Mean-field theory of spin-liquid states with finite energy gap and topological orders

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The mean-field theory of a T- and P-symmetric spin-liquid state is developed. The quasiparticle excitations in the spin-liquid state are shown to be spin-$\frac{1}{2}$ neutral fermions (the spinons) and charge e spinless bosons (the holons). The spin-liquid state is shown to be characterized by a nontrivial topological order. Although our discussions are based on the mean-field theory, the concept of the topological order and the associated universal properties (e.g., the quantum number of the quasiparticles) are expected to be valid beyond the mean-field theory. We also discuss the dynamical stability of the mean-field theory.

I. INTRODUCTION

The existence of the Mott insulator in two and higher dimensions remains one of the unresolved problems in theoretical physics. Here, by the Mott insulator, we mean an insulator with an *odd* number electrons per unit cell. In the past few years, this problem has attracted a lot of attention because of its relation to high-T_c superconductors.[1]

In one dimension the Mott insulator can be shown to exist thanks to the exact result of the one-dimensional (1D) Hubbard model.[2] However, in higher dimensions no exact results are available. It is not clear whether the Mott insulator can exist or not. Recent mean-field results strongly suggest that a Mott insulator—chiral spin state[3]—may exist in two dimensions. This Mott insulator breaks time-reversal symmetry (T) and parity (P).

In this paper we are going to argue that a T- and P-symmetric Mott insulator may exist in two and higher dimensions based on the mean-field approach to spin-liquid states. The Mott insulator corresponds to the short-ranged resonanting-valence-bond (s-RVB) state conjectured before.[4] We will show that the Mott insulators in higher dimensions are closely related to the two known incompressible liquid states of electron systems—the quantum Hall and superconducting states. The quantum Hall states are directly related to the chiral spin states, and as we will see, the superconducting states are closely related to the s-RVB state.

We will also discuss in detail the effect of the gauge fluctuations in mean-field theory. We show that the gauge fluctuations are very important and in many cases cause the infrared divergence in mean-field states. Those mean-field states are not self-consistent. To construct self-consistent mean-field states, one needs to find a way to control the infrared divergence caused by the gauge-field fluctuations. As we will see later, the mean-field theory of the T- and P-symmetric Mott insulator (or s-RVB state) has very good infrared stability because of the Higgs mechanism and is self-consistent. This strongly suggests that the T- and P-symmetric Mott insulator is a generic state and is supported by some Hamiltonians.

Of course, the mean-field theory of spin-liquid states is not reliable because of quantum fluctuations. Mean-field theory can only provide some qualitative results. However, mean-field theory deserves further developments because of the following reasons: (a) Mean field theory does provide some insights about *possible* spin-liquid states. Although it cannot determine specifically which spin Hamiltonians actually support the spin liquids constructed, mean-field theory does provide some clue about what type of interactions may favor the spin liquids under consideration. It also tells us the characteristic properties of those spin-liquid states, so that if they are discovered in numerical calculations, we are able to recognize them. (b) One of the main problems we are going to address in this paper is the self-consistency of mean-field theory. In order for mean-field theory to have any chance to describe the real spin liquids, it must have a certain stability in the infrared limits, by which we mean that a good mean-field ground state should be a generic state, i.e., be stable against any small perturbations. If such a stability exists, the mean-field ground state may have some chance to survive the quantum fluctuations and to describe a real spin liquid (qualitatively). Many mean-field ground states studied before do not have this infrared stability. In this paper we are going to outline sufficient conditions under which the mean-field ground states are dynamically stable. In summary, mean-field theory as a primitive and the only analytic approach to spin liquids in higher dimensions needs further developments to bring it closer to reality.

In general, the Mott insulator described by the s-RVB states and the chiral spin states represent new kinds of universality classes of insulators. Those new universality classes of insulators are characterized by topological orders. The holes in the new insulators have unusual properties (e.g., the unusual statistics). Thus the doped insulators become some sort of strange metal (e.g., boson metals or semion metals). It would be very interesting to see whether those strange metals can explain the unusual normal-state properties observed in the high-T_c samples.

The paper is arranged as follows. In Sec. II we will briefly review the mean-field approach to spin-liquid states and discuss infrared dynamical stability of mean-field theory. In Sec. III a mean-field theory of the s-RVB state is discussed. The spin excitations are found to be spin-$\frac{1}{2}$ fermions (the spinons). In Sec. IV we study dynamical properties of the doped holes in the s-RVB state. We show that the holes are charge-e bosons (the holons). The flux is shown to be quantized in units of $hc/2e$ even in the charge-e holon condensed state. In Sec. V we discuss how to characterize the s-RVB state. It is shown that the s-RVB state contains nontrivial topological orders.

II. MEAN-FIELD THEORY OF SPIN-LIQUID STATES AND DYNAMICAL STABILITY OF MEAN-FIELD THEORY

In this section I will briefly review the mean-field approach to the spin-liquid state developed in Refs. 5–7.

At half-filling the Hubbard model reduces to the Heseinberg model

$$H=\sum_{\langle ij\rangle} J_{ij}\mathbf{S}_i\cdot\mathbf{S}_j \ , \tag{1}$$

with possible frustrations. To obtain the mean-field ground state of the spin liquids, we introduce the slave-fermion operators $s_{i\alpha}$, $\alpha=1,2$. The spin operator $\mathbf{S}_i$ can be expressed as

$$\mathbf{S}_i=s^\dagger_{i\alpha}\sigma_{\alpha\beta}s_{i\beta} \ . \tag{2}$$

In terms of the slave-fermion operators, the Hamiltonian (1) can be rewritten as

$$H=\sum_{\langle ij\rangle} -2J_{ij}s^\dagger_{i\alpha}s_{j\alpha}s^\dagger_{j\beta}s_{i\beta} \ . \tag{3}$$

Note that the Hilbert space of (3) is larger than that of (1). The equivalence between (1) and (3) is valid only in the subspace where there is exactly one slave fermion per site. Therefore, in order to use (3) to describe the spin state, we need to impose the following constraint:

$$s^\dagger_{i\alpha}s_{i\alpha}=1, \quad s_{i\alpha}s_{i\beta}\epsilon_{\alpha\beta}=0 \ . \tag{4}$$

The second constraint is actually a consequence of the first one.

Introducing the Hubbard-Stratonovich (HS) field

$$\begin{aligned}\eta_{ij}&=\epsilon^{\alpha\beta}s_{i\alpha}s_{j\beta}=\eta_{ji} \ ,\\ \chi_{ij}&=s^\dagger_{i\alpha}s_{j\alpha}=\chi^\dagger_{ji} \ ,\end{aligned} \tag{5}$$

we may rewrite the Hamiltonian (3) as

$$H_{\text{mean}}=\sum_{\langle ij\rangle} J_{ij}[|\eta_{ij}|^2+|\chi_{ij}|^2-(\chi_{ji}s^\dagger_{i\alpha}s_{j\alpha}+\eta_{ij}s_{i\alpha}s_{j\beta}\epsilon^{\alpha\beta}+\text{H.c.})]+\sum_i\{a^3_0(s^\dagger_{i\alpha}s_{i\alpha}-1)+[(a^1_0+ia^2_0)s_{i\alpha}s_{i\beta}\epsilon_{\alpha\beta}+\text{H.c.}]\} \ . \tag{6}$$

The original Hamiltonian (3) can be recovered by integrating out the HS fields η_{ij} and χ_{ij}. a^l_0 are the Lagrangian multipliers to enforce the constraint (4).

The Hamiltonian (6) and the constraints (4) have a local SU(2) symmetry.[7] The local SU(2) symmetry becomes explicit if we introduce

$$(\psi_{a\alpha})=\begin{bmatrix} s_1 & s_2\\ s^\dagger_2 & -s^\dagger_1\end{bmatrix} , \tag{7}$$

$$8U_{ij}=\begin{bmatrix} -\chi^\dagger_{ij} & \eta_{ij}\\ \eta^\dagger_{ij} & \chi_{ij}\end{bmatrix} . \tag{8}$$

Using (7) and (8), we can rewrite (6) as

$$H_{\text{mean}}=\sum_{\langle ij\rangle} J_{ij}\text{Tr}[8U^\dagger_{ij}U_{ij}+(\psi^\dagger_i U_{ij}\psi_j+\text{H.c.})]+\sum_i a^l_0\text{Tr}(\psi^\dagger_i\tau^l\psi_i) \ , \tag{9}$$

where τ^l, $l=1,2,3$, are the Pauli matrices. From (9) we can see clearly that the Hamiltonian is invariant under a local SU(2) gauge transformation W_i:

$$\begin{aligned}\psi_i&\to W_i\psi_i \ ,\\ U_{ij}&\to W_iU_{ij}W^\dagger_j \ .\end{aligned} \tag{10}$$

A mean-field ground state at "zeroth" order is obtained by making the following approximations. First, we replace constraint (4) by its vacuum average

$$\langle s^\dagger_{i\alpha}s_{i\alpha}\rangle=1, \quad \langle s_{i\alpha}s_{i\beta}\epsilon_{\alpha\beta}\rangle=0 \ . \tag{11}$$

Such a constraint can be enforced by the *site-dependent* Lagrangian multipliers $a^l_0(i)$ in the Hamiltonian. At the zeroth order we ignore the fluctuations of a^l_0, i.e., assuming that a^l_0 is time independent. If we included the fluctuations of the a^l_0 the constraint, (11) would become the original constraint (4). Second, we assign specific values to η_{ij} and χ_{ij} and ignore their quantum fluctuations. In this approximation the dynamics of the slave fermions is governed by the mean-field Hamiltonian

$$H_{\text{mean}}=\sum_{\langle ij\rangle} J_{ij}\text{Tr}(\psi^\dagger_i U^{(\text{mean})}_{ij}\psi_j+\text{H.c.})+\sum_i a^l_0\text{Tr}(\psi^\dagger_i\tau^l\psi_i) \ , \tag{12}$$

where $U^{(\text{mean})}_{ij}$ is the mean-field solution, which satisfies the self-consistency condition

$$\chi^{(\text{mean})}_{ij}=\langle s^\dagger_{i\alpha}s_{j\alpha}\rangle, \quad \eta^{(\text{mean})}_{ij}=\langle s_{i\alpha}s_{j\beta}\epsilon_{\alpha\beta}\rangle \ . \tag{13}$$

a^l_0 in (12) are chosen such that (11) is satisfied.

If the fluctuations of the mean-field order parameters χ_{ij} and η_{ij} had finite-energy gaps, it would be qualitatively correct to ignore those fluctuations, at least at low energies. However, because of the SU(2) gauge symmetry, the "phase" fluctuations of U_{ij},

$$U_{ij}=U^{(\text{mean})}_{ij}e^{iA_{ij}} \ , \tag{14}$$

are gapless, where $A_{ij}=a^l_{ij}\tau^l$ is 2×2 traceless Hermitian matrix. We must include those phase fluctuations in order to obtain even qualitatively correct results for spin-liquid states at low energies. This leads us to "first"-order mean-field theory. At first order we include the "phase" fluctuations of U_{ij}, i.e., A_{ij}, and the fluctuations of the Lagrangian multipliers a_0^l. Those fluctuations describe a SU(2) lattice gauge field.[6,7] It can be shown that the SU(2) gauge-field fluctuations enforce the operator constraint (4). At first order the slave-fermion dynamics is governed by

$$H_{\text{mean}} = \sum_{\langle ij \rangle} J_{ij} \text{Tr}(\psi_i^\dagger U_{ij}^{(\text{mean})} e^{i a_{ij}^l \tau^l} \psi_j + \text{H.c.}) + \sum_i a_0^l \text{Tr}(\psi_i^\dagger \tau^l \psi_i) \; . \tag{15}$$

Equation (15) describes a fermion system interaction with SU(2) lattice gauge theory.

Given the above mean-field formalism, we would like to have a general discussion about the mean-field theory of spin-liquid states. At zeroth order the density of the slave fermions still fluctuates. Some times the slave-fermion density fluctuations may even be gapless. In those cases the density fluctuation is so strong that the constraint (4) is badly violated. Zeroth-order mean-field theory (12) may fail to provide even a qualitative description of the low-energy properties of a true spin-liquid state (which contains no charge fluctuations). For example, at zeroth order the quasiparticle excitations are spin-$\frac{1}{2}$ fermions. However, after including the gauge fluctuations and imposing the constraint (4), those quasiparticles may not be able to survive and appear in the physical spectrum of true spin-liquid states.

After taking into account the SU(2) gauge interaction, the density fluctuations of the slave fermions should at least have a finite-energy gap and the slave-fermion gas should be incompressible. In this case the constraint (4) is satisfied by (15) at least in the infrared limit.

In general, the gauge field will mediate a confining interaction between the quasiparticles in zeroth-order mean-field theory. As a result, the quasiparticles obtain an infinite self-energy and are confined. The confinement prevents those quasiparticles from appearing in the physical spectrum of true spin-liquid states.[8] This indicates that the quantum fluctuations are extremely important, which may even qualitatively change the physical picture arisen from zeroth-order mean-field theory. When the gauge interaction are confining, the zeroth order of mean-field theory even fails to give a qualitative description of spin-liquid states. In order to obtain a qualitative description of spin-liquid states from mean-field theory, we must include gauge fields.

However, a system of fermions with SU(2) gauge interaction usually is so complicated that it is very difficult to obtain the low-energy properties of the system. The difficulty comes from the gaplessness of the fermion excitations and the gauge excitations. The system has severe infrared divergence. The infrared properties of the system can hardly be determined. Mean-field theory essentially "maps" a complicated spin system into a (probably more) complicated fermion system with SU(2) gauge interaction. The low-energy properties of both systems are hard to obtain. In this case we have to say that mean-field theory is not so useful in the sense that mean-field theory fails provide qualitative information about the low-energy properties of spin-liquid states.

But there are exceptions to the above. There are three possible cases in which the low-energy properties of (15) can be relatively reliably determined.

(A) The mean-field solution U_{ij} breaks the translation symmetry and the slave fermions described by (12) (at zeroth order) form a band insulator. (Here we have assumed that $\eta_{ij}=0$.) In this case even at zeroth order without the SU(2) gauge-field fluctuations, the slave-fermion gas is already incompressible. Because of the finite-energy gap, we can safely integrate out the electrons, thus reducing (15) into a pure lattice gauge theory. The low-energy properties of the original spin system can be obtained from the low-energy properties of effective lattice gauge theory, which in many cases is still nontrivial. In general, the gauge field mediates a confining interaction and the spin-$\frac{1}{2}$ slave fermion cannot appear in the physical spectrum. Only slave-fermion pairs can appear as physical quasiparticles which carry integer spins. The spin excitations in general have finite gap. The spin-Peierls state discuss in Ref. 9 is an example of this type of spin state. Here we would like to point out that the translation noninvariance of the unphysical quantities U_{ij} does not necessarily mean that the physical spin state [obtained after integrating out the SU(2) gauge field or after doing a Gutzwiller projection] violates the translation symmetry. The necessary and sufficient conditions for the physical spin state to be translational invariant is that U_{ij} and the translated $U'_{ij}=U_{i+\tau,j+\tau}$ are gauge equivalent; i.e., there exists a gauge transformation W_i such that

$$U_{ij}=W_i U'_{ij} W_j^{-1} \; . \tag{16}$$

Therefore, even though zeroth-order mean-field theory violates translation symmetry, the corresponding spin state might still be a spin-liquid state which respects the translation symmetry. It would be interesting to find such examples.

(B) The mean-field solution $U_{ij}^{(\text{mean})}$ generates a flux (again assuming $\eta_{ij}=0$). The slave fermions described by (12) behave as if they are moving in a magnetic field. When the filling fraction is right, an integral number of Landau levels are completely filled. The slave-fermion gas is incompressible because of the finite gap between Landau levels. Again, the slave-fermion density fluctuations are absent even at zeroth order of mean-field theory. The time-reversal symmetry and parity are spontaneously broken. After integrating out the slave fermions, effective lattice gauge theory contains a Chern-Simons term due to T and P breaking. Because of the Chern-Simons term, gauge-field fluctuations have a finite-energy gap and can only mediate short-range interactions. The slave fermions are not confined. The quasiparticles (spinons) are dressed slave fermions which carry spin $\frac{1}{2}$ and have fractional statistics. The mean-field chiral spin state studied in Refs. 3 and 10–12 is a typical example of this possibility.

(C) The third way to obtain a reliable mean-field theory is to break the SU(2) gauge symmetry. In this case the gauge fluctuations obtain a finite-energy gap as a result of the Anderson-Higgs mechanism, which solves the infrared problem. Note that there is no gapless excitations in the Higgs phase. The slave-fermion gas is again incompressible. The above possibility can be achieved by requiring that the plaquette variable of the mean-field solution

$$P_{ijk\cdots l}=U_{ij}U_{jk}\cdots U_{li} \tag{17}$$

completely break the SU(2) symmetry. In this case η_{ij} must be nonzero. To see how the gauge bosons obtain a finite-energy gap (or a finite mass), let us note that the free energy of (9) in general contains the following gauge-invariant term:

$$F=\mathrm{Tr}(P_{ij\cdots k}e^{ia^l_{ii'}\tau^l}P^\dagger_{i'j'\cdots k'}e^{ia^l_{i'i}\tau^l}) . \tag{18}$$

In the continuum limits (18) reduces to the mass term for the a^l_μ field:

$$F\propto\int d^2x\,|\mathrm{Tr}[P,a^l_\mu\tau^l]|^2 , \tag{19}$$

where $P=P_{ij\cdots k}$. If P does not commute with the SU(2) transformation (i.e., $P\neq\pm1$), the free energy (19) will generate mass terms for the SU(2) gauge-field fluctuations around the mean-field solution. But note that the mass term (19) can only break SU(2) down to U(1). The gauge symmetry that commutes with P remains unbroken. For example, when $P\propto\tau^3$, we have $F\propto(a^1)^2+(a^2)^2$. The gauge field a^3 remain gapless and the gauge symmetry generated by τ^3 remains unbroken. To break the SU(2) gauge symmetry completely, we need another plaquette variable $P'=P'_{ij\cdots k}$ that does not commute with P. The total-mass terms now become

$$F=\int d^2x\,|\mathrm{Tr}[P,a^l_\mu\tau^l]|^2+\int d^2x\,|\mathrm{Tr}[P',a^l_\mu\tau^l]|^2 . \tag{20}$$

We see that at least two mass terms are necessary to completely break the SU(2) gauge symmetry. Once the SU(2) gauge symmetry is broken, the infrared problems are well under control as a result of the finite-mass term. In this case the low-energy properties of mean-field theory and the corresponding spin-liquid state can reliably derived.

In all the cases discussed above, the infrared problem of the slave fermions with the SU(2) gauge interaction is resolved by opening energy gaps. Because of those energy gaps, the infrared behavior of mean-field theory is well under control. We can obtain the low-energy properties of mean-field theory quite reliably. Because of the infrared stability of the theory, those low-energy properties are expected to be robust, at least qualitatively, against small perturbations. Therefore, it is reasonable to assume that the low-energy properties of mean-field theory qualitatively describe the low-energy properties of the spin-liquid state. Because of this reason, we will say that mean-field theories satisfying (A), (B), and (C) are dynamically stable. Dynamically stable mean-field theories can lead to a reliable description of the low-energy properties of the spin-liquid state. Many mean-field theories studied before are not dynamically stable. It is thus dangerous to extract the low-energy properties of the spin-liquid state, e.g., the quantum numbers of the quasiparticles, from those dynamically unstable mean-field theories.

In the mean-field theories discussed in (A)–(C), the slave fermions form incompressible states. The dynamical stability of the mean-field states and the incompressibility of the slave fermions are closely related. Given a slave-fermion system with a gauge interaction, one may ask when the slave fermions are incompressible. There are only three well-understood incompressible states for a fermion system. The fermions may form a band insulator, which corresponds to case (A). If a "magnetic" field is dynamically generated, the fermions may form an incompressible quantum Hall liquid, which corresponds to case (B). The fermions can also form a superconducting state, which is again incompressible. Note that a superconducting state, in contrast to a superfluid state, contains no gapless excitations. It certainly contains no gapless density fluctuations. This corresponds to case (C).

The spin-liquid states corresponding to case (B) have been studied in detail in Ref. 3. In the next section, as an example, we will study a T- and P-symmetric spin-liquid state corresponding to case (C).

III. *T*- AND *P*-SYMMETRIC SPIN-LIQUID STATE: MEAN-FIELD THEORY OF THE s-RVB STATE

Again, we will consider the frustrated Hamiltonian (1). The T- and P-symmetric spin-liquid state is given by the following mean-field ansatz:

$$\begin{aligned}&\chi_{i,i+\hat x}=\chi_{i,i+\hat y}=\chi ,\\ &\eta_{i,i+\hat x+\hat y}=\eta^\dagger_{i,i-\hat x+\hat y}=\eta ,\\ &\chi_{ij}=\eta_{ij}=0,\quad\text{otherwise} ,\\ &a^1_0\neq0,\quad a^2_0=a^3_0=0 ,\end{aligned} \tag{21}$$

where χ is a real parameter and η is a complex parameter. The corresponding SU(2) link variables U_{ij} are given by

$$\begin{aligned}&U_{i,i+\hat x}=U_{i,i+\hat y}=-\chi\tau^3 ,\\ &U_{i,i+\hat x+\hat y}=\mathrm{Re}(\eta)\tau^1+\mathrm{Im}(\eta)\tau^2 ,\\ &U_{i,i-\hat x+\hat y}=\mathrm{Re}(\eta)\tau^1-\mathrm{Im}(\eta)\tau^2 .\end{aligned} \tag{22}$$

From the mean-field ansatz we can easily obtain the following results.

(1) The mean-field ansatz is invariant under the translation and the spin rotation. Therefore, the mean-field ansatz describes a spin-liquid state. Or, more precisely, the ground state of H in (12), after the Gutzwiller projection, gives rise to a wave function of the spin-liquid state (with translation symmetry).

(2) $U_{ij}=U^\dagger_{ji}=U_{ji}$. Thus the links are nonoriented.

(3) Around triangles we have

$$\begin{aligned}U_{23}U_{31}U_{12}&=U_{32}U_{31}U_{14}\\ &=-\chi^2[\mathrm{Re}(\eta)\tau^1+\mathrm{Im}(\eta)\tau^2] ,\\ U_{12}U_{24}U_{41}&=U_{32}U_{24}U_{43}\\ &=-\chi^2[\mathrm{Re}(\eta)\tau^1-\mathrm{Im}(\eta)\tau^2] .\end{aligned} \tag{23}$$

Because τ^1 and τ^2 do not commute, the SU(2) gauge symmetry is completely broken if $\mathrm{Im}\eta \neq 0$. All gauge bosons obtain finite masses (or energy gaps).[13]

(4) The mean-field ansatz is invariant under the parity P: $x \leftrightarrow y$. Therefore, the spin-liquid state is P invariant.

(5) Later, we will show that the constraint (11) can be completely satisfied by properly choosing the value of a_0^1.

However, the mean-field ansatz is not invariant under 90° rotation, and one might naively expect that the spin-liquid state breaks the 90° rotation symmetry. But remember that U_{ij} are not physically observable. Only the SU(2) gauge-equivalent classes of U_{ij} are physical variables. Therefore, the noninvariance of U_{ij} under 90° rotation does not imply the noninvariance of the gauge-equivalent classes of U_{ij}. In the following we will show that the gauge-equivalent classes of the mean-field ansatz (21) [or (22)] is invariant under 90° rotation. Hence the corresponding spin-liquid state has the 90° rotation symmetry.

Under 90° rotation $U_{ij} \to U'_{ij}$, where

$$\begin{aligned} U'_{i,i+\hat{x}} &= U'_{i,i+\hat{y}} = -\chi\tau^3 , \\ U'_{i+\hat{x}+\hat{y}} &= \mathrm{Re}(\eta)\tau^1 - \mathrm{Im}(\eta)\tau^2 , \\ U'_{i-\hat{x}+\hat{y}} &= \mathrm{Re}(\eta)\tau^1 + \mathrm{Im}(\eta)\tau^2 . \end{aligned} \tag{24}$$

U_{ij} and U'_{ij} can be shown to be gauge equivalent:

$$U_{ij} = W_i U'_{ij} W_i^\dagger , \tag{25}$$

where

$$W_i = (-1)^i \tau^1 . \tag{26}$$

The gauge transformation W_i also leaves a_0^1 invariant. Therefore, the mean-field ansatz actually describes a spin-liquid state which has the 90° rotation symmetry.

Under T the mean-field ansatz U_{ij} is changed to $U'_{ij} = U^*_{ij}$. Using the same gauge transformation (26), we can show that U_{ij} and U'_{ij} are gauge equivalent. Thus our spin-liquid state also has time-reversal symmetry.

Let $|\Phi\rangle$ be the ground state of the mean-field Hamiltonian (12). We would like to view $|\Phi\rangle$ as a trial wave function. The mean-field ground-state energy is given by $E = \langle \Phi | H | \Phi \rangle$, where H is given by (3). Note that for the mean-field ansatz (21) the mean-field Hamiltonian H_{mean} can be written as

$$\begin{aligned} \frac{H_{\mathrm{mean}}}{J_1\chi} &= \sum_{\mathrm{NN}} s^\dagger_{i\alpha} s_{j\alpha} + \sum_{\mathrm{NNN}} (\zeta s_{i\alpha} s_{j\beta} \epsilon^{\alpha\beta} + \mathrm{H.c.}) \\ &\quad + \sum_i (\tilde{a}_0^1 s_{i\alpha} s_{i\beta} \epsilon_{\alpha\beta} + \mathrm{H.c.}) , \end{aligned} \tag{27}$$

where $\zeta = J_2\eta / J_1\chi$ and $\tilde{a}_0^1 = a_0^1 / J_1\chi$. From (27) we see that $|\Phi\rangle$ only depends on ζ [$\tilde{a}_0^1$ is determined by (11)]. Thus the mean-field energy $E(\zeta)$ is a function of ζ.

When $J_1/J_1 = 0$, we find that $E(\zeta)$ is minimized at $\zeta = 0$. The mean-field solution is given by $\chi = 0.41$ and $\eta = 0$. When $0 < J_2/J_1 < 1.4$, $E(\zeta)$ is minimized at a nonzero ζ with $\mathrm{Im}\zeta$, $\mathrm{Re}\zeta \neq 0$. Hence the mean-field solution also satisfies $\mathrm{Im}\eta$, $\mathrm{Re}\eta \neq 0$, and the SU(2) gauge symmetry is completely broken. If J_2/J_1 is not too small (e.g., $J_2/J_1 > 0.3$), we have $\mathrm{Im}\eta/\mathrm{Re}\eta \sim 1$. When $J_2/J_1 > 1.4$, we find that $E(\zeta)$ is minimized at $|\zeta| = \infty$ and $\mathrm{Im}\zeta/\mathrm{Re}\zeta = 1$. In this case the mean-field solution is given by $\chi = 0$ and $\eta = 0.48 e^{i\pi/4}$.

We would like to emphasize that the self-consistency of our mean-field theory requires $\eta \neq 0$ [only in this case can we break the SU(2) gauge symmetry]. From the above discussion we see that the frustration (i.e., nonzero J_2) is crucial for the existance of our T- and P-symmetric spin-liquid state with finite-energy gap.

We would like to remark that although the mean-field ansatz with $\mathrm{Im}\eta$, $\mathrm{Re}\eta \neq 0$ minimizes the mean-field ground-state energy for $0 < J_2/J_1 < 1.4$, this does not imply that the spin Hamiltonian (1) supports such a spin-liquid state. This is because the quantum fluctuations are large and the mean-field calculation is not reliable. The above calculation is just to demonstrate that the mean-field ansatz is self-consistent. In the following we will assume that the mean-field solution given by (21) with $\mathrm{Im}\eta$, $\mathrm{Re}\eta \neq 0$ is stable. In this case the SU(2) gauge symmetry is completely broken. If the simplest frustrated Heisenberg model (1) does not give rise to this result, we will assume such a result can be realized in some other spin models.

Assuming the stability of the mean-field ground state, we can study the properties of the quasiparticle excitations. In zeroth-order mean-field theory, the dynamics of the quasiparticles is described by the mean-field Hamiltonian (12). Note that the mean-field ansatz actually describes a BCS "superconductor." We may directly apply the results in mean-field theory of the BCS superconductor to our case. We would like to point out that the superconducting order parameter in our case has an $s + id$ symmetry. The order parameter does not have a definite angular momentum. This is possible because the gap equation is nonlinear. From (12) we see that the superconducting gap is given by

$$\Delta_k = 2J_2[\eta\cos(k_x + k_y) + \eta^\dagger \cos(-k_x + k_y)] + a_0^1 + i a_0^2 , \tag{28}$$

and the slave-fermion spectrum is given by

$$\varepsilon_k = 4J_1\chi(\cos k_x + \cos k_y) + a_0^3 . \tag{29}$$

$$\chi_{ii} = 1, \quad \eta_{ii} = 0 . \tag{30}$$

χ_{ij} and η_{ij} can be determined from Δ_k and ε_k:

$$\begin{aligned} \chi_k &= \sum_j \chi_{ij} e^{i\mathbf{k}\cdot(\mathbf{j}-\mathbf{i})} = \left| 1 - \frac{\varepsilon_k}{E_k} \right| , \\ \eta_k &= \sum_j \eta_{ij} e^{i\mathbf{k}\cdot(\mathbf{j}-\mathbf{i})} = \frac{1}{2}\frac{\Delta_k}{E_k} , \end{aligned} \tag{31}$$

where

$$E_k = (\varepsilon_k^2 + |\Delta_k|^2)^{1/2} . \tag{32}$$

The constraint (30) reduces to

$$\chi_{i,i}=\frac{1}{N}\sum_k\left[1-\frac{\varepsilon_k}{E_k}\right],$$
$$\eta_{i,i}=\frac{1}{N}\sum_k\frac{1}{2}\frac{\Delta_k}{E_k}. \tag{33}$$

From (33) we find that the constraint (11) can be satisfied if we choose $a_0^3=a_0^2=0$. But, in general, $a_0^1\neq 0$.

From BCS theory we see that the spectrum of the quasiparticle excitations is given by E_k. The quasiparticles are fermions with spin $\frac{1}{2}$ and zero electrical charge. Note that the quasiparticle excitations has a finite-energy gap, as one can see from (32).

We would like to stress that the mean-field state studied above is *not* an (electrical) superconductor. It is only a superconductor for the dynamical (sometimes is called fictitious) gauge field a_μ^3 which arises from the phase fluctuations of χ_{ij}. For the electromagnetic field our mean-field state is an insulator.

In "first"-order mean-field theory, the collective fluctuations of the phases of χ_{ij} and η_{ij} are included. From the previous discussion we see that these fluctuations correspond to an SU(2) gauge field. Sometimes the SU(2) gauge fluctuations may drastically change the structure of ground state. The quasiparticles in zeroth-order mean-field theory may disappear from the physical spectrum as a result of possible confinement of the SU(2) gauge field. If this happens, the zeroth-order mean-field results provides little information about the physical properties of the true spin-liquid state.

However, in our mean-field state given by (21), the SU(2) gauge symmetry is broken. The gauge bosons obtain finite masses. The infrared behavior of the SU(2) gauge fluctuations are well under control. Those fluctuations do not qualitatively change the structure of the zeroth-order mean-field ground state. The massive SU(2) gauge fluctuations can only mediate short-range interactions between quasiparticles in zeroth-order mean-field theory, and there is no confinement. Therefore, the quasiparticles in zeroth-order mean-field theory also appear as quasiparticles in first-order mean-field theory. Those quasiparticles qualitatively describe the excitations in the corresponding spin-liquid state. In particular, the spin-liquid state considered here supports neutral spin-$\frac{1}{2}$ fermionic excitations. These excitations are spinons. The energy spectrum of the spinons is qualitatively given by E_k and has finite-energy gap.

To summarize, we have described a mean-field theory of a spin-liquid state. The spin-liquid state is invariant under the translation and 90° rotation. It also has T and P symmetry. The spin excitations in the spin-liquid state are found to be neutral spin-$\frac{1}{2}$ spinons. The spinons obey Fermi statistics and have a finite-energy gap. We would like to emphasize that the mean-field theory discussed above is dynamically stable. Therefore, we expect that mean-field theory qualitatively describes the properties of the actual spin-liquid state supported by some spin Hamiltonian.

IV. DOPED s-RVB STATE

In this section we are going to study the t-J model in the low doping limit assuming that at zero doping the ground state of the spin system is the spin-liquid state described by the mean-field theory in the last section. Introducing the slave boson b, we may write the t-J model as

$$H=\sum_{i,j}J_{ij}s_{i\alpha}^\dagger s_{j\alpha}s_{j\beta}^\dagger s_{j\beta}+\sum_{i,j}ts_{i\alpha}^\dagger s_{j\alpha}b_j^\dagger b_i\ , \tag{34}$$

with the constraint

$$s_{i\alpha}^\dagger s_{i\alpha}+b_i^\dagger b_i=1\ , \tag{35}$$

on the state in the physical Hilbert space. Introducing the HS fields χ_{ij} and η_{ij}, we obtain the mean-field Hamiltonian at nonzero doping:

$$H_{\text{mean}}=\sum_{i,j}J_{ij}[\chi_{ij}s_{i\alpha}^\dagger s_{j\alpha}+(\eta_{ij}s_{i\alpha}^\dagger s_{j\beta}^\dagger\epsilon^{\alpha\beta}+\text{H.c.})]$$
$$+\sum_{i,j}t\chi_{ij}b_j^\dagger b_i+\sum_i a_0^3(s_{i\alpha}^\dagger s_{i\alpha}+b_i^\dagger b_i-1)+\sum_i[(a_0^1+ia_0^2)s_{i1}^\dagger s_{i2}^\dagger+\text{H.c.}](1-b_i^\dagger b_i)\ . \tag{36}$$

At zeroth order the fluctuations of χ_{ij} and η_{ij} are ignored. From the above discussions we see that for the particular mean-field state (21) the gauge fluctuations do not cause the infrared divergence and zeroth-order mean-field theory gives a correct qualitative description of the spin-liquid state. Therefore, we expect that (36) qualitatively describes the quasiparticle excitations in the doped spin-liquid state.

In (36), b describes a charge-e spinless boson which is called holon. At zeroth order the hopping rate of the holons is given by $t\chi$. Because χ is order $O(1)$, the effective mass of the holon is of order $1/t$, which is close to the electron-band mass. But in (36) we have ignored the spinon-holon interaction. After including the interaction we expect the effective mass of the holon to be much larger than $1/t$.

The properties of the spinons and holons have been studied by many people based on mean-field theory and assuming that the interaction between the spinons and holons are weak.[14] However, many mean-field theories studied before contain strong gauge-field fluctuations which mediate a confining interaction between the spinons and holons. In this paper we find a mean-field spin-liquid state in which the SU(2) gauge fields are massive and the interaction between the spinons and holons is relatively weak. Thus many results of the spinons and holons obtained in the previous studies may apply to our spin-liquid state.

In addition to the spinons and holons, there is another quasiparticle excitation in our spin-liquid state. This excitation appears as topological soliton in mean-field theory. Note that in our mean-field state the SU(2) gauge symmetry is broken by the Higgs fields in the adjoint representation of the SU(2). In this case the SU(2) gauge symmetry is broken down to Z_2 gauge symmetry. The quasiparticles in the Z_2 gauge theory are the Z_2 vortexes. These Z_0 vortexes are the new quasiparticle excitations mentioned above. In mean-field theory a Z_2 vortex is described by the following ansatz:

$$\tilde{U}_{ij}=e^{-i\theta_i\tau^3/2}U_{ij}e^{i\theta_i\tau^3/2}\ , \tag{37}$$

where θ_i is the angle of the ith site relative to the center of the Z_2 vortex (x,y):

$$\tan\theta_i=\frac{i_x-x}{i_y-y}\ . \tag{38}$$

Because $e^{i\theta\tau^3/2}|_{\theta=2\pi}=-1$, the holon (spinon) wave function will obtain a minus sign as a holon (spinon) moves around the Z_2 vortex. Therefore, the Z_2 vortex behaves like a π flux vortex to the spinons and holons.

Because of the Z_2 vortex, the $hc/2e$ flux has a finite energy even in the charge-e holon condensed state.[15] In the superconducting state a Z_2 vortex and a bare $hc/2e$ magnetic vortex have infinite energy since the holon wave functions change sign as a holon goes around the Z_2 vortex or $hc/2e$ magnetic vortex. But the bound state of a Z_2 vortex and a $hc/2e$ magnetic vortex has a finite energy. The holon wave function does not change sign as a holon moves around such a bound state. Therefore, even in the charge-e holon condensed state, the minimum flux quantum is still $hc/2e$. The charge-e holon superconducting state does not contradict the experimental results. We also note that binding a Z_2 cortex to a holon changes the statistics of the holon from bosonic to fermionic. Similarly, the bound state of a spinon and a Z_2 vortex behave like a boson. This phenomenon has been discussed in detail in Ref. 16.

V. TOPOLOGICAL ORDERS IN SPIN-LIQUID STATES

In this paper and in Ref. 3, the s-RVB and chiral spin states are constructed by using mean-field theory. Both spin-liquid states have a finite-energy gap and have the translation symmetry. At half-filling such spin-liquid states are Mott insulators or, more precisely, an insulator with *odd* number of electrons per unit cell.

However, spin-liquid states are constructed by using mean-field theory, which involves unphysical fields (e.g., the slave-fermion field s_i and slave-boson field b_i). Our description and characterization of spin-liquid states are also in term of those unphysical quantities. It is very important to understand how to characterize spin-liquid states (or the Mott insulators) using physical properties. Especially, we would like to know whether there is any physical order parameter which characterizes the spin-liquid phase studied in this paper.

In this section we are going to argue that the spin-liquid states studied here cannot be completely characterized by their symmetry properties and by the order parameters associated with broken symmetries. Our spin-liquid states contain nontrivial topological orders and are characterized by those topological orders.

First, let us discuss what are the topological orders.[17] Consider a rigid state containing no gapless quasiparticle excitations. Because of the finite-energy gap, such a system is almost trivial at low energies. The only nontrivial feature at low energies comes from degenerate ground states. One may naively think the degenerate ground state always come from broken discrete symmetries and conclude that a rigid state is characterized by its symmetry properties. However, this naive expectation is not correct. It has been shown that the fractional quantum Hall (FQH) fluid supports a degenerate ground state even when the Hamiltonian contains no discrete symmetries.[18] The ground-state degeneracy is shown to be robust against arbitrary perturbations despite there being no symmetry to protect it. Furthermore, the number of the degenerate ground states depends on the topology of the space. All of these results point to one thing: Rigid systems, such as FQH states, may have nontrivial infrared fixed points which cannot be characterized by broken symmetries. Those unusual infrared fixed points are said to be characterized by topological orders.

Both the chiral spin state and the s-RVB state are rigid states. In Ref. 17 and 19 the chiral spin states are shown to have nontrivial topological orders. In this section we will show that the s-RVB state also contains nontrivial topological orders.

As in the FQH states, the nontrivial topological orders can be detected by measuring the ground-state degeneracy of the system on compactified spaces. The ground-state degeneracy and its relation to the Z_2 vortex has been studied in Ref. 16 within the nearest-neighbor dimer model.[4] It was shown that the s-RVB state has 2^{2g} degenerate ground states on genus-g Riemann surfaces. In the following we will describe the above result in terms of our mean-field theory.

In mean-field theory the 2^{2g} degenerate ground states can be constructed by adding zero or one unit of the Z_2 flux through the $2g$ noncontractable loops on the genus-g Riemann surface. On the torus the mean-field ansatz for the four degenerate ground states is given by

$$U_{ij}^{(m,n)}=\exp\left[-i\left[m\frac{i_x}{L_x}+n\frac{i_y}{L_y}\right]\tau^3\pi\right]U_{ij} \times\exp\left[i\left[m\frac{i_x}{L_x}+n\frac{i_y}{L_y}\right]\tau^3\pi\right]\ , \tag{39}$$

where $m,n=0,1$ and L_x and L_y are the size of torus in the x and y directions. $U_{ij}^{(m,n)}$ with different m and n are *locally* gauge equivalent. Therefore, the free energies for different ansatz are expected to be the same (in the thermodynamical limit). Thus $U_{ij}^{(m,n)}$ describes the degenerate ground states of the system. However, $U_{ij}^{(m,n)}$ with different m and n are not gauge equivalent in the global sense. A spinon propagating all the way around the torus in the x (y) direction obtain a phase $e^{im\pi}$ ($e^{in\pi}$). Therefore, $U_{ij}^{(mn)}$ describes different ground states. In other words, the ground-state wave functions for different m and n are orthogonal to each other.

On a finite compactified lattice, the only way for the system to tunnel from one ground state to another is through the following tunneling process. At first a pair of the Z_2 vortexes is created. One of the Z_2 vortexes propagates all the way around the torus and then annihilates with the other Z_2 vortex. Such a process effectively adds a unit of the Z_2 flux to the hole of the torus and changes m or n by 1. The different ground states can not tunnel into each other through any local fluctuations. As a direct consequence of this result, the energy split between different ground states on a finite lattice is expected to be of order $e^{-L/\xi}$.

In mean-field theory the degeneracy of ground states is a consequence of the gauge symmetry. The gauge symmetry remains to be exact even after we include an arbitrary perturbation to the original spin Hamiltonian:

$$\delta H=\sum_{i,j}\delta J_{ij}\mathbf{S}_i\cdot\mathbf{S}_j+\cdots \quad . \tag{40}$$

δH may break translation symmetry, rotation symmetry, etc. The above arguments are still valid even for the modified Hamiltonian. The mean-field ground states remain to be fourfold degenerate. We expect this result remains true even beyond mean-field theory. The ground-state degeneracy of our spin-liquid state cannot be changed by any perturbation as long as the perturbation is weak enough not to drive a phase transition. Therefore, the ground-state degeneracy can be regarded as a quantum number characterizing the spin-liquid state. A more complete characterization of the spin liquid can be obtained by studying the non-Abelian Barry's phase associated with the twisted Hamiltonians.[17]

From the above discussion we conclude that the ground-state degeneracy of the s-RVB state is not due to broken discrete symmetries. This is because (1) the ground-state degeneracy depends on the topology of the compactified space; (2) the ground-state degeneracy is robust against arbitrary perturbations, even those perturbations which break all the symmetries in the Hamiltonian; and (3) the energy split of the ground states is of order $e^{-L/\xi}$ for a finite system of size L. If the ground-state degeneracy was due to broken discrete symmetry, the energy split would be at most of order E^{-L^2/ξ^2}. This result suggests that the s-RVB state studied in Sec. III contains nontrivial topological orders.

The above discussion is essentially an application of the standard Z_2 gauge theory to our mean-field state. We hope to clarify the following points through the discussion in this section. (a) The stability of the Z_2 vortex is connected to the breaking of the SU(2) gauge symmetry. We know that the stability of the Z_2 vortex or the existance of the Z_2 gauge structure is crutial for the stability of the ground-state degeneracy. It was shown that the dimer fluctuations in the nearest-neighbor RVB state mediate a confining interaction between holons.[20] This indicates that the Z_2 vortex in the nearest-neighbor RVB state is unstable. Therefore, it is not clear whether the nearest-neighbor RVB state is a generic state or not. What we have learned from the above discussion is that the stability of the s-RVB state is ensured by the gauge symmetry breaking. Our results suggest that the inclusion of the longer bonds in the dimer model may help to stabilize the Z_2 vortex and to make the s-RVB state a generic state. (b) The ground-state degeneracy is due to the gauge symmetry in mean-field theory. The degeneracy is robust against any small perturbations.

The topological order is a very useful concept. Let us consider the following question: What is the difference between the spin-Peierls state and the s-RVB state? One may immediately say that the two states have different symmetries. But if we modify our Hamiltonians to break the translation and rotation symmetries, then the two states will have the same symmetries. In this case we can still ask whether the two states are the same or not in the sense of whether one state can be continuously deform into the other without phase transitions. When the translation and rotation symmetries are broken, the spin-Peierls state only support a nondegenerate ground state, while the s-RVB state still has four degenerate ground states on the torus. Therefore, the spin-Peierls and s-RVB states are different even when they have the same symmetries. The two states differ by having different topological orders.

VI. DISCUSSION

In this paper we studied some mean-field theories of spin-liquid states. In those mean-field theories the charge excitations have a finite gap. Because the system is already incompressible at the mean-field level, the Gutzwiller projection does not drastically change the correlations in the field theories. Many properties of spin-liquid states can be obtained from mean-field theories. In particular, for cases (B) and (C) discussed in Sec. II, the gauge fluctuations in mean-field theories have a finite gap and do not mediate any confining interactions. In those cases the quasiparticles in mean-field theories directly correspond to the quasiparticles in spin-liquid states. Such mean-field theories qualitatively describe all the properties of the spin-liquid state and are said to be dynamically stable. The chiral spin state is one of the dynamically stable mean-field states. The gauge bosons are massive as a result of the Chern-Simons term. In this paper we propose another dynamically stable mean-field state which has the T and P symmetries. The gauge bosons obtain finite masses from the Higgs mechanism. The quasiparticles in the corresponding spin-liquid state are found to be the spinons with Fermi statistics and holons with Bose statistics. Such a state corresponds to the s-RVB state. The s-RVB state is also shown to contain nontrivial topological orders. We would like to stress that the results obtained in this paper are not limited in two dimensions. They apply equally well to higher dimensions.

Many properties of our mean-field state are closely related to those obtained in the dimer model.[4,16] However, in the simplest dimer model which contains only bonds connecting nearest neighbors, the dimer fluctuations are shown to induce a confining interaction between the holons and spinons.[20] Also, in this dimer model, the holons and spinons on the even sublattice do not mix with the holons and spinons on the odd sublattice. The studies in this paper suggest that the confining interactions in the dimer model may be removed by including the dimers connecting the next-nearest neighbors. Those long dimers may also help to prevent the formation of the spin-Peierls state.

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[1]P. W. Anderson, in *Frontiers and Borderline of Many-Particle Physics* (CIV, Italy, 1988).

[2]E. H. Lieb and F. Y. Wu, Phys. Rev. Lett. **20**, 1445 (1968).

[3]X. G. Wen, F. Wilczek, and A. Zee, Phys. Rev. B **39**, 11 413 (1989).

[4]S. A. Kivelson, D. S. Rokhsar, and J. P. Sethna, Phys. Rev. B **35**, 8865 (1987); D. S. Rokhsar and S. A. Kivelson, Phys. Rev. Lett. **61**, 2376 (1988).

[5]G. Baskaran, Z. Zou, and P. W. Anderson, Solid State Commun. **63**, 873 (1987).

[6]G. Baskaran and P. W. Anderson, Phys. Rev. B **37**, 580 (1988).

[7]I. Affleck, Z. Zou, T. Hsu, and P. W. Anderson, Phys. Rev. B **38**, 745 (1988); E. Dagotto, E. Fradkin, and A. Moreo, *ibid.* **38**, 2926 (1988).

[8]X. G. Wen, Phys. Rev. B **39**, 7223 (1989).

[9]C. K. Majumdar and D. K. Ghosh, J. Math. Phys. **10**, 1388 (1969); I. Affleck, T. Kennedy, E. H. Lieb, and H. Tasaki, Phys. Rev. Lett. **59**, 799 (1987); Commun. Math. Phys. **115**, 477 (1988); N. Read and S. Sachdev, Phys. Rev. Lett. **62**, 1694 (1989).

[10]X. G. Wen and A. Zee, Phys. Rev. Lett. **63**, 461 (1989).

[11]V. Kalmeyer and R. B. Laughlin, Phys. Rev. Lett. **59**, 2095 (1987).

[12]R. B. Laughlin and Z. Zou, Phys. Rev. B **41**, 664 (1990).

[13]G. Baskaran also studied some mean-field theories which completely break the SU(2) gauge symmetry. G. Baskaran (private communication).

[14]P. W. Anderson and Z. Zou, Phys. Rev. Lett. **60**, 132 (1988); C. Kallin, A. J. Berlinsky, and W. K. Wu, Phys. Rev. B **39**, 4267 (1989).

[15]X. G. Wen, Phys. Rev. B **38**, 12 004 (1988).

[16]N. Read and B. Chakraborty, Phys. Rev. B **40**, 7133 (1989).

[17]X. G. Wen, Int. J. Mod. Phys. B **4**, 239 (1990).

[18]X. G. Wen and Q. Niu, Phys. Rev. B **41**, 9377 (1990).

[19]X. G. Wen, Phys. Rev. B **40**, 7387 (1989).

[20]L. B. Ioffe and A. I. Larkin, Phys. Rev. B **40**, 6941 (1989); E. Fradkin and S. Kivelson, Mod. Phys. Lett. B **4**, 225 (1990).