Linearized gap equation for a superconductor in a strong magnetic field

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A linearized gap equation for a superconductor in the presence of a strong magnetic field incorporating the Landau-level spectrum, the Pauli term, as well as the dynamical interaction is given. The most commonly used model of a BCS-type pairing interaction of strength $\bar{V}$ is a special case of the dynamical interaction, and in this case an exact solution to this linearized gap equation was obtained by us before. From this solution, several new results recently obtained by Tesanovic *et al.* are recovered. By expressing the solution in terms of the Landau states, the Cooper pairing is shown to involve many Landau levels.

In a recent paper, Tesanovic *et al.*[1] (henceforth referred to as TRX) obtained very interesting results concerning the superconductivity properties in a strong magnetic field when the Landau level spectrum is incorporated but within the usual BCS-type pairing interaction of strength $\bar{V}$ which is local in space. In this Brief Report we present a linearized gap equation which takes into account the dynamical interactions as well as makes explicit use of the Landau level structure, so as to suitably incorporate the changes in the dielectric properties of the electron gas in such strong fields which may in turn alter the phonons and their density of states. Equipped with this more general theory we may introduce a BCS-type pairing interaction by suitably averaging over the dynamical interaction near the Fermi surface as is usually done in the conventional dynamical theory of superconductivity.[2] Our derivation, unlike TRX, is without Grassmann integrals and is thus more intuitive. Following well-known steps,[2] starting with the Gorkov anomalous Green function, the linearized gap equation in coordinate representation is found to be

$$\Delta(\mathbf{r}_1\mathbf{r}_2)=\int\int K(\mathbf{r}_1\mathbf{r}_2;\mathbf{r}_3\mathbf{r}_4)\Delta(\mathbf{r}_3\mathbf{r}_4)d^3r_3d^3r_4 , \tag{1}$$

where

$$\begin{aligned}K(\mathbf{r}_1\mathbf{r}_2;\mathbf{r}_3\mathbf{r}_4)=&-\tfrac{1}{2}\sum_{\alpha_1\alpha_2}\cdots\sum_{\alpha_3\alpha_4}\frac{1}{E_{\alpha_{3\uparrow}}+E_{\alpha_{4\downarrow}}}\int\int d^3r_{3'}d^3r_{4'}\\&\times[V_{\mathrm{sup}}(|E_{\alpha_{1\uparrow}}|+|E_{\alpha_{4\downarrow}}|;\mathbf{r}_{3'}\mathbf{r}_{4'})\tanh\beta E_{\alpha_{4\downarrow}}/2+V_{\mathrm{sup}}(|E_{\alpha_{2\downarrow}}|+|E_{\alpha_{3\uparrow}}|;\mathbf{r}_{3'}\mathbf{r}_{4'})\tanh\beta E_{\alpha_{3\uparrow}}/2]\\&\times[\Psi^*_{\alpha_1}(\mathbf{r}_1)\Psi_{\alpha_1}(\mathbf{r}_{3'})][\Psi^*_{\alpha_2}(\mathbf{r}_2)\Psi_{\alpha_2}(\mathbf{r}_{4'})][\Psi^*_{\alpha_3}(\mathbf{r}_{3'})\Psi_{\alpha_3}(\mathbf{r}_3)][\Psi^*_{\alpha_4}(\mathbf{r}_{4'})\Psi_{\alpha_4}(\mathbf{r}_4)] .\end{aligned} \tag{2}$$

Here α stands for the quantum numbers Landau level n, degeneracy index k, and momentum p in the z direction and

$$\Psi_\alpha(\mathbf{r})\equiv\Psi_{nkp\sigma_z}(\mathbf{r})=\hat{\sigma}_z\left[\frac{1}{\pi l_0^2}\right]^{1/4}\left[\frac{1}{2^n n!}\right]^{1/2}e^{1/2ixy/l_0^2}e^{-iky}e^{ipz}e^{-(x-kl_0^2)^2/2l_0^2}H_n\left[\frac{x-kl_0^2}{l_0}\right] . \tag{3}$$

σ_z is the spinor $\binom{1}{0}$ for up spin and $\binom{0}{1}$ for down spin, are the solutions of the usual free particle Hamiltonian in the symmetric gauge $\mathbf{A}\equiv(-\frac{1}{2}yH,\frac{1}{2}xH,0)$ with the corresponding energy eigenvalues

$$E_\alpha\equiv E_{np\sigma_z}=(n+\tfrac{1}{2})\omega_c+\frac{p^2}{2m_3}\pm\frac{1}{2}g\mu_BH . \tag{4}$$

Here μ_B is the Bohr magneton, g is the g factor, equals 2 for free electron, $\omega_c=1/ml_0^2$ is the Larmor frequency, l_0^2 is the square of the Larmor radius, c/eH, and + goes with $\hat{\sigma}_z=\uparrow$ and − with $\hat{\sigma}_z=\downarrow$.

$H_n(x)$ is the usual Hermite polynomial. It should be noted here that this choice of the solution is made so that we have a harmonic oscillator in the x direction centered at kl_0^2 so that the summation over the degeneracy index k can be carried out subsequently in a convenient way [see Eq. (6)]. The position-dependent phase factors in Eq. (3) appear naturally symmetrically in this procedure and are a manifestation of the property of the system under translation and concomitant gauge transformation in the (x,y) plane. The dynamical interaction here appears in the form $V_{\rm sup}$ and is related to the screened interaction via[2]

$$V_{\rm sup}(|E|;\mathbf{r}_1\mathbf{r}_2)=V_c(\mathbf{r}_1\mathbf{r}_2)+2\int_0^\infty \frac{d\Omega}{\pi}\frac{\operatorname{Im}V(\Omega;\mathbf{r}_1\mathbf{r}_2)}{\Omega+|E|} . \tag{5}$$

$\operatorname{Im}V(\Omega;\mathbf{r}_1\mathbf{r}_2)$ is related to the imaginary part of the inverse of the total dynamical dielectric function containing contributions from phononic as well as electronic excitations. V_c is the unscreened Coulomb potential. Since E_α is degenerate with respect to the k_α index, the respective bracketed terms can be summed over k_α thus

$$\sum_{k_\alpha}\rightarrow\int_{-\infty}^{\infty}\frac{dk_\alpha}{2\pi}\Psi^*_{n_\alpha k_\alpha p_\alpha}(\mathbf{r}_1)\Psi_{n_\alpha k_\alpha p_\alpha}(\mathbf{r}_{3'})=\frac{1}{2\pi l_0^2}L_{n_\alpha}(|\rho_{13'}|^2/2l_0^2)e^{-|\rho_{13'}|^2/4l_0^2}e^{ip_\alpha(z_1-z_{3'})}e^{-i(x_3y_1-x_1y_3)/2l_0^2} . \tag{6}$$

The last factor in Eq. (6) is an important phase factor exhibiting a subtle, combined translation and gauge transformation property in the presence of a magnetic field mentioned earlier. Also $\beta=1/k_BT$, T is the temperature close to the critical temperature T_c, and $L_n(x)$ is the usual Laguerre polynomial, and $\rho=(x,y)$. This equation contains all the dynamical contributions to the Cooper pair formation. Even though Im V may contain resonances due to phonons, plasmons, magnetoplasmons, excitons, etc., $V_{\rm sup}$ is a smooth function of energy E and can thus be parametrized in the usual way as a square well in the energy space with a cutoff as in the BCS scheme.[2] The strength of the interaction is taken to be that associated with those states near the Fermi energy so that $|E|=0$ in these units contribute most. Thus only a subset of states $(\alpha_1\alpha_2\alpha_3\alpha_4)$ participate in the pairing where the effective potential is most attractive.

Thus, a BCS-type model may be considered for further study of the gap equation, wherein $V_{\rm sup}$ is taken to be energy independent nonzero only for $|E|<\Omega$, the energy cutoff, and local in space so that in Eq. (2) we may take

$$V_{\rm sup}=\bar{V}\delta(\mathbf{r}_{3'}-\mathbf{r}_{4'})\quad\text{for}\,|E|<\Omega,\qquad\text{zero elsewhere} . \tag{7}$$

We may then perform the sums on α_1 and α_2 as they now become completeness statements on the Ψ_α's yielding $\delta(\mathbf{r}_1-\mathbf{r}_{3'})\delta(\mathbf{r}_2-\mathbf{r}_{3'})$ and hence the $\mathbf{r}_{3'}$ integration leads to

$$K(\mathbf{r}_1\mathbf{r}_2;\mathbf{r}_3\mathbf{r}_4)=-\tfrac{1}{2}\bar{V}\delta(\mathbf{r}_1-\mathbf{r}_2)\sum_{\alpha_3}\sum_{\alpha_4}\left[\frac{\tanh\beta E_{\alpha_{3\uparrow}}/2+\tanh BE_{\alpha_{4\downarrow}}/2}{E_{\alpha_{3\uparrow}}+E_{\alpha_{4\downarrow}}}\right][\Psi^*_{\alpha_3}(\mathbf{r}_1)\Psi_{\alpha_3}(\mathbf{r}_3)][\Psi^*_{\alpha_4}(\mathbf{r}_1)\Psi_{\alpha_4}(\mathbf{r}_4)] . \tag{8}$$

Using Eqs. (6) and (1) we obtain

$$\Delta(\mathbf{r}_1\mathbf{r}_2)=\delta(\mathbf{r}_1-\mathbf{r}_2)\Delta_c(\mathbf{r}_1) , \tag{9}$$

where now $\Delta_c(\mathbf{r}_1)$ obeys the integral equation

$$\Delta_c(\mathbf{r}_1)=-\int\tilde{K}(\mathbf{r}_1\mathbf{r}_2)\Delta_c(\mathbf{r}_2)d^3r_2 \tag{10}$$

with $\tilde{K}(\mathbf{r}_1\mathbf{r}_2)$ given by

$$\tilde{K}(\mathbf{r}_1\mathbf{r}_2)=\frac{\bar{V}}{4\pi^2l_0^4}\sum_{n_3}\sum_{n_4}\int\int_{-\infty}^{\infty}\frac{dp_3dp_4}{(2\pi)^2}\left[\frac{\tanh\beta E_{n_3p_{3\uparrow}}/2+\tanh\beta E_{n_4p_{4\downarrow}}/2}{E_{n_3p_{3\uparrow}}+E_{n_4p_{4\downarrow}}}\right]e^{i(p_3+p_4)(z_1-z_2)}$$
$$\times L_{n_3}(|\rho_{12}|^2/2l_0^2)L_{n_4}(|\rho_{12}|^2/2l_0^2)e^{-|\rho_{12}|^2/2l_0^2}e^{-i(x_2y_1-x_1y_2)/l_0^2} . \tag{11}$$

It should be noted that this kernel, apart from the important position dependent phase factor [last exponential in Eq. (11)], has the translational and cylindrical symmetries. The phase factor is the outcome of the subtle property of the electron system in a magnetic field mentioned earlier. It is this phase factor which led us to the exact solution of the form[3]

$$\Delta_c(\mathbf{r}_1)=\Delta_0e^{-|\rho_1|^2/2l_0^2} \tag{12}$$

with the condition that T and H obey the self-consistency relation[3]

$$1=\frac{\bar{V}}{4\pi l_0^2}\sum_{n_3}\sum_{n_4}\frac{(n_3+n_4)!}{n_3!n_4!}\left[\frac{1}{2}\right]^{n_3+n_4}I_{n_3n_4} \tag{13}$$

with

$$I_{n_3n_4}=\int_{-\infty}^{\infty}\frac{dp_3}{2\pi}\left[\frac{\tanh\beta E_{n_3p_{3\parallel}}/2+\tanh\beta E_{n_4p_{4\parallel}}/2}{E_{n_3p_{3\parallel}}+E_{n_4p_{4\parallel}}}\right],$$

where we have used the integral[4]

$$\int_0^{\infty}dx\,e^{-2x}L_{n_3}(x)L_{n_4}(x)=\frac{(n_3+n_4)!}{n_3!n_4!}\left[\frac{1}{2}\right]^{n_3+n_4+1}. \tag{14}$$

Equation (13) is an explicit version of the gap equation given by TRX, their Eq. (3), in the Landau level representation. Several results of TRX follow from Eq. (13).

(a) If only the lowest Landau level is occupied, then $T_c(H>H_{QL})$ is given by

$$1=\frac{\bar{V}}{4\pi l_0^2}I_{0,0}$$

with

$$I_{0,0}\cong N_{1,0}\ln\left[\frac{1.14\Omega}{T_c(H>H_{QL})}\right] \tag{15}$$

in the notation of TRX, and we obtain their result for $T_c(H>QL)$.

(b) The quantum-limit approximation (QLA) of TRX implies that only the terms with $n_3=n_4$ contribute in Eq. (13), and using

$$I_{n_3n_3}\cong N_{1,n_3}(0)\ln\left[\frac{1.14\Omega}{T_c(\mathrm{QLA})}\right] \tag{16}$$

we obtain Eq. (5) of TRX.

(c) In the low-field regime we may use the Poisson summation formula to perform the sum over the Landau levels leading to the *dHvA* oscillations.[3] Retaining only the first oscillatory terms, the result of TRX for T_c in the low field regime is recovered.

We may finally draw attention to the nature of the exact solution, Eq. (12), of the gap equation, Eq. (10), by explicitly computing the matrix element $\Delta_{\alpha\beta}$ associated with it. In general, we have

$$\Delta_{\alpha\beta}\equiv\int\int\Psi_\alpha(\mathbf{r}_1)\Psi_\beta(\mathbf{r}_2)\Delta(\mathbf{r}_1\mathbf{r}_2)d^3r_1d^3r_2 \tag{17}$$

and in the $\bar{V}$ model, this takes the form when explicitly written out using Eqs. (3), (9), and (12):

$$\Delta_{nkp;n'k'p'}=\Delta_0\frac{2\pi\delta(p+p')\sqrt{\pi l_0^2}}{2^{n+n'}(n!n'!)^{1/2}}e^{-(k^2+k'^2)l_0^2/2}\sum_{m=0}^{[n,n']}2^m(-1)^{n-m}\frac{H_{n-m}\left[\left[\frac{k-k'}{\sqrt{2}}\right]l_0\right]H_{n'-m}\left[\left[\frac{k-k'}{\sqrt{2}}\right]l_0\right]}{m!(n-m)!(n'-m)!}, \tag{18}$$

where $[n,n']$ is the smaller of n,n'. It should be pointed out that this has a factor which is not translationally invariant in k, again exhibiting the special feature of the system which requires a gauge transformation when a translation in the (x,y) plane is made for the system Hamiltonian to be invariant under translations. Under such transformatons the wave functions acquire position-dependent phase factors which make important contributions in evaluating the integral in Eq. (17). This shows that "pairing" in the Landau-level representation is maximal for $p'=-p$ (expected as with the usual Cooper pair) but among all the locations (k,k') of the centers of the orbits and (n,n') pairs of Landau levels connected by intermediate states m in the sum in Eq. (18). An examination of a few cases shows that

$$\Delta_{nkp;ok'p'}=\Delta_0\frac{2\pi\delta(p+p')\sqrt{\pi l_0^2}}{2^n(n!)^{1/2}}e^{-(k^2+k'^2)l_0^2/2}(-1)^nH_n\left[\left[\frac{k-k'}{\sqrt{2}}\right]l_0\right] \tag{19}$$

and

$$\Delta_{okp;ok'p'}=\Delta_0 2\pi(p+p')\sqrt{\pi l_0^2}\,e^{-(k^2+k'^2)l_0^2/2}. \tag{20}$$

The gap parameter in the ground Landau orbit appears to be the largest with the off-diagonal ones such as the one in Eq. (19) falling off rapidly with increasing n as well as on a scale $kl_0\gg1$ and $(k-k')l_0\gg1$. Thus the pairing mechanism in the presence of the magnetic field involves in general a large number of Landau states.

We may summarize this work by pointing out that the effects of magnetic field on the superconducting properties are contained in Eq. (1) which takes account of the degenerate Landau states in the presence of the magnetic field, and possible modifications of the phonons and the electron screening. A simplified BCS-like model may be deduced from this under certain conditions and the resulting gap equation and its solution are studied in some detail.

Note added in proof. The sum in Eq. (18) can be performed and the result is[5]

$$(-1)^nH_{n+n'}\left[\left[\frac{k-k'}{\sqrt{2}}\right]l_0\right].$$

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