Quantum-spin-chain realizations of conformal field theories

Diptiman Sen
Centre for Theoretical Studies, Indian Institute of Science, Bangalore, 560012, India
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We present a series of one-dimensional quantum Hamiltonians that, at certain critical points, realize the minimal series of conformal field theories with central charge less than one. The models consist of ferromagnetically coupled SU(2) spins in a transverse magnetic field. The infinite-spin (free-boson) limit is especially studied after performing the Holstein-Primakoff transformation. The analysis can be generalized to SU(3) quantum chains. These realize a different series of conformal field theories at criticality.

I. INTRODUCTION

Soon after the advent of conformally invariant field theories (CFT) in two dimensions,[1] statistical-mechanics models were identified that can realize such theories at critical points describing second-order phase transitions.[2–6] By now a very large number of exactly solvable statistical models are known to provide examples of CFT.[7–11]

Many CFT can be obtained by coset constructions from Lie groups.[12] The minimal unitary series with the central charge c (in the Virasaro algebra) given by

$$c=1-\frac{6}{(K+2)(K+3)}\ , \tag{1}$$

can be written as $[\mathrm{SU}(2)_K\times \mathrm{SU}(2)_1]/\mathrm{SU}(2)_{K+1}$. Here K is a positive integer and $\mathrm{SU}(2)_K$ denotes a level-K-affine Kac-Moody algebra based on the classical Lie group SU(2).[13] The theories in (1) are realized by the statistical models in Refs. 5 and 6 where there is an integer variable which sits on each site of a square lattice and takes values from 1 to $K+2$. Integers on neighboring sites must differ by one. The states on the lattice links can therefore take $K+1$ different values. These states will be identified with spins in our models.

For the cosets $[\mathrm{SU}(3)_K\times \mathrm{SU}(3)_1]/\mathrm{SU}(3)_{K+1}$, the central charge is

$$c=2\left[1-\frac{12}{(K+3)(K+4)}\right]\ . \tag{2}$$

These CFT can occur in statistical models where two integer variables sit on each lattice site.[7,8] The states then correspond to the points in the weight diagrams of various representations of SU(3).[14] Other coset models known from statistical mechanics include $[\mathrm{SU(N)}_K \times \mathrm{SU(N)}_L]/\mathrm{SU(N)}_{K+L}$, and the parafermion theories[9] with Z_K symmetry $\mathrm{SU}(2)_K/\mathrm{U}(1)$ having $c=2(K-1)/(K+2)$. The CFT $\mathrm{SU}(2)_K$ with $c=3K/(K+2)$ are realized by quantum spin chains with certain isotropic polynomial interactions between nearest-neighbor spins, with $K=2S$ for the spin-S model.[10,11]

In spite of the impressive success of exactly solvable statistical models in providing concrete examples of CFT, it is instructive to study the subject from other points of view. Zamolodchikov's Landau-Ginzburg (LGZ) theory is one example of an alternative approach.[15] The minimal series in (1) is described in terms of the $(K+1)$-fold multicritical point of a scalar field theory with an effective Lagrangian

$$\mathcal{L}=\tfrac{1}{2}(\partial_\mu\phi)^2+g\phi^{2(K+1)}\ . \tag{3}$$

Here ϕ is a function of two coordinates t and x. This theory has a multicritical point where $K+1$ phases have merged and become indistinguishable from each other. [Below the critical temperature, the potential $V(\phi)$ has $K+1$ distinct minima with the same energy. These correspond to that many equilibrium phases with the same free energy.] Note that in the limit $K\to\infty$, the multicritical theory describes a free massless boson since the potential vanishes in the neighborhood of $\phi=0$. Such a boson constitutes a CFT with $c=1$ which is also the infinite-K limit of (1). The LGZ approach to CFT will often be used below.

The facts that the series in (1) is given by cosets of the group SU(2) which has rank one, and that the LGZ description of the series in (3) contains one scalar field are related. Further, the LGZ potential $V(\phi)=g\phi^{2(K+1)}$ at the critical point has the Weyl symmetry of SU(2).[14] Denote the three generators of SU(2) by (S_+,S_-,S_z) where S_z forms the Cartan subalgebra and $S_\pm$ are the ladder operators. The Weyl group has two elements. The nontrivial element reflects (S_+,S_-,S_z) to $(S_+,S_-,-S_z)$. We identify the expectation values of S_z in the various phases with the minima of $V(\phi)$ below the critical temperature. At the critical temperature, all the phases coalesce and $V(\phi)$ has a single minimum at $\phi=0$. Under the Weyl reflection $\phi\to-\phi$, $V(\phi)$, and $(\partial_x\phi)^2$ in (3) are both invariant. In Sec. II the kinetic term $(\partial_t\phi)^2$ will be identified with $S_x=S_++S_-$. The sum of the ladder operators is certainly Weyl symmetric.

For the SU(3)-coset models in (2), the LGZ description uses two scalar fields ϕ_1 and ϕ_2. SU(3) has two generators S_1 and S_2 in its Cartan subalgebra, and the ladder operators E_{a+} and their Hermitian conjugates E_{a-}, with $a=1$, 2, and 3. The expectation values of S_1 and S_2 in the various phases of a statistical model are denoted by ϕ_1 and ϕ_2, respectively. The LGZ Lagrangian is

$$\mathcal{L}=\frac{1}{2}(\partial_\mu\phi_1)^2+\frac{1}{2}(\partial_\mu\phi_2)^2-V(\phi_1,\phi_2) . \tag{4}$$

The Weyl group now has six elements. The group is generated by $2\pi/3$ rotations in the (ϕ_1,ϕ_2) plane, and a reflection to $(-\phi_1,\phi_2)$. By Weyl symmetry, the potential can only be a function of $\alpha_\phi=\phi_1^2+\phi_2^2$ and $\beta_\phi=\phi_2^3-3\phi_2\phi_1^2$. The term $(\partial_x\phi_1)^2+(\partial_x\phi_2)^2$ is Weyl symmetric. It will be shown in Sec. III that $(\partial_t\phi_1)^2+(\partial_t\phi_2)^2$ arises from the generator

$$T=\sum_{a=1}^{3}(E_{a+}+E_{a-}) \tag{5}$$

which is also Weyl symmetric. The first member $(K=1)$ of the SU(3) series has $c=\frac{4}{5}$ and corresponds to the three-state Potts model. The LGZ Lagrangian for it was written long ago and it has the required Weyl symmetry.[16]

In this paper we express the two series in (1) and (2) in terms of quantum-spin-chain models. SU(2) is studied in detail in Sec. II, while Sec. III analyzes SU(3) more briefly. The spin chain models arise in the quantum Hamiltonian limit of the row-to-row transfer matrix $\hat{T}$ of appropriate two-dimensional statistical models.[17] In this limit, the couplings between the spins in one row and the next are taken to infinity while the couplings between spins in the same row go to zero in proportion to a small parameter τ. As $\tau\rightarrow 0$, the transfer matrix takes the form $\hat{T}=\exp(\tau H_Q)$ where H_Q is the Hamiltonian of a quantum chain. The ground state (or states) of this model corresponds to the equilibrium phase (or phases) of the statistical model.

The one-dimensional quantum systems we will study are not exactly solvable, but they have the virtue of being amenable to well-known methods of perturbative and numerical analysis. We have the additional motivations that CFT have not so far been extensively studied from this point of view, and that the connection to the LGZ description is more transparent for the quantum chains than for the exactly solvable statistical models.

A key role in our analysis is played by the Holstein-Primakoff (HP) transformation from the generators of SU(N) to bosonic creation and annihilation operators. This is well known and widely used for SU(2) spins.[18,19] We generalize HP to the symmetric representations of SU(3) in Sec. III. HP is particularly useful for studying the infinite-K (free boson) limit of the SU(N) cosets in Eqs. (1) and (2).

II. SU(2) QUANTUM SPIN CHAINS

At a second-order critical point, quantum models have a vanishing energy gap. On considering the spectra of low-momentum excitations (with wavelengths much larger than the lattice spacing), we find one or more relativistic massless dispersion relations $\omega(k)=vk$, where v is the "velocity of light" and k is the momentum. In this way, a CFT is recovered. Consider, for example, the quantum model which is related to the two-dimensional Ising model with nearest-neighbor ferromagnetic interactions. The Hamiltonian for a chain with N sites is

$$H_Q=\sum_{i=1}^{N}[\tfrac{1}{2}(S_{iz}-S_{i+1,z})^2+\gamma S_{ix}] . \tag{6}$$

A spin-$\frac{1}{2}$ object sits at each site. The operators S_{ia} are given by the three Pauli matrices $\sigma_{ia}/2$. The transverse magnetic field γ plays the role of temperature in the corresponding statistical model. For small fields $0\leq\gamma<\gamma^*$ (where $\gamma^*=\frac{1}{2}$), the ground state is doubly degenerate in the thermodynamic limit $N\rightarrow\infty$. Both states are ordered and have nonzero expectation values

$$\langle S_z\rangle=\frac{1}{N}\sum_i\langle S_{iz}\rangle .$$

(The values in the two states have opposite signs by Weyl symmetry.) For $\gamma>\gamma^*$, there is a unique disordered ground state with $\langle S_z\rangle=0$. In both these regimes, the low momentum spectrum has a gap and correlation functions decay exponentially at large distances. As a relativistic field theory, one has a free massive Majorana fermion.[20] Exactly at the critical point γ^*, the energy gap vanishes and the long-distance correlations decay algebraically. We then get a massless Majorana fermion which constitutes a CFT with $c=\frac{1}{2}$. This is the first number $(K=1)$ of the SU(2) coset series.

We will generalize (6) to higher spins and argue that these models have a multicritical point realizing a CFT with c given in (1). The integer K is related to the spin by $K=2S$. Our models are the simplest possible ones which are Weyl symmetric and have a critical point where $K+1$ phases simultaneously become indistinguishable. In addition to the terms in (6), the models contain an on-site interaction given by a finite polynomial in S_{iz}^2. The degree of the polynomial is $[S]$, the largest integer less than or equal to S. The Hamiltonian is

$$H_Q=\sum_i\left[\frac{1}{2}(S_{iz}-S_{i+1,z})^2+\gamma S_{ix}+\sum_{n=1}^{[S]}a_{2n}S_{iz}^{2n}\right] . \tag{7}$$

Consider the phase diagram of (7) in the $([S]+1)$-dimensional parameter space $(\gamma,\mathbf{a}_{2n})$. At the origin $(0,\mathbf{0})$, the ground state is $(K+1)$-fold degenerate. In each state, the spins at all sites have the same value of $S_z=m$ where $m=-S,-S+1,\ldots,S$ distinguishes between the various phases. [We will use the notation $\langle S_z\rangle=m(\gamma)$ and $m=m(0)$ below.] As γ is increased from zero, the coefficients a_{2n} must be correspondingly tuned in order to maintain the same degree of degeneracy for the ground state. In any state m, the spins at some sites start fluctuating to values of S_z different from m and $\langle S_z\rangle$ begins to approach zero. When γ reaches a critical value γ^*, and the a_{2n}'s correspondingly go to a_{2n}^*, the $K+1$ states become indistinguishable and one has a unique disordered ground state with $\langle S_z\rangle=0$. [We will not investigate what happens beyond γ^*. The $S>\frac{1}{2}$ models do not enjoy the duality symmetry of Eq. (6) whereby γ and $1/(4\gamma)$ are related.] The number $[S]$ of the a_{2n}'s is exactly sufficient so as to be able to tune $K+1$ states to the same energy. We therefore have a line A parametrized by γ which runs from the origin (0,0) to the multicritical point $P=(\gamma^*,\mathbf{a}_{2n}^*)$.

At the point P, one expects a CFT with c given by (1). This is because our models have the same (Weyl) symmetry as the ones solved exactly in Refs. 5 and 6, and have the same degree of multicriticality $K+1$. By universality, they must both describe the same CFT at that critical point. (For $S=\frac{1}{2}$ and 1 we know from earlier works that CFT with $c=\frac{1}{2}$ and $\frac{7}{10}$ are obtained.[2,20,21])

Our calculations will proceed as follows. First, we find an equation for the line A by a perturbative expansion in small γ. (This is analogous to a low temperature expansion in statistical mechanics.) Along that line in the phase diagram we study how the expectation value $m(\gamma)$ in a given phase m approaches zero. By applying a simple ratio test we estimate the location of the critical point P. (For illustrative purposes we will present the complete phase diagrams for spin 1 and spin $\frac{3}{2}$.) Finally, we argue that it is consistent for the $S\rightarrow\infty$ limit to reduce to a free massless boson at criticality. We give heuristic arguments to show that the infinite S limit can fit into the LGZ approach.

At the beginning of the line A where $\gamma=0$, all the $K+1$ states have zero energy. In any state m, excitations consist of one or more spins differing from m. Using Rayleigh-Schrödinger perturbation theory we determine how the ground-state energies $E(m,\gamma,\mathbf{a}_{2n})$ change with increasing γ, and how the a_{2n}'s must be adjusted so that $E(m,\gamma,\mathbf{a}_{2n})$ may continue to remain independent of m. The perturbative expansion parameter turns out to be γ^2 rather than γ, essentially because all representations of SU(2) are real rather than complex [namely, the symmetry $(S_x,S_y,S_z)\rightarrow(-S_x,S_y,-S_z)$ can be implemented unitarily]. We take S to be arbitrary and keep all the a_{2n}'s from $n=1$ to ∞ for our calculations. For any particular S, one can then truncate keeping only the first $[S]$ of the a_{2n}'s by using the fact that S_z^{2n} for $n>[S]$ can be expressed in terms of the first $[S]$ powers of S_z^2 and the unit matrix $\underline{I}$.

To order γ^4 we discover that only a_2 and a_4 need to be changed from zero, that is,

$$a_2=-\tfrac{1}{2}\gamma^2+[\tfrac{5}{48}S(S+1)-\tfrac{25}{96}]\gamma^4\ ,$$
$$a_4=-\tfrac{5}{96}\gamma^4\ . \tag{8}$$

It can be shown in general that the series expansion for a_{2n} begins at order γ^{2n}. This will prove to be important later.

The line $A=(\gamma,\mathbf{a}_{2n}(\gamma))$ is actually a first-order transition line on which $K+1$ different phases coexist. Take $S=1$, for example. The Hamiltonian in (7) is a particular case of the Blume-Emery-Griffiths model,[21] which is known to have a tricritical point (realizing a CFT with $c=\frac{7}{10}$). Figure 1 exhibits the phase diagram. In the ordered region O lying below the line A, the phases $m=1$ and -1 coexist with $\langle S_z\rangle>0$ and $\langle S_z\rangle<0$, respectively. Above A, the disordered ground state D has $\langle S_z\rangle=0$. As A approaches the tricritical point P, the values of $\langle S_z\rangle$ in the phases $m=\pm1$ go to zero. The solid line B in

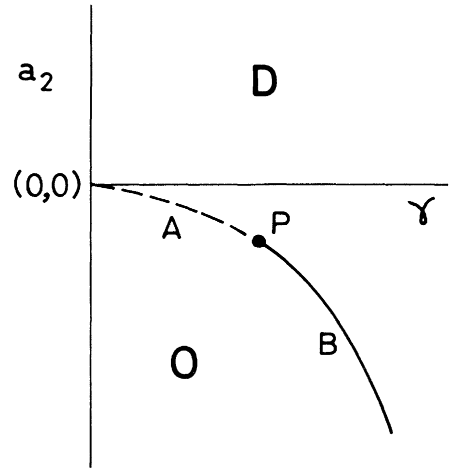

FIG. 1. Phase diagram of the spin-1 model. The dashed and solid lines A and B indicate lines of first-order and second-order transitions, respectively. O and D denote ordered and disordered regimes. P is a tricritical point.

the figure is a line of second-order-phase-transition points of the usual Ising type (a CFT with $c=\frac{1}{2}$). In Ref. 21 the numerical estimate of the coordinates of P is $(0.416,-0.090)$ which is consistent with Eq. (8).

As another example, the $S=\frac{3}{2}$ phase diagram is shown in Fig. 2. This has two ordered phases O_1 and O_2. In O_1, the phases $m=\pm\frac{3}{2}$ coexist while in O_2, $m=\pm\frac{1}{2}$ coexist. At point P, all four phases coalesce to produce a CFT with $c=\frac{4}{5}$. (The arrangements of first- and second-order transition lines shown in Figs. 1 and 2 are the simplest examples of the configurations allowed by Landau's

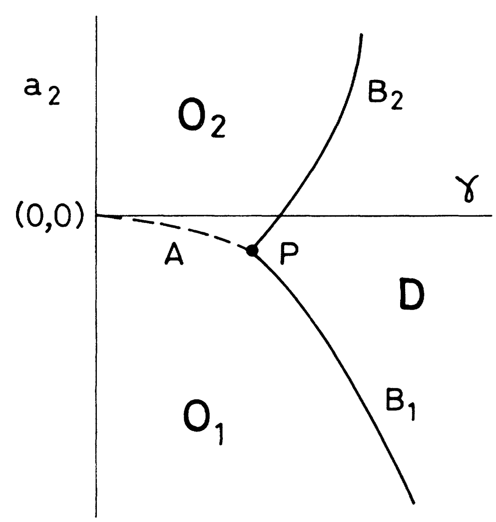

FIG. 2. Phase diagram for spin $\frac{3}{2}$. There are two ordered regimes O_1 and O_2, and two second-order transition lines B_1 and B_2. P is a multicritical point.

general theory of phase transitions.[22])

The phase diagrams become increasingly complicated as S increases. The multicritical point P always lies at an intersection of the first-order line A and several second-order surfaces. Although the complete phase diagram is hard to visualize as $S\to\infty$, we will see that the model becomes quite simple exactly at P.

We now consider the expectation value $\langle S_z \rangle$. One could study this in any one of the $K+1$ phases distinguished by m. Following Ref. 17 we begin with the Hamiltonian

$$H_Q(\theta)=H_Q+\theta\sum_i S_{iz} , \tag{9}$$

where H_Q is given in (7) and θ denotes a longitudinal magnetic field. Let $E(m,\gamma,\theta)$ be the expectation value of $H_Q(\theta)$ in the state $m(\gamma,\theta)$ at some point on the line A. We calculate $E(m,\gamma,\theta)$ perturbatively in γ but exactly in θ. The expectation value $\langle S_z \rangle$ is given by

$$m(\gamma)=\frac{1}{N}\frac{\partial E(m,\gamma,\theta)}{\partial\theta}\bigg|_{\theta=0} . \tag{10}$$

From the expansion in (8) we can calculate $m(\gamma)$ to order γ^4. We find

$$\frac{m(\gamma)}{m(0)}=1-\frac{\gamma^2}{2}-\gamma^4\left[\tfrac{17}{16}S(S+1)-\tfrac{17}{16}m^2+\tfrac{11}{32}\right] . \tag{11}$$

We now apply a ratio test to determine the critical point γ^* and an exponent β. We assume that $m(\gamma)$ fits the formula $(1-\gamma^2/\gamma^{*2})^\beta$ and expand the latter to order γ^4. A comparison with (11) yields two equations for γ^{*^2} and β. These give

$$\frac{1}{\gamma^{*^2}}=\tfrac{17}{4}[S(S+1)-m^2]+\tfrac{15}{8} \tag{12}$$

and $\beta=\gamma^{*^2}/2$. (Note that for $S=\frac{1}{2}$ and $m=\pm\frac{1}{2}$, these values of γ^* and β agree with the exact results.[17]) Unfortunately, the estimate for γ^* in (12) depends on the variable m. Since we are eventually interested in the infinite S limit and want results which are independent of S in that limit, we should consider values of m much smaller than S. We thus get $\gamma^{*^2}\sim\frac{4}{17}S^{-2}$. However, this value should not be taken too seriously because we only used two terms (the minimum number necessary) to apply the ratio test. The only information we can extract with some confidence from the above is that $\gamma^*\sim 1/S$ as $S\to\infty$. Since a_{2n} starts with γ^{2n}, it is reasonable to suppose that $a^*_{2n}\sim S^{-2n}$.

Now we do a spin-wave analysis at large S.[18,19] At $S\to\infty$, the configuration of spins can be viewed classically to lowest order in $1/S$. Since S_{iy} is absent in (7), the ground state must have all the spins lying in the $(\hat{x},\hat{z})$ plane. At $\gamma=0$, the spins all point in the same direction and the ground-state energy is zero regardless of that direction. As the magnetic field γ increases, the spins start tilting toward the negative $\hat{x}$ direction. At γ^*, they all have the classical values $(S_x,S_y,S_z)=(-S,0,0)$.

To the next order in $1/S$, the operators S_y and S_z undergo quantum fluctuations about zero. To quantify this we perform the HP transformation at each site:

$$\begin{aligned} S_y+iS_z&=a^\dagger(2S-N)^{1/2} , \\ S_y-iS_z&=(2S-N)^{1/2}a , \\ S_x&=-S+N , \end{aligned} \tag{13}$$

where $[a,a^\dagger]=1$ and N is the number operator $a^\dagger a$. [One can verify from (13) that $S_x^2+S_y^2+S_z^2=S(S+1)$.] We expand (13) for large S and keep only the lowest-order terms. On defining the canonically conjugate variables $q=i/\sqrt{2}(a-a^\dagger)$ and $p=1/\sqrt{2}(a+a^\dagger)$, we find that

$$q_i=\frac{s_{iz}}{\sqrt{S}},\quad p_i=\frac{S_{iy}}{\sqrt{S}} , \tag{14}$$

satisfy $[q_i,p_j]=i\delta_{ij}$. From Eqs. (13) and (14), we get the exact expression

$$S_{ix}=-S-\tfrac{1}{2}+\tfrac{1}{2}(q_i^2+p_i^2) . \tag{15}$$

For $S\to\infty$ let us take $\gamma^*=\alpha S^{-1}$ where α is a positive constant and $a_{2n}=\alpha_{2n}S^{-2n}$. Then (7) reduces to

$$H_Q=\tfrac{1}{2}\sum_i\left[\frac{\alpha}{S}P_i^2+S(q_i-q_{i+1})^2\right] , \tag{16}$$

where higher-order terms in $1/S$ and a constant have been dropped. On Fourier transforming we obtain the spin-wave spectrum

$$\omega(k)=2\alpha^{1/2}\left|\sin\frac{k}{2}\right| , \tag{17}$$

where the momentum k lies in the range $[-\pi,\pi]$. In the continuum limit $k\to 0$, this is the relativistic dispersion for a free massless boson with "velocity" $v=\alpha^{1/2}$.

We end this section by looking at the infinite S limit from the point of view of the LGZ Hamiltonian

$$H=\tfrac{1}{2}(\partial_t\phi)^2+\tfrac{1}{2}(\partial_x\phi)^2+V(\phi) . \tag{18}$$

We identify $\phi(t,x)$ with $S_{iz}=q_i\sqrt{S}$ so that $(S_{iz}-S_{i+1,z})^2$ in (7) becomes $(\partial_x\phi)^2$ in the continuum limit. The kinetic term $(\partial_t\phi)^2=\alpha p_i^2S^{-1}$ comes from the magnetic field operator γS_{ix}. We now argue that it is possible for the potential $V(\phi)$ to vanish as $S\to\infty$. Classically, $V(\phi)$ can be obtained from (18) by setting all the S_{iy} equal to zero so that $\partial_t\phi=0$, and all the S_{iz} equal to each other so that $\partial_x\phi=0$. Then the potential

$$V(\phi)=-\gamma S_x+\sum_n a_{2n}S_z^{2n} \tag{19}$$

from Eq. (7). We eliminate S_x in favor of S_z through $S_x=[S(S+1)-S_z^2]^{1/2}$, and express (19) in terms of $\phi=S_z$.

According to Zamolodchikov,[15] the potential must become independent of ϕ as $S\to\infty$. If we substitute $\gamma=\alpha S_c^{-1}$ and $a_{2n}=\alpha_{2n}S_c^{-2n}$ (the difference between $S_c=[S(S+1)]^{1/2}$ and S can be ignored for large S), we find that

$$V(\phi)=-\alpha\left[1-\left[\frac{S_z}{S_c}\right]^2\right]^{1/2}+\sum_n \alpha_{2n}\left[\frac{S_z}{S_c}\right]^{2n} \quad (20)$$

is required to be independent of S_z. Suppose that the coefficient $\beta_2 S_c^{-2}$ of $S_z^2=\phi^2$ in (20) is not zero. Then the LGZ action is

$$I=\int d^2x\left[\frac{1}{2}(\partial_\mu\phi)^2+\beta_2\frac{\phi^2}{S_c^2}\right] . \quad (21)$$

On rescaling the coordinates $(t,x)\rightarrow(S_c t, S_c x)$ we find an action which does not describe a free massless boson. So we must have $\beta_2=0$. Similarly, one can successively show that the coefficient of each power S_z^2 in (20) must vanish if one wants to obtain a CFT with $c=1$.

This implies certain relations between the numbers α and α_{2n}. For example, $\alpha_2=-\frac{1}{2}\alpha$, $\alpha_4=-\frac{1}{8}\alpha$, and so on. The main point in all this is that there exists a unique point P in the space of couplings $(\gamma,\mathbf{a}_{2n})$ where we get the Hamiltonian of a free massless boson in the infinite S limit.

III. SU(3) QUANTUM CHAINS

We now extend the above analysis to the case of SU(3). However, instead of discussing the magnetic-field perturbation theory as in Sec. II, we will proceed directly to the infinite K limit. We use the HP transformation to show that two free massless bosons emerge in that limit thereby giving a CFT with $c=2$.

We first review some group theory. We will consider the symmetric representations of SU(3) since the HP transformation is only known for these at the moment. Such a representation is denoted by $(K,0)$ where K is an integer. As a Young tableaux, it is shown by a single row of K boxes. The dimensionality of the representation is $d(K)=\frac{1}{2}(K+1)(K+2)$. In the weight diagram, these $d(K)$ points form an equilateral triangle of side $2K$. The vertices of the triangle have the eigenvalues of (S_1,S_2) equal to $(-K,K/\sqrt{3})$, $(K,K/\sqrt{3})$, and $(0,-2K/\sqrt{3})$. In Fig. 3 we show the $K=2$ representation for illustration. The double-headed arrows marked 1, 2, and 3 denote the directions along which the ladder operators $E_{1\pm}$, $E_{2\pm}$, and $E_{3\pm}$ move the six points. The quadratic Casimir invariant is given by

$$C_K=S_1^2+S_2^2+2\sum_a \{E_{a+},E_{a-}\}=\tfrac{4}{3}K(K+3) . \quad (22)$$

The obvious generalization of Sec. II is to consider a chain with a spin lying in the $(K,0)$ representation of SU(3) sitting at each site. As pointed out in Sec. I, the operators T in (5), $\alpha_s=S_1^2+S_2^2$ and $\beta_S=S_2^3-3S_2S_1^2$ are all Weyl invariant. So a candidate quantum Hamiltonian is

$$H_Q=\sum_i [\tfrac{1}{2}(S_{i1}-S_{i+1,1})^2+\tfrac{1}{2}(S_{i2}-S_{i+1,2})^2 -\gamma T_i+P(\alpha_{is},\beta_{is})] , \quad (23)$$

where $P(\alpha_s,\beta_s)$ is a polynomial in α_s and β_s of finite degree for any given value of K. As before, we can argue on grounds of symmetry and universality that a CFT with c given in (2) will emerge at the $d(K)$-fold multicritical point of the above model.

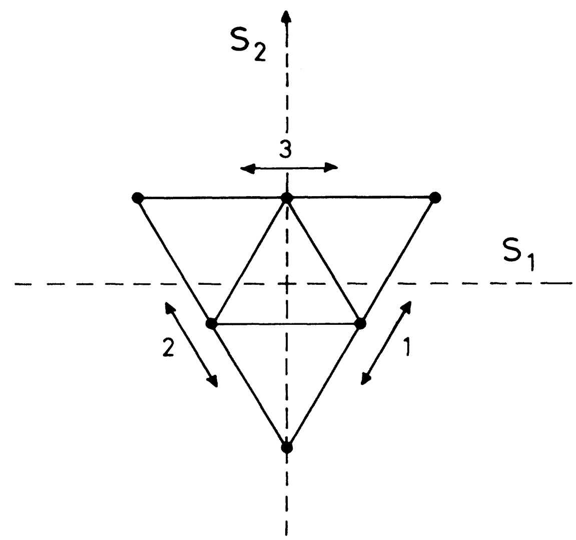

FIG. 3. The **6** representation of SU(3). S_1 and S_2 generate the Cartan subalgebra. The arrows marked 1, 2, and 3 indicate the directions along which the three pairs of ladder operators $E_{a\pm}$ act.

For $K=1$ (the fundamental representation denoted by 3), α_s has the same value for each of the three possible states, as does β_S. Hence the polynomial $P(\alpha_s,\beta_s)$ can be dropped [we ignored a polynomial in S_z^2 in (6) for spin $\frac{1}{2}$ for the same reason]. The remaining pieces in H_Q precisely give the quantum Hamiltonian of the three-state Potts model.[23] At $\gamma=0$, this has three phases denoted by three points in a plane (ϕ_1,ϕ_2). As γ increases to a critical value $\gamma^*=\frac{2}{3}$, these points converge to the origin (0,0) where a $c=\frac{4}{5}$ CFT resides.

We now consider the infinite K limit. Guided by the results for SU(2) we simply assume that the coefficients of the operator γT and of the terms in $P(\alpha_s,\beta_s)$ scale in the right way as $K\rightarrow\infty$ so as to produce a theory of two massless bosons. We will examine how the LGZ Hamiltonian

$$H=\tfrac{1}{2}[(\partial_t\phi_1)^2+(\partial_t\phi_2)^2+(\partial_x\phi_1)^2+(\partial_x\phi_2)^2] \quad (24)$$

can arise from H_Q. It is immediately clear that the last two terms in (24) may be identified with the first two terms in (23) since $\phi_1=\langle S_{i1}\rangle$ and $\phi_2=\langle S_{i2}\rangle$. We must now show that the operators T_i can produce the kinetic terms $(\partial_t\phi_1)^2+(\partial_t\phi_2)^2$.

Before doing that, it is convenient to rotate the generators S_1, S_2, and $E_{a\pm}$ to different ones S_1', S_2', and $E_{a\pm}'$ by a unitary transformation of H_Q. The reason for doing this is that at the critical point γ^*, all the spins line up (in a classical sense as $K\rightarrow\infty$) in the direction of the eigenvector of T with the largest eigenvalue $2K$. [This is entirely similar to the way in which the SU(2) spins in Sec. II line up in the direction of the eigenvector of S_x with the largest eigenvalue $(-S)$ as S goes to ∞.] We note that in any representation K, the set of eigenvalues of T and $-\sqrt{3}S_2$ are identical. Further, just as in Sec. II, S_1 and S_2 should go over (as $K\to\infty$) to the operators $q_1 \sim E_{1+}+E_{1-}$ and $q_2 \sim E_{2+}+E_{2-}$, as E_+ and E_- become the creation and annihilation operators $a^\dagger$ and a. From these considerations we can show that the rotation which must be performed at each site of the chain is of the form

$$T=-\sqrt{3}S_2' ,$$
$$S_1=(\tfrac{2}{3})^{1/2}(E'_{1+}+E'_{1-})+(\tfrac{1}{3})^{1/2}(E'_{3+}+E'_{3-}) , \quad (25)$$
$$S_2=(\tfrac{2}{3})^{1/2}(E'_{2+}+E'_{2-})+(\tfrac{1}{3})^{1/2}S'_1 .$$

The five other relations needed to describe completely the passage from the unprimed generators to the primed ones will not be displayed here (they are lengthy and also irrelevant for studying the infinite K limit).

We now present the Holstein-Primakoff transformation from the primed generators to two independent sets of bosonic operators $a_1(a_1^\dagger)$ and $a_2(a_2^\dagger)$:

$$\begin{aligned} S_1' &= a_1^\dagger a_1 - a_2^\dagger a_2 , \\ S_2' &= -\frac{2K}{\sqrt{3}}+\sqrt{3}N , \\ E'_{3+} &= a_1^\dagger a_2 , \\ E'_{3-} &= a_2^\dagger a_1 , \\ E'_{1+} &= a_1^\dagger (K-N)^{1/2} , \\ E'_{1-} &= (K-N)^{1/2} a_1 , \\ E'_{2+} &= a_2^\dagger (K-N)^{1/2} , \\ E'_{2-} &= (K-N)^{1/2} a_2 , \end{aligned} \quad (26)$$

where N is the total number operator $a_1^\dagger a_1 + a_2^\dagger a_2$. One can check that (26) satisfies all the commutation relations of SU(3), as well as giving the correct value of the Casimir in (22). Note that the generators $E'_{3\pm}$ and S'_1 which are purely quadratic in the bosonic operators form an SU(2) algebra.[14]

The operators a and $a^\dagger$ describe quantum fluctuations about the ground state in which S'_{i2} has the expectation value $-2K/\sqrt{3}$. On transforming from a and $a^\dagger$ to the conjugate operators q and p, and assuming that the polynomial interaction $P(\alpha_S,\beta_S)$ becomes negligible as $K\to\infty$, Eq. (23) reduces to

$$H_Q=\sum_i \left[\frac{2K}{3}(q_{i1}-q_{i+1,1})^2+\frac{2K}{3}(q_{i2}-q_{i+1,2})^2 + \frac{3\gamma^*}{2}(P_{i1}^2+P_{i2}^2) \right] . \quad (27)$$

Here terms of higher order in $1/K$ and a constant have been dropped. In the continuum limit, relativistic dispersion relations for two massless bosons follow with the "velocity" $v=[4K\gamma^*]^{1/2}$.

To complete the story for SU(3) in the same way as for SU(2), the above discussion needs to be supplemented by a perturbative analysis in γ (which will identify the value of γ^* and the relative magnitudes of the various terms in H in orders of $1/K$), and by an analysis of the LGZ potential $V(\alpha_\phi,\beta_\phi)$. We will not pursue this here.

IV. OUTLOOK

The analysis in this paper can be generalized in several directions. One can study the symmetric representations of higher SU(N) almost immediately since the HP transformation can be easily extended to these. HP transformations for other representations of either SU(3) or the higher SU(N) are not known to me.

One may ask whether coset models of the form $(G_K\times G_1)/G_{K+1}$ can be studied in a similar way for other Lie groups G. Suppose that we are only interested in the infinite K limit (which may correspond to representations of G which grow large in a particular manner). Then the central charge tends to $c(G_1)$. If a LGZ description in terms of rank-(G) free massless scalar fields is to be valid in that limit, one must have

$$c(G_1)=\text{rank}(G) . \quad (28)$$

This equality holds only for the groups $A_N=\text{SU}(N+1)$, $D_N=\text{SO}(2N)$, and $E_N(N=6, 7, \text{and } 8)$. We may therefore consider generalizing our analysis to various representations of these groups. For this purpose, it would be useful to construct appropriate HP transformations.

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