is described by a nonanalytic "Cantorus" even if the periodic potential is of an arbitrarily small strength.

(5) It follows from property 2 that $\mathrm{Int}(l)<x_{n+1}-x_n<\mathrm{Int}(l)+1$.

After the $\{x_n\}$ optimized for some l is substituted into Eq. (6) the $G(l,\mu)$ obtained defines the magnetization curve. Indeed, bearing in mind that μ is linearly connected with H by Eq. (7) and the induction $B=\phi/ld$, with ϕ given in Eq. (2), is linear in $1/l$, one sees that minimizing $G(l,\mu)$ over l gives $l(\mu)$ and, therefore, $B(H)$. The graph of $B(H)$ is the devil's staircase (Fig. 3). It is monotonic and continuous, but not analytic. It has plateaus at all rational (l/a)'s, which correspond to commensurate vortex configurations. Incommensurate states characterized by irrational vortex concentrations (l/a) exist on the Cantor set of the H axis. The plateaus widths are determined by the strength of the periodic potential, which can be easily controlled by changing the temperature.[3,5] At relatively high temperatures V is small and does not exceed V_c for the majority of l's. The plateaus on the $B(H)$ curve are

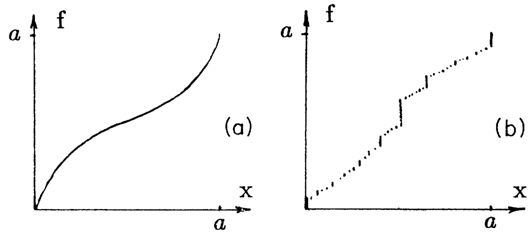

FIG. 2. Aubry functions (Ref. 11). (a) Kolmogorov-Arnold-Moser torus and (b) Cantorus.

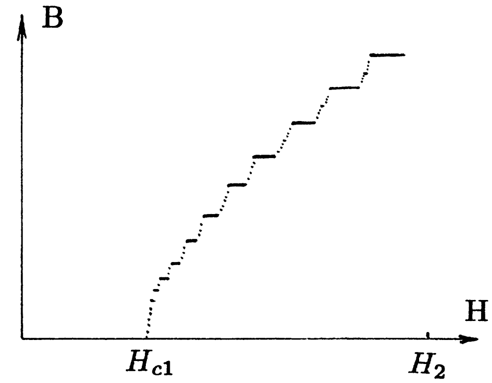

FIG. 3. Magnetic induction as a function of applied magnetic field.

tiny and, moreover, although their number is infinite their *total width* is small (an incomplete staircase). This statement can be rephrased by saying that the complementary Cantor set of H's corresponding to the incommensurate states has positive measure. As the amplitude of V rises, the plateaus become larger and the complementary Cantor set shrinks. Finally, as V exceeds the maximum of V_c over all l's the measure of the Cantor set becomes zero. Such a devil's staircase is called complete.

When V is much stronger than the typical vortex-vortex interaction, all the vortices are situated almost in the minima of the periodic potential, i.e., x_n take only integer values. This limiting case is equivalent to the lattice gas. The corresponding discrete model has been exactly solved in Ref. 14. The Cantor set on the H axis, corresponding to the incommensurate vortex configurations, is not only of zero measure but also of zero fractal dimension.[14] This means that in physical terms the magnetization curve $B(H)$ consists of plateaus only.

Below, I give the qualitative estimations showing that the plateaus in $B(H)$ are substantial for low temperatures and for all the magnetic fields $H_{c1}<H<H_2$. Since H_2 is of the order of H_{c1}, the intervortex spacing $l\gg 1$. There are two competing distances in the system: a, the period of $V(x)$, and the spacing between the vortices which would establish in the absence of V. If the latter distance is integer there is no competition: all vortices are situated in the minima of $V(x)$, so both the sum of $V(x_j)$ and the rest of G are minimized. This commensurate situation occurs at an infinite set of chemical potentials $\mu=\mu_s^*$, $S\in\mathbb{Z}$:

$$\mu_s^*=\sum_{n=1}^{\infty}[U(nSa)-nSaU'(nSa)] . \quad (8)$$

The set becomes dense near H_{c1}. The primary, widest steps occur near these μ_s^*'s. Their widths can be estimated by introducing vortex displacements from the minima of $V(x)$: $x_n=nSa+u_n$ and by expanding $U(x)$ in the power series near $x=Sa$, which gives the canonical Frenkel-Kontorova model for u_n. Denoting its energy as a function of l by $F_s(l)$, the $\mu(l)$ dependence near μ_s^* becomes

$$\mu(l)=\mu_s^*+F_s(l)-l\partial F_s(l)/\partial l , \quad (9)$$

$F_s(l)$ is known to have cusps at all rational l;[11] the values of these cusps define through Eq. (9) the corresponding secondary plateaus in $B(H)$. The primary plateaus are related to the cusp of $F_s(l)$ at the origin (namely, $l=Sa$). Substituting $K=U''(Sa)$ as the springs' strength into the well-known estimation $\Delta\mu \propto (KV)^{1/2}$ (Refs. 11, 15, and 16), the plateau becomes

$$\mu_s^+ - \mu_s^- \propto Sa[U''(Sa)V]^{1/2}\,, \tag{10}$$

where U can be estimated using Eq. (4); V strongly depends on $\xi_\perp/a$. If the coherence length $\xi_\perp(T) \ll a$, the interlayer spacing, the periodic potential V is quite strong: $V \propto H_c^2\xi^2/8\pi \propto (\Phi_0/4\pi\lambda)^2$;[3] in the opposite case $\xi_\perp(T) \gg a$ then V is exponentially weak: $V \propto \exp(-8\xi^2/a^2)$.[5] At low temperatures $\xi_\perp \ll a$ holds for $YBa_2Cu_3O_7$, $Bi_2Sr_2CaCu_2O_8$, and $Tl_2Sr_2CaCu_2O_8$. Comparing the plateau width (10) with the distance between the subsequent plateaus μ_s^* and μ_{s+1}^*, one finds that at low temperatures the $B(H)$ devil's staircase consists mostly of plateaus for all magnetic fields in which the one-dimensional vortex chain is stable. At higher temperatures or in less discrete compounds the plateaus are smaller and the magnetization curve $B(H)$ looks like a smooth function, perhaps with several tiny plateaus.

The $B(H)$ curve has been calculated by minimizing the Gibbs energy (6), i.e., in the zero-temperature limit. Nevertheless, the result holds also for nonzero temperatures. This statement can be justified at least at relatively low temperatures by the following well-known speculation. All the above used energies are the energies per unit length of a vortex. Statistic weights of different vortex configurations are $\exp(-GL_z/T)$, where G is given in Eq. (6) and L_z is the size of the sample in the z direction. True thermodynamic limit $L_z \rightarrow \infty$ has the same effect on the statistic weights as $T \rightarrow 0$ limit. Thus, fluctuations of vortices as wholes are suppressed by infinite L_z. However, temperature affects bulk variables like λ, ξ, H_{c1}, etc. Some deviations of the vortex cores from the straight lines also occur, which is believed to renormalize the energies, but not to result in drastic changes, at least at moderate temperatures.

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