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Exact solutions for Ising-model odd-number correlations on planar lattices

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Simple procedures are first used to obtain exact solutions of highly-localized odd-number Ising correlations on the kagomé, square, and honeycomb lattices. To extend these results, a systematic and unifying method is then developed and demonstrated for finding exact solutions of n-site (with n an odd integer) Ising correlations on various planar lattices. The method combines five transformation or mapping theorems and linear-algebraic correlation identities of the triangular Ising model supplemented by a foreknowledge of its spontaneous magnetization and three select triplet correlations. In particular, considering a select seven-site cluster of the triangular Ising model, the knowledge of all its eleven odd-number correlations defined upon this cluster is shown sufficient for determining exactly *all* honeycomb, decorated-honeycomb, and kagomé Ising odd-number correlations upon their correspondingly select 10-, 19-, and 9-site clusters, respectively. The direct applicability of the catenated mapping theorems and relative ease of the calculational procedures are highlighted by the resulting large numbers of multisite correlation solutions (e.g., approximately 80 and 50 for the honeycomb and kagomé Ising models, respectively), the large $n_{\max}$ values ($n_{\max}=9$, 9, and 19, respectively, for the honeycomb, kagomé, and decorated-honeycomb Ising models), and convenient prescriptions for extracting critical amplitudes. The results also offer examples of correlation degeneracies and other linear-algebraic correlation identities that do not depend explicitly upon the interaction parameters.

I. INTRODUCTION

Our fundamental understanding of anomalous thermodynamic and transport behaviors in a many-body cooperative system stems in large measure from concomitant knowledge and applications of its thermal equilibrium correlations. The familiar representations of macroscopic observables in terms of their underlying correlations, e.g., specific heat and magnetic susceptibility as energy and magnetization fluctuations, respectively, Kubo formulas, temperature-dependent Green's functions, fluctuation-dissipation theorems, and so forth, constitute many of the most instructive basic relationships and useful formulations in statistical mechanics and bestride virtually all areas of theoretical investigation in phase transitions, critical and multicritical phenomena. Indeed, the impetus for the modern synoptic view of critical phenomena was the recognition of the essential role that the anomalously long-ranged spatial correlations played near a critical point, resulting in scaling theories and renormalization-group approaches towards problems in phase transitions and particle physics. In general, since correlation functions are structured using thermal expectation values of *products* of localized variables, they clearly offer a more detailed description than thermodynamic for the order and symmetry present in the system, and a precise presentation of the correlation solutions becomes highly desirable.

The Ising model remains the most studied and, arguably, the most significant lattice-statistical model in theoretical and computational science (any Hamiltonian having a finite density of finitely discrete commuting local variables can be cast as an Ising model). The model is unusually rich in its physical applications, being used not only to represent certain kinds of highly anisotropic magnetic crystals but also, e.g., as a lattice model for fluids, alloys, adsorbed monolayers, equilibrium polymerization, for biological and chemical systems, and in field theories of elementary particles (lattice gauge theories describing the quark structure of hadrons). Notably, within the wide variety of applications, the Ising model often provides a unified understanding of seemingly diverse problems.

The two-dimensional ($d=2$) simple Ising model in zero magnetic field is the only realistic microscopic model of cooperative phenomena for which many correlation solutions have been exactly found. From this perspective, the Ising model is thus set apart from other exactly solved models in statistical physics, e.g., the six-vertex, eight-vertex, and hard-hexagon models[1] for which very little is known concerning their correlations, certainly away from criticality. Planar Ising model *even-number* localized correlations have traditionally been calculated largely through the direct use of Pfaffian techniques[2] involving rather complicated expressions of elliptic integrals and Toeplitz determinants, with exact and explicit solutions, therefore, being restricted, in practice, to small, even numbers of closely neighboring lattice sites. Recently, however, a systematic and unifying method[3] was developed and demonstrated for obtaining the exact solutions of localized, even-number, multisite Ising correlations on various planar lattices. The scheme, which is exceedingly simpler than solely using Pfaffian techniques, embodied a series of five mapping theorems in alliance with algebraic correlation identities, where the triangular Ising model served an overarching role in the theoretical framework.

On the other hand, with no general method likened to the Pfaffian procedures available, exact solutions for the *odd-number* correlations of planar Ising models appear less frequently in the literature.[4] Since theoretical analyses of certain physical quantities and phenomena require such solutions (e.g., the joint configurational probabilities of the Ising spins are represented in terms of *both* even- and odd-number multispin correlations), one is motivated to similarly develop systematic procedures for finding exact solutions of the localized odd-number multisite Ising correlations on various planar lattices. As shown previously by Baxter,[5] the anisotropic free-fermion model is generic, being equivalent to the checkerboard Ising model and, therefore, containing the anisotropic square, triangular, and honeycomb Ising models as special cases. Recently, Baxter and Choy[4] have calculated local three-spin correlations of the anisotropic free-fermion model and, by employing the unifying concept of Z invariance,[6] they proved that a whole class of local three-spin correlations on various lattices are all given by the same universal function. Thus, Baxter and Choy were able, in particular, to obtain exact solutions for many three-spin correlations of the anisotropic free-fermion and planar Ising models. Some similar results were also obtained by Lin and Wu.[4] Within the developments of the present paper, supplemental use is made of these exact solutions for triplet correlations specialized to the isotropic triangular Ising model thereby enabling the theorems and many of the same strategies employed by Barry, Khatun, and Tanaka[3] for obtaining the even-number correlations of planar Ising models to be invoked for now finding the companion odd-number correlations.

II. EXACT SOLUTIONS FOR HIGHLY LOCALIZED CORRELATIONS OF THE KAGOMÉ, SQUARE, AND HONEYCOMB ISING MODELS

In this section, exact solutions for the highly localized odd-number correlations will first be found in a comparatively straightforward manner for the kagomé, square, and honeycomb Ising models. The practical procedures involve simple and direct use of linear-algebraic correlation identities having interaction-dependent coefficients, supplemented only by *a priori* knowledge of the spontaneous magnetizations.

The kagomé lattice (Japanese woven bamboo pattern) is a $d=2$ periodic array of equilateral triangles and regular hexagons (see Fig. 1), thus also called the 3-6 lattice. The lattice is regular (all sites equivalent, all bonds equivalent) and may be termed "close packed" since it contains elementary polygons having an odd number of sides (viz., triangles). One recognizes that the kagomé lattice has the same coordination number 4 as the square lattice, the latter being "loose packed." One defines the kagomé Ising-model ferromagnet on such a lattice of N_k sites as the (dimensionless) Hamiltonian

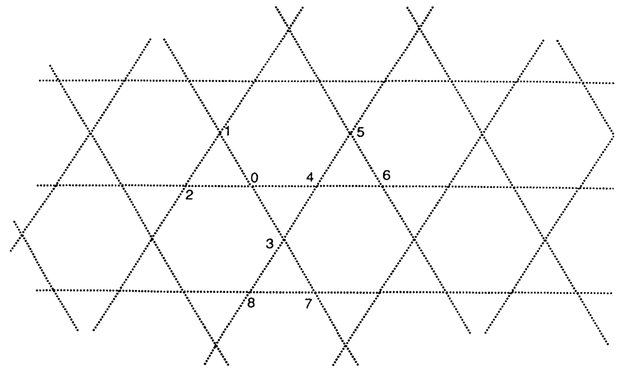

FIG. 1. The kagomé lattice. Nine sites are specifically enumerated for later use.

$$\mathcal{H}_k=-Q\sum_{\langle m,n\rangle}\mu_m\mu_n\ , \tag{2.1}$$

where each site-localized Ising variable $\mu_s=\pm1, \sum_{\langle m,n\rangle}$ designates a summation over all distinct nearest-neighbor pairs of lattice sites, and $Q>0$ is the (dimensionless) strength parameter of the ferromagnetic interaction. Letting the set of all Ising variables $\{\mu_0,\mu_1,\ldots,\mu_{N_k-1}\}\equiv\mu$, the magnetic canonical partition function Z_k is given by the usual trace formula over all degrees of freedom of the system:

$$Z_k=\mathrm{Tr}_\mu(e^{-\mathcal{H}_k})\ . \tag{2.2}$$

A class of correlation identities considered in the present paper is a set of linear-algebraic equations with coefficients dependent only upon the (dimensionless) interaction parameters.[7] To develop such identities systematically, one now proceeds to derive their basic generating equations. In the present case, let $[h]$ be any function of the kagomé Ising variables $\mu_1,\mu_2,\ldots,\mu_{N_k-1}$ (*excluding* μ_0, the origin-site variable in Fig. 1). Similarly, letting $\mathcal{H}'_k,\mathrm{Tr}'_\mu$ denote a *restricted* (dimensionless) Hamiltonian and trace operation, respectively, which *exclude* μ_0, one can construct the canonical thermal average $\langle\mu_0[h]\rangle$ as

$$\begin{aligned}Z_k\langle\mu_0[h]\rangle&=\mathrm{Tr}_\mu(e^{-\mathcal{H}_k}\mu_0[h])\\&=\mathrm{Tr}_\mu\exp[-\mathcal{H}'_k+Q\mu_0(\mu_1+\mu_2+\mu_3+\mu_4)]\mu_0[h]\\&=\mathrm{Tr}'_\mu(e^{-\mathcal{H}'_k}[h])\mathrm{Tr}_{\mu_0}(\exp[Q\mu_0(\mu_1+\mu_2+\mu_3+\mu_4)]\mu_0)\\&=\mathrm{Tr}_\mu(e^{-\mathcal{H}_k}[h])\frac{\mathrm{Tr}_{\mu_0}\exp[Q\mu_0(\mu_1+\mu_2+\mu_3+\mu_4)]\mu_0}{\mathrm{Tr}_{\mu_0}\exp[Q\mu_0(\mu_1+\mu_2+\mu_3+\mu_4)]}\end{aligned}$$

thereby yielding

$$\langle \mu_0[h] \rangle = \left\langle [h] \frac{\mathrm{Tr}_{\mu_0} \exp[Q\mu_0(\mu_1+\mu_2+\mu_3+\mu_4)]\mu_0}{\mathrm{Tr}_{\mu_0} \exp[Q\mu_0(\mu_1+\mu_2+\mu_3+\mu_4)]} \right\rangle, \quad \mu_0 \notin [h] \tag{2.3}$$

having written the standard definition of a canonical thermal average which initiated the above development. To further develop the last expression (2.3), one utilizes the following finite series expansion:

$$\frac{\mathrm{Tr}_{\mu_0} \exp[Q\mu_0(\mu_1+\mu_2+\mu_3+\mu_4)]\mu_0}{\mathrm{Tr}_{\mu_0} \exp[Q\mu_0(\mu_1+\mu_2+\mu_3+\mu_4)]} = \tanh[Q(\mu_1+\mu_2+\mu_3+\mu_4)]$$

$$= A_Q(\mu_1+\mu_2+\mu_3+\mu_4) + B_Q(\mu_1\mu_2\mu_3+\mu_1\mu_2\mu_4+\mu_1\mu_3\mu_4+\mu_2\mu_3\mu_4) , \tag{2.4}$$

where the expansion coefficients

$$A_Q = \tfrac{1}{8}[\tanh(4Q)+2\tanh(2Q)] , \tag{2.5a}$$

$$B_Q = \tfrac{1}{8}[\tanh(4Q)-2\tanh(2Q)] . \tag{2.5b}$$

To obtain the finite series form (2.4), use was made of the facts that any Ising variable μ_l satisfies $\mu_l^{2n+1} = \mu_l$, $\mu_l^{2n} = 1$, $n = 0, 1, 2, \ldots$. The coefficient expressions (2.5) were then determined by considering all possible realizations of the Ising variables μ_1, μ_2, μ_3, and μ_4 in the identity (2.4). Substituting (2.4) into (2.3), one obtains

$$\langle \mu_0[h] \rangle = A_Q \langle (\mu_1+\mu_2+\mu_3+\mu_4)[h] \rangle + B_Q \langle (\mu_1\mu_2\mu_3+\mu_1\mu_2\mu_4+\mu_1\mu_3\mu_4+\mu_2\mu_3\mu_4)[h] \rangle, \quad \mu_0 \notin [h] . \tag{2.6}$$

Equation (2.6) is the *basic generating equation* for developing linear-algebraic identities among Ising multisite correlations upon the kagomé lattice.

Using the basic generating equation (2.6), the fact that $\mu_l^2 = 1$ for any Ising variable and symmetry-group operations of the system Hamiltonian (2.1), one considers the five previously enumerated sites 0, 1, 2, 3, and 4 in Fig. 1 and obtains

$$y_{1k} = 4A_Q y_{1k} + 4B_Q y_{5k} , \tag{2.7a}$$

$$y_{2k} = y_{3k} = y_{4k} = 2(A_Q + B_Q)(y_{1k} + y_{5k}) , \tag{2.7b}$$

$$y_{6k} = 4A_Q y_{5k} + 4B_Q y_{1k} , \tag{2.7c}$$

where the localized kagomé correlations are defined by

$$\begin{aligned} y_{1k} &= \langle \mu_0 \rangle , \\ y_{2k} &= \langle \mu_0\mu_1\mu_2 \rangle , \\ y_{3k} &= \langle \mu_0\mu_1\mu_3 \rangle , \\ y_{4k} &= \langle \mu_0\mu_1\mu_4 \rangle , \\ y_{5k} &= \langle \mu_1\mu_2\mu_4 \rangle , \\ y_{6k} &= \langle \mu_0\mu_1\mu_2\mu_3\mu_4 \rangle . \end{aligned} \tag{2.8}$$

Simple inspection of the equations (2.7) reveals that the ratios of the correlations, $\eta_i \equiv y_{ik}/y_{1k}$, $i = 2,3,4,5,6$ can be obtained in terms of the known interaction-dependent coefficients A_Q, B_Q. Since A_Q, B_Q [Eqs. (2.5)] are analytic functions of temperature for all finite temperatures, the exact solutions for the ratio quantities η_i, $i = 2,3,4,5,6$ are each found to be similarly analytic:

$$\eta_5 = (4B_Q)^{-1}(1-4A_Q) , \tag{2.9a}$$

$$\eta_2 = \eta_3 = \eta_4 = 2(A_Q + B_Q)[1 + (4B_Q)^{-1}(1-4A_Q)] , \tag{2.9b}$$

$$\eta_6 = 4B_Q + A_Q B_Q^{-1}(1-4A_Q) . \tag{2.9c}$$

Owing to the fact that each odd-number correlation is thus constructed as a product of a suitably analytic ratio solution and the singular (branch point) spontaneous magnetization (appearing in Table I), the odd-number correlations manifestly possess the same critical temperature T_c and critical exponent $\frac{1}{8}$ as the spontaneous magnetization but differing *critical amplitudes* A_{ik}, $i = 1, 2, \ldots$, where $y_{ik} \sim A_{ik}\epsilon_k^{1/8}$, $T \to T_c -$,

$$\epsilon_k \equiv (T_c - T)/T_c = (Q - Q_c)/Q_c$$

being the fractional deviation of the temperature from its critical value T_c.

In order to explicitly calculate the amplitudes A_{jk}, $j = 2,3,4,5,6$, one first formally multiplies the ratio solutions (2.9) by the spontaneous magnetization y_{1k} and then substitutes the values of the critical coefficients A_{Q_c}, B_{Q_c} and critical amplitude A_{1k} (appearing in Table I):

$$A_{5k} = (4B_{Q_c})^{-1}(1-4A_{Q_c})A_{1k} = 1.011\,96\cdots , \tag{2.10a}$$

$$\begin{aligned} A_{2k} &= A_{3k} \\ &= A_{4k} \\ &= 2(A_{Q_c} + B_{Q_c})[1 + (4B_{Q_c})^{-1}(1-4A_{Q_c})]A_{1k} \\ &= 1.072\,70\cdots , \end{aligned} \tag{2.10b}$$

TABLE I. Spontaneous magnetizations and various critical constants of the triangular (t), square (sq), kagomé (k), honeycomb (hc), and decorated-honeycomb (dhc) Ising models. (The $\cosh^{-1}$ notation symbolizes arccosh).

	Spontaneous magnetizations	Critical constants		
t	$v_1=\left\{1-\dfrac{(1-\tanh R)^3(1-\tanh^2 R)^3}{16(\tanh^3 R)(1+\tanh^3 R)}\right\}^{1/8}$	$R_c=\frac{1}{4}\ln 3$ $=0.274\,65\ldots$	$V_1=(4\ln 3)^{1/8}$ $=1.203\,26\ldots$	$C_c=\frac{29}{140}=0.207\,14\ldots$ $D_c=-\frac{1}{56}=-0.017\,85\ldots$ $E_c=\frac{1}{140}=0.007\,14$
sq	$y_{1s}=[1-\sinh^{-4}(2S)]^{1/8}$	$S_c=\frac{1}{2}\ln(1+\sqrt{2})$ $=0.440\,68\ldots$	$A_{1s}=[4\sqrt{2}\ln(1+\sqrt{2})]^{1/8}$ $=1.222\,40\ldots$	$A_{S_c}=5\sqrt{2}/24=0.294\,62\ldots$ $B_{S_c}=-\sqrt{2}/24=-0.058\,92\ldots$
k	$y_{1k}=g(Q)[1-l(Q)]^{1/8}$, where $g(Q)=[(e^{4Q}-1)(e^{4Q}+3)]^{1/2}/(e^{4Q}+1)$ and $l(Q)=128(e^{8Q}+3)(e^{4Q}+1)^3/(e^{4Q}-1)^6(e^{4Q}+3)^2$	$Q_c=\frac{1}{4}\ln(3+2\sqrt{3})$ $=0.466\,56\ldots$	$A_{1k}=[2\sqrt{2}(2\sqrt{3}-3)]^{1/2}$ $\times[\ln(3+2\sqrt{3})]^{1/8}$ $=1.238\,65\ldots$	$A_{Q_c}=(12+7\sqrt{3})/(40+23\sqrt{3})$ $=0.302\,16\ldots$ $B_{Q_c}=-(5+3\sqrt{3})/(80+46\sqrt{3})$ $=-0.063\,85\ldots$
hc	$y_1=\left\{1-4\dfrac{\cosh(3K)\cosh^3(K)}{\sinh^6(2K)}\right\}^{1/8}$	$K_c=\frac{1}{2}\ln(2+\sqrt{3})$ $=0.658\,47\ldots$	$A_{1h}=\left\{\dfrac{8\sqrt{3}}{3}\ln(2+\sqrt{3})\right\}^{1/8}$ $=1.253\,17\ldots$	$A_c=-2B_c=2\sqrt{3}/9=0.384\,90\ldots$
dhc	$y_{1d}=\frac{2}{5}(1+3M)y_1(K)$, where $K=\frac{1}{2}\ln\cosh(2L)$ and $M=\frac{1}{2}\tanh(2L)$	$L_c=\frac{1}{2}\cosh^{-1}(2+\sqrt{3})$ $=0.995\,82\ldots$	$A_{1d}=\frac{2}{5}[1+3(\sqrt{3}-\frac{3}{2})^{1/2}]$ $\times\left\{\dfrac{16\sqrt{3}}{3}(\sqrt{3}-\frac{3}{2})^{1/2}\right.$ $\left.\times\cosh^{-1}(2+\sqrt{3})\right\}^{1/8}$ $=1.284\,72\ldots$	$M_c=\frac{1}{2}\tanh(2L_c)$ $=(\sqrt{3}-\frac{3}{2})^{1/2}=0.481\,71\ldots$

$$A_{6k}=[4B_{Q_c}+A_{Q_c}B_{Q_c}^{-1}(1-4A_{Q_c})]A_{1k}=0.906\,75\cdots . \tag{2.10c}$$

The result (2.9b) immediately grants the exact solutions[8]

$$\begin{aligned} y_{2k}&=y_{3k}\\ &=y_{4k}\\ &=2(A_Q+B_Q)[1+(4B_Q)^{-1}(1-4A_Q)]y_{1k} , \end{aligned} \tag{2.11}$$

which evidently is the first explicit solution of a threefold essential degeneracy to appear in the literature for Ising model correlations on any lattice. Other examples for the existence of a twofold essential degeneracy between septet correlations

$$\langle \mu_0\mu_1\mu_2\mu_3\mu_5\mu_6\mu_7\rangle=\langle \mu_0\mu_1\mu_2\mu_3\mu_5\mu_6\mu_8\rangle \tag{2.12a}$$

and a threefold essential degeneracy among quintet correlations

$$\langle \mu_0\mu_1\mu_3\mu_5\mu_7\rangle=\langle \mu_0\mu_1\mu_3\mu_5\mu_8\rangle=\langle \mu_0\mu_1\mu_4\mu_5\mu_7\rangle \tag{2.12b}$$

can be proven by using each correlation in (2.12) as a left-hand-side generator in the basic generating equation (2.6) and comparing the resulting correlation identities with aid of the kagomé lattice symmetry in Fig. 1.

The familiar square lattice (shown in Fig. 2) is regular (all sites equivalent, all bonds equivalent) and, as remarked earlier, possesses the same coordination number 4 as the kagomé lattice. One defines the square Ising-model ferromagnet on such a lattice of N_s sites as the (dimensionless) Hamiltonian

$$\mathcal{H}_s=-S\sum_{\langle i,j\rangle} v_i v_j , \tag{2.13}$$

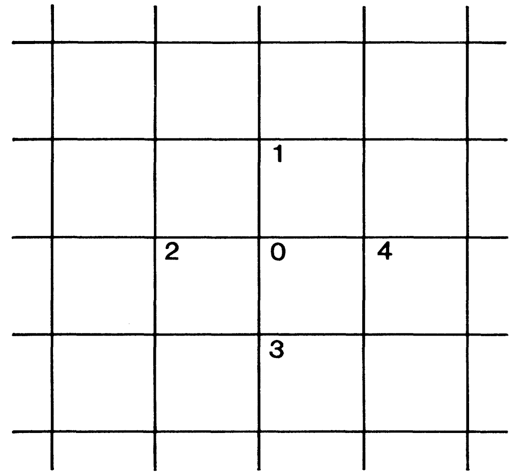

FIG. 2. The square lattice where five sites are specifically enumerated, namely, the origin site and its four nearest-neighboring sites.

where, as before, each site-localized Ising variable $v_l=\pm 1$, $\sum_{\langle i,j\rangle}$ designates the summation over all distinct nearest-neighbor pairs of lattice sites, and $S>0$ is the (dimensionless) strength parameter of the ferromagnetic interaction.

Since the square and kagomé Ising models are each defined upon four-coordinated lattice structures, the basic generating equations for developing their correlation identities are formally the same except for the interchanging of their respective (dimensionless) interaction parameters. Specifically, for the case of the square Ising model (2.13), the basic generating equation (2.6) is transcribed to read

$$\langle v_0[e]\rangle=A_S\langle (v_1+v_2+v_3+v_4)[e]\rangle+B_S\langle (v_1v_2v_3+v_1v_2v_4+v_1v_3v_4+v_2v_3v_4)[e]\rangle,\quad v_0\notin[e] \tag{2.14}$$

with coefficients

$$A_S=\tfrac{1}{8}[\tanh(4S)+2\tanh(2S)] , \tag{2.15a}$$

$$B_S=\tfrac{1}{8}[\tanh(4S)-2\tanh(2S)] . \tag{2.15b}$$

In a similar fashion as previously, the *basic generating equation* (2.14) is used to develop the following linear-algebraic identities among the odd-number correlations defined upon sites 0,1,2,3,4 of the square lattice in Fig. 2:

$$y_{1s}=4A_Sy_{1s}+4B_Sy_{3s} , \tag{2.16a}$$

$$y_{2s}=y_{4s}=2(A_S+B_S)(y_{1s}+y_{3s}) , \tag{2.16b}$$

$$y_{5s}=4A_Sy_{3s}+4B_Sy_{1s} , \tag{2.16c}$$

where

$$\begin{aligned} y_{1s}&=\langle v_0\rangle ,\\ y_{2s}&=\langle v_0v_1v_2\rangle ,\\ y_{3s}&=\langle v_1v_2v_3\rangle ,\\ y_{4s}&=\langle v_0v_1v_3\rangle ,\\ y_{5s}&=\langle v_0v_1v_2v_3v_4\rangle . \end{aligned} \tag{2.17}$$

As before, a simple inspection of (2.16) reveals that the ratios of the correlations $\xi_i=y_{is}/y_{1s}$, $i=2,3,4,5$, can be obtained in terms of the known interaction-dependent coefficients A_S, B_S whereupon the correlations y_{3s}, $y_{2s}=y_{4s}$, and y_{5s} are then obtained by multiplying the corresponding ratio solutions by the spontaneous magnetization y_{1s} of the square Ising model in Table I. These exact correlation solutions (Dekeyser and Rogiers[4] and Khatun[8]) are thus given by

$$y_{3s}=(4B_S)^{-1}(1-4A_S)y_{1s}\ , \tag{2.18a}$$

$$y_{2s}=y_{4s}=2(A_S+B_S)[1+(4B_S)^{-1}(1-4A_S)]y_{1s}\ , \tag{2.18b}$$

$$y_{5s}=[4B_S+A_SB_S^{-1}(1-4A_S)]y_{1s}\ , \tag{2.18c}$$

where (2.18b) demonstrates an essential doubly degenerate solution. In the same manner as previously, the solutions (2.18) may be used to calculate the *critical amplitudes* A_{is}, $i=2,3,4,5$:

$$A_{3s}=(4B_{S_c})^{-1}(1-4A_{S_c})A_{1s}=0.925\,802\cdots\ , \tag{2.19a}$$

$$A_{2s}=A_{4s}=2(A_{S_c}+B_{S_c})[1+(4B_{S_c})^{-1}(1-4A_{S_c})]A_{1s}=1.012\,677\cdots\ , \tag{2.19b}$$

$$A_{5s}=[4B_{S_c}+A_{S_c}B_{S_c}^{-1}(1-4A_{S_c})]A_{1s}=0.802\,944\cdots\ , \tag{2.19c}$$

where the critical values A_{S_c}, B_{S_c}, and A_{1s} in Table I have been substituted into (2.19).

Consider the honeycomb lattice structure ($d=2$ periodic array of regular hexagons) shown in Fig. 3. The honeycomb Ising-model ferromagnet is defined on such a lattice of N_h sites as the (dimensionless) Hamiltonian

$$\mathcal{H}_h=-K\sum_{\langle i,j\rangle}\sigma_i\sigma_j\ , \tag{2.20}$$

where, again, each site-localized Ising variable $\sigma_r=\pm1$, $\sum_{\langle i,j\rangle}$ designates the summation over all distinct nearest-neighbor pairs of lattice sites, and $K>0$ is the (dimensionless) strength parameter of the ferromagnetic interaction. Letting the set of all Ising variables $\{\sigma_0,\sigma_1,\ldots,\sigma_{N_h-1}\}\equiv\sigma$, the magnetic canonical partition function Z_h is given as before by the trace formula over all degrees of freedom of the system:

$$Z_h=\mathrm{Tr}_\sigma(e^{-\mathcal{H}_h})\ . \tag{2.21}$$

The *basic generating equation* for developing linear-algebraic identities among the multisite correlations of the honeycomb Ising model is derived, using earlier-type arguments, to be

$$\langle\sigma_0[f]\rangle=A\langle(\sigma_1+\sigma_2+\sigma_3)[f]\rangle+B\langle\sigma_1\sigma_2\sigma_3[f]\rangle,\quad \sigma_0\notin[f]\ , \tag{2.22}$$

where

$$A=\tfrac{1}{4}[\tanh(3K)+\tanh(K)]\ , \tag{2.23a}$$

$$B=\tfrac{1}{4}[\tanh(3K)-3\tanh(K)]\ . \tag{2.23b}$$

Following similar procedures used above for the kagomé and square Ising models, Barry, Múnera, and Tanaka[4] show that, aided by *a priori* knowledge of only the spontaneous magnetization y_1 of the honeycomb Ising ferromagnet given in Table I, exact solutions can be found for the following seven (exhaustive in number) odd-number correlations (and their corresponding critical amplitudes) defined upon the cluster of six sites 0,1,2,3,4,5 in Fig. 3:

$$\begin{aligned} y_2&=\langle\sigma_0\sigma_1\sigma_2\rangle\ ,\\ y_3&=\langle\sigma_0\sigma_4\sigma_5\rangle\ ,\\ y_4&=\langle\sigma_0\sigma_1\sigma_5\rangle\ ,\\ y_5&=\langle\sigma_0\sigma_1\sigma_4\rangle\ ,\\ y_6&=\langle\sigma_1\sigma_2\sigma_4\rangle\ ,\\ y_7&=\langle\sigma_0\sigma_1\sigma_2\sigma_4\sigma_5\rangle\ ,\\ y_8&=\langle\sigma_0\sigma_1\sigma_2\sigma_3\sigma_4\rangle\ . \end{aligned} \tag{2.24}$$

The essential double degeneracy

$$y_4=y_5 \tag{2.25}$$

was also established in these same investigations.

To review, assuming a foreknowledge of only the spontaneous magnetizations, the present section has demonstrated that the *highly localized* odd-number correlations can be obtained by relatively direct procedures for the kagomé, square, and honeycomb Ising-model ferromagnets. These simple methods do not appear promising, however, for finding additional numbers of exact solutions for odd-number localized correlations upon the above lattice structures or upon the only other $d=2$ regular lattice, the triangular lattice with its larger coordination number 6. The difficulties encountered for extending the number of exact solutions arise especially in the search for linear

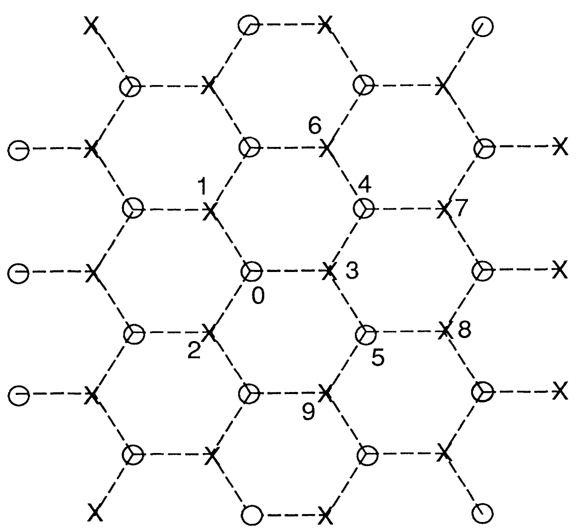

FIG. 3. The honeycomb lattice and its decomposition into two interlacing sublattices ("circled" ○ sites and "crossed" × sites, respectively) where each sublattice is triangular. Ten honeycomb sites are specifically enumerated for use throughout the paper.

independence in the systems of algebraic correlation identities which, from experience, is a more elusive algebraic property than closure. To address these shortcomings, the remainder of the paper will develop and demonstrate a systematic and unifying method for obtaining the exact solutions of n-site (n odd integer) Ising correlations on various planar lattices where the triangular Ising model investigated in the next section will later play an enveloping role in the theory.

III. EXACT SOLUTIONS FOR ODD-NUMBER LOCALIZED CORRELATIONS OF THE TRIANGULAR ISING MODEL

Consider the case of a triangular lattice ($\times$ sites, say, in Fig. 3). The triangular Ising model ferromagnet is defined upon such a lattice of N_t ($=\frac{1}{2}N_h$) sites as the (dimensionless) Hamiltonian

$$\mathcal{H}_t = -R \sum_{\langle k,l \rangle} \sigma_k \sigma_l \ , \tag{3.1}$$

where, again, each site-localized Ising variable $\sigma_q = \pm 1$, $\sum_{\langle k,l \rangle}$ denotes summation over all distinct nearest-neighbor pairs of lattice sites, and $R > 0$ is the (dimensionless) strength parameter of the ferromagnetic interaction. The magnetic canonical partition function Z_t of the triangular Ising model is now defined by

$$Z_t = \mathrm{Tr}_\times (e^{-\mathcal{H}_t}) \ , \tag{3.2}$$

where the notation $\mathrm{Tr}_\times$ signifies that the trace operation is taken over the degrees of freedom of all N_t $\times$-site Ising variables.

In Fig. 3, calling site 3 the origin site, the sites 1, 2, 6, 7, 8, 9 then become its six nearest-neighboring sites. Using the same manner of derivation as previously, the *basic generating equation* for developing linear-algebraic identities among Ising multisite correlations upon the triangular lattice is given by

$$\begin{aligned} \langle 3[g] \rangle = {} & C\langle (1+2+6+7+8+9)[g] \rangle \\ & + D\langle (126+127+128+129+167+168+169+178+179+189 \\ & \quad +267+268+269+278+279+289+678+679+689+789)[g] \rangle \\ & + E\langle (12\,678+12\,679+12\,689+12\,789+16\,789+26\,789)[g] \rangle , \quad 3 \notin [g] \ , \end{aligned} \tag{3.3}$$

where, for notational simplicity, only the numeric site labels of the Ising variables are entered within the thermal average symbols, and where the coefficients are given by

$$C = \tfrac{1}{32}[\tanh(6R) + 4\tanh(4R) + 5\tanh(2R)] \ , \tag{3.4a}$$

$$D = \tfrac{1}{32}[\tanh(6R) - 3\tanh(2R)] \ , \tag{3.4b}$$

$$E = \tfrac{1}{32}[\tanh(6R) - 4\tanh(4R) + 5\tanh(2R)] \ . \tag{3.4c}$$

Considering the spatially compact cluster of triangular lattice sites 1,2,3,6,7,8,9 (see $\times$ sites in Fig. 3), the set of odd-number thermal averages v_i, $i = 1, \ldots, 11$, defined below and diagrammatically depicted in Fig. 4 exhausts all such (nonequivalent) possibilities defined upon this seven-site cluster:

$$\begin{aligned} & v_1 = \langle 3 \rangle , \quad v_2 = \langle 312 \rangle , \quad v_3 = \langle 317 \rangle , \\ & v_4 = \langle 318 \rangle , \quad v_5 = \langle 31267 \rangle , \\ & v_6 = \langle 31268 \rangle , \\ & v_7 = \langle 3126789 \rangle , \quad v_8 = \langle 127 \rangle , \\ & v_9 = \langle 179 \rangle , \quad v_{10} = \langle 12678 \rangle , \quad v_{11} = \langle 31278 \rangle . \end{aligned} \tag{3.5}$$

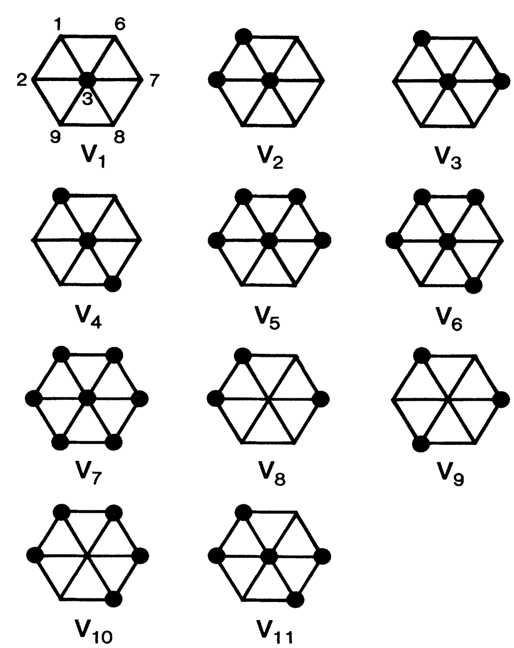

FIG. 4. Diagrammatic representation of the triangular lattice odd-number correlations v_i, $i = 1, 2, \ldots, 11$. These correlations exhaust all such (nonequivalent) possibilities defined upon sites 1,2,3,6,7,8,9 enumerated in Fig. 3 or 7 and constitute a spanning set in the context of the present paper.

Using the basic generating equation (3.3) and the definitions (3.5), the following system of exact, linear, algebraic, homogeneous correlation identities are derived with cognizance of the triangular lattice symmetry (see Fig. 4):

$$v_1=6Cv_1+2D(3v_3+6v_8+v_9)+6Ev_{10}\ , \tag{3.6a}$$

$$v_2=2C(v_1+v_3+v_8)+2D(2v_1+v_3+4v_8+v_9+2v_{10})+2E(v_3+v_8+v_{10})\ , \tag{3.6b}$$

$$v_3=C(2v_1+v_3+2v_8+v_9)+4D(v_1+v_3+2v_8+v_{10})+E(v_3+2v_8+v_9+2v_{10})\ , \tag{3.6c}$$

$$v_4=2C(v_1+2v_8)+2D(2v_1+3v_3+2v_8+v_9+2v_{10})+2E(2v_8+v_{10})\ , \tag{3.6d}$$

$$v_5=2C(v_3+v_8+v_{10})+2D(2v_1+v_3+4v_8+v_9+2v_{10})+2E(v_1+v_3+v_8)\ , \tag{3.6e}$$

$$v_6=C(v_3+2v_8+v_9+2v_{10})+4D(v_1+v_3+2v_8+v_{10})+E(2v_1+v_3+2v_8+v_9)\ , \tag{3.6f}$$

$$v_7=6Cv_{10}+2D(3v_3+6v_8+v_9)+6Ev_1\ , \tag{3.6g}$$

$$v_{11}=2C(2v_8+v_{10})+2D(2v_1+3v_3+2v_8+v_9+2v_{10})+2E(v_1+2v_8)\ . \tag{3.6h}$$

The results of Baxter and Choy[4] for the anisotropic triangular Ising model include, as a specialization, the following exact solutions for the *isotropic* case under investigation:

$$v_1=\left[1-\frac{(1-\tanh R)^3(1-\tanh^2 R)^3}{16(\tanh^3 R)(1+\tanh^3 R)}\right]^{1/8}\ , \tag{3.7a}$$

$$v_2=\left[1+2\frac{e^{4R}-2e^{-4R}+1-(e^{4R}+3)^{1/2}(e^{4R}-1)^{1/2}}{(e^{2R}-e^{-2R})^2}\right]v_1\ , \tag{3.7b}$$

$$v_3=v_4=\left[1-\frac{[e^{4R}+1-(e^{4R}+3)^{1/2}(e^{4R}-1)^{1/2}]^2}{(e^{2R}-e^{-2R})^2}\right]v_1\ . \tag{3.7c}$$

Subtracting (3.6b) from (3.6c), one obtains

$$v_9=v_3+(C-2D+E)^{-1}(v_3-v_2) \tag{3.8}$$

and, since the exact solutions (3.7b) and (3.7c) are known for v_2 and v_3, respectively, the relation (3.8) determines the exact solution for v_9.

Using the *degeneracy relation* $v_3=v_4$ stated in (3.7c), one subtracts (3.6d) from (3.6c) giving

$$(C-2D+E)(v_3-2v_8+v_9)=0\ . \tag{3.9}$$

Since $C-2D+E\neq 0$ (except at $K=0$), one concludes from (3.9) that

$$v_8=\tfrac{1}{2}(v_3+v_9)\ , \tag{3.10}$$

i.e., the exact solution for v_8 is found to be the arithmetic mean of the previously known solutions v_3 and v_9. Subtracting (3.6h) from (3.6f), one obtains

$$v_6-v_{11}=(C-2D+E)(v_3-2v_8+v_9)=0 \tag{3.11}$$

using (3.9). Therefore, the *degeneracy relation*

$$v_6=v_{11} \tag{3.12}$$

is established.

Since v_1, v_3 $(=v_4)$, v_8, and v_9 are now determined, the identity (3.6a) yields the exact solution for v_{10}:

$$v_{10}=(6E)^{-1}[(1-6C)v_1-2D(3v_3+6v_8+v_9)]\ . \tag{3.13}$$

The identities (3.6e)–(3.6g) can finally be employed to immediately grant the exact solutions for the remaining correlations v_5, v_6, $(=v_{11})$, and v_7, respectively. A major goal of the present paper has thus been achieved, namely, for the case of the isotropic triangular Ising model, exact solutions have been found for the 11 odd-number correlations $v_1,v_2,\ldots,v_{11}$ defined upon the spatially compact cluster of seven lattice sites 1,2,3,6,7,8,9 (see the × sites in Fig. 3). Substituting as needed the critical values R_c, C_c, D_c, E_c, and V_1 from Table I into the exact solution expressions (3.7b), (3.7c), (3.8), (3.10), (3.13), (3.6e), (3.6f), and (3.6g), one directly evaluates the *critical amplitudes* V_j, $j=2,3,\ldots,11$, where, as usual, $v_i\sim V_i\epsilon_t^{1/8}$, $R\to R_c+$, $i=1,2,\ldots,11$, $\epsilon_t\equiv(R-R_c)/R_c$ being the fractional deviation of the temperature from its critical value:

$$V_1=1.203\,269\cdots\ , \tag{3.14a}$$

$$V_2=0.967\,245\cdots\ , \tag{3.14b}$$

$$V_3=V_4=0.944\,097\ldots\ , \tag{3.14c}$$

$$V_5=0.782\,058\ldots\ , \tag{3.14d}$$

$$V_6=V_{11}=0.758\,910\ldots\ , \tag{3.14e}$$

$$V_7=0.647\,709\ldots\ , \tag{3.14f}$$

$$V_8=0.897\,800\ldots\ , \tag{3.14g}$$

$$V_9=0.851\,503\ldots\ , \tag{3.14h}$$

$$V_{10}=0.740\,302\ldots\ . \tag{3.14i}$$

One notes that, besides the double-degeneracy relations (3.7c) and (3.12), other linear-algebraic correlation identities which do *not* explicitly contain the interaction parameters can be derived solely using the system of identities (3.6) (but not vice versa), for example, in addition to (3.10),

$$v_1-3v_2+3v_5-v_7=0 , \tag{3.15a}$$

$$v_2-v_3-v_5+v_6=0 . \tag{3.15b}$$

IV. ENVELOPING ROLE OF THE TRIANGULAR ISING MODEL

In the earlier companion article of Barry, Khatun, and Tanaka,[3] the investigations and analyses of exact solutions for Ising-model *even-number* correlations on planar lattices employed a series of five mapping or extended transformation theorems (extended in the sense that the theorems apply beyond partition functions to multisite correlations) which successively mapped unknown planar Ising correlations upon linear combinations of those Ising correlations already known on other planar lattices. The catenated mapping theorems were direct and systematic in their applications and, in alliance with algebraic correlation identities, the method demonstrated that only a few localized Ising correlations on the triangular lattice actually needed to be calculated by traditional Pfaffian procedures in order to obtain large numbers of exact solutions for localized Ising correlations upon the honeycomb and kagomé lattices as well as upon other irregular (bond-decorated) planar lattices. In that paper, the theoretical approach was emphasized to be exceedingly simpler than using solely Pfaffian procedures which are known to become progressively lengthly and arduous as either the number of sites under consideration or the distances between these sites increase.

In the theoretical studies of the present paper upon exact solutions of planar Ising-model *odd-number* correlations, the same five mapping theorems mentioned above remain completely valid and the enveloping strategy using the triangular Ising model can thus be repeated. Since discussions and proofs of these five mapping theorems can be found in the earlier companion paper,[3] only the statements and leading ideas of the theorems will be retold here along with their new applications throughout later sections in determining exact solutions of planar Ising-model odd-number correlations. For emphasis, writing corresponding (dimensionless) interaction parameters explicitly as subscripts on Hamiltonians, partition functions, and thermal averages will be a useful and frequent notation in this and the following sections.

(1) Theorem 1. Consider the honeycomb lattice decomposition depicted in Fig. 3 and let $[r]$ be any function of Ising variables containing *only* "crossed" $\times$ sites (or *only* "circled" $\circ$ sites). Then,

$$\langle [r] \rangle_{h,K} = \langle [r] \rangle_{t,R} ,$$

where $\langle [r] \rangle_{h,K}$ and $\langle [r] \rangle_{t,R}$ denote canonical thermal averages pertaining to the honeycomb and triangular Ising-model (dimensionless) Hamiltonians $\mathcal{H}_{h,K}$ and $\mathcal{H}_{t,R}$, respectively, with their (dimensionless) interaction parameters related by $2\cosh 2K = e^{4R}+1$.

The above registry theorem will have significant consequences throughout the remainder of the studies. In general, any honeycomb lattice thermal average associated with a configuration of sites which are in registry with the sites of the triangular lattice can now be equated to the corresponding triangular lattice thermal average, and vice versa, in the sense of the star-triangle (Y-Δ) relationships of theorem 1. It should thus be clear that the thermal-average symbol for any correlation that satisfies the registry theorem 1 does not actually require subscripts $\langle \cdots \rangle_{h,K}$ or $\langle \cdots \rangle_{t,R}$ since either context is correct.

As an extension, the next theorem which utilizes theorem 1 will enable *any* honeycomb Ising correlation to be systematically expanded into a linear combination of triangular Ising correlations.

(2) Theorem 2. Any honeycomb Ising-model correlation can be represented as a linear combination of triangular Ising-model correlations.

As an example which illustrates the above simple superposition procedure, considering the following five-site correlation where, again, for notational brevity, only the site labels appear within thermal average symbols, one finds that

$$\begin{aligned}
\langle 03489 \rangle_{h,K} &= \langle 04389 \rangle_{h,K} \\
&= \langle [A(1+2+3)+B123][A(3+6+7)+B367]389 \rangle_{h,K} \\
&= A^2\langle 189+289+389+689+789+13\,689+13\,789+23\,689+23\,789 \rangle_{h,K} \\
&\quad + AB\langle 12\,389+12\,689+12\,789+16\,789+26\,789+36\,789 \rangle_{h,K} + B^2\langle 1\,236\,789 \rangle_{h,K} \\
&= A^2\langle 189+289+389+689+789+13\,689+13\,789+23\,689+23\,789 \rangle_{t,R} \\
&\quad + AB\langle 12\,389+12\,689+12\,789+16\,789+26\,789+36\,789 \rangle_{t,R} + B^2\langle 1\,236\,789 \rangle_{t,R} \\
&= A^2(v_2+2v_3+v_5+2v_6+2v_8+v_{11})+2AB(v_5+2v_{10})+B^2 v_7 ,
\end{aligned} \tag{4.1}$$

where all $\circ$ sites were systematically eliminated using the basic generating equation (2.22) for honeycomb Ising correlation identities whereupon every ensuing honeycomb Ising correlation then directly corresponds by theorem 1 to some individual triangular Ising correlation defined in (3.5) and diagrammatically represented in Fig. 4.

Towards proving that kagomé Ising correlations can be mapped upon linear combinations of honecomb Ising correlations, one first introduces the *decorated-*

honeycomb lattice, which is the lattice formed by the previous honeycomb lattice supplemented with lattice points at the centers of all bonds (see Fig. 5). The resulting bond-decorated lattice is irregular since all sites are no longer equivalent. The decorated-honeycomb Ising model ferromagnet is then defined by the (dimensionless) Hamiltonian

$$\mathcal{H}_d = -L \sum_{\langle p,q \rangle} \sigma_p \mu_q ; \tag{4.2}$$

where σ_p, μ_q are Ising variables localized on an original honeycomb site p and "solid-circled" decoration site q, respectively, $\sum_{\langle p,q \rangle}$ designates a summation over all distinct nearest-neighbor pairs of lattice sites, and $L > 0$ is the (dimensionless) strength parameter of the ferromagnetic interaction. The magnetic canonical partition function Z_d is given as customary by the trace formula

$$Z_d = \mathrm{Tr}_{\mu,\sigma}(e^{-\mathcal{H}_d}) . \tag{4.3}$$

Small portions of the honeycomb, decorated-honeycomb, and kagomé Ising models are depicted in Fig. 6, and these models are connected by the decoration-iteration (I) and star-triangle (Y-Δ) transformations,[9] respectively. Using the decorated-honeycomb Ising model in a mediating role, one next states three theorems which, taken together, will enable any correlation of the kagomé Ising model to be mapped upon a linear combination of honeycomb Ising-model correlations.

(3) Theorem 3. $\langle \mu_l \mu_m \cdots \mu_r \rangle_{k,Q} = \langle \mu_l \mu_m \cdots \mu_r \rangle_{d,L}$, where the corresponding (dimensionless) interaction parameters are related by $e^{4Q} = 2\cosh 2L - 1$.

(4) Theorem 4. $\langle \mu_l \mu_m \cdots \mu_r \rangle_{d,L} = M^n \langle (\sigma_j + \sigma_k)(\sigma_q + \sigma_s) \cdots (\sigma_t + \sigma_u) \rangle_{d,L}$, where the left-hand-side (lhs) μ product contains n factors, $M = \frac{1}{2}\tanh 2L$, and on the rhs, σ_j, σ_k are the nearest-neighbor Ising variables of μ_l; σ_q, σ_s are the nearest-neighboring Ising variables of μ_m, and so forth.

(5) Theorem 5.

$$\langle \sigma_i \sigma_n \sigma_p \cdots \sigma_v \rangle_{d,L} = \langle \sigma_i \sigma_n \sigma_p \cdots \sigma_v \rangle_{h,K} ,$$

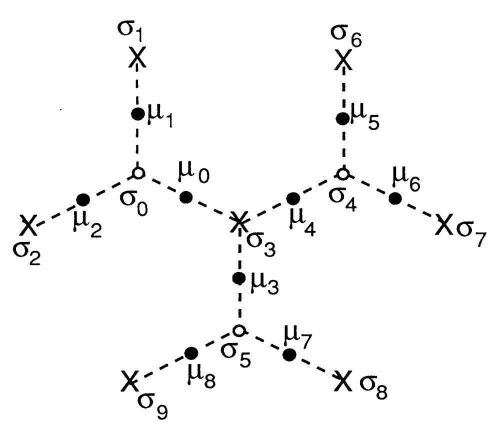

FIG. 5. A portion of the decorated-honeycomb lattice where ten σ variables $\sigma_0, \sigma_1, \ldots, \sigma_9$ and nine μ variables $\mu_0, \mu_1, \ldots, \mu_8$ are specified for later use.

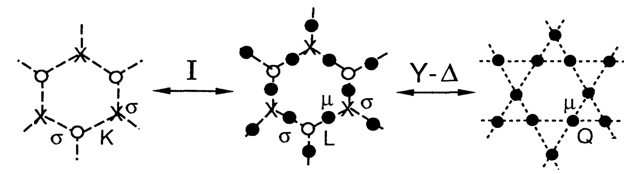

FIG. 6. The honeycomb, decorated-honeycomb, and kagomé Ising models are connected by transformation theory. The decoration-iteration (I) and star-triangle (Y-Δ) transformations equate, aside from known multiplicative constants, the partition functions $Z_{d,L}$ to $Z_{h,K}$ and $Z_{d,L}$ to $Z_{k,Q}$, respectively, where their (dimensionless) interaction parameters K, L, and Q are simply related.

where the corresponding (dimensionless) interaction parameters are related by $\cosh(2L) = e^{2K}$.

The unifying scheme of the present section should now become clear. Systematically, theorem 3 maps a kagomé Ising correlation upon a μ-type decorated-honeycomb Ising correlation, whereupon theorem 4 then maps this latter correlation upon a linear combination of σ-type decorated-honeycomb Ising correlations. Each of the latter correlations is then equated to a honeycomb Ising correlation by theorem 5. Since theorems 1 and 2 previously established that any honeycomb Ising correlation can itself be mapped upon a linear combination of triangular Ising correlations, one sees that the triangular Ising model plays the role of a "canopy or umbrella" in the sense that knowing *all* its correlations on a select cluster of sites is sufficient to determine *all* honeycomb, decorated-honeycomb, and kagomé Ising correlations upon their respective sites which are appropriately located within or upon the selected "canopy" cluster of triangular lattice sites (see Fig. 7). Specifically in Fig. 7,

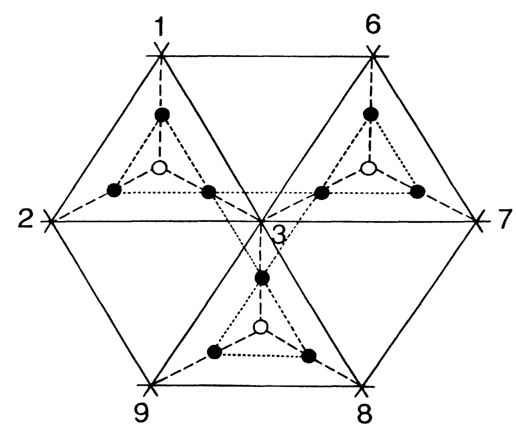

FIG. 7. For the calculation of Ising correlations, the triangular lattice (solid bonds) may be viewed as enveloping the honeycomb (dashed bonds), kagomé (dotted bonds), and decorated-honeycomb (dashed bonds) lattices. Seven sites 1,2,3,6,7,8,9 of the triangular lattice are specifically enumerated for select applications of the present theory.

knowledge of all triangular lattice correlations $v_1, v_2, \ldots, v_{11}$ upon its 7-site cluster is sufficient to determine all honeycomb lattice correlations upon its 10-site cluster, all decorated-honeycomb lattice correlations upon its 19-site cluster, and all kagomé lattice correlations upon its 9-site cluster. This fact, that all such Ising odd-number correlations can now be simply and systematically mapped upon triangular Ising-model odd-number correlations, underscores the desirability of having available exact solutions for the latter and will be demonstrated in the next section.

V. SOME SELECT RESULTS

This section illustrates the procedures of the present theory by finding exact solutions for select odd-number correlations of the kagomé Ising model in terms of the known spanning correlations $v_1, \ldots, v_{11}$ of the triangular Ising model.

Considering, first, the spontaneous magnetization (ordering parameter), one has

$$\begin{aligned}\langle \mu_0 \rangle_{k,Q} &= \langle \mu_0 \rangle_{d,L} \quad \text{(by theorem 3)}\\ &= M \langle \sigma_0 + \sigma_3 \rangle_{d,L} \quad \text{(by theorem 4)}\\ &= M \langle \sigma_0 + \sigma_3 \rangle_{h,K} \quad \text{(by theorem 5)}\\ &= 2M \langle \sigma_0 \rangle_{h,K}\\ &= 2M \langle \sigma_0 \rangle_{t,R} \quad \text{(by theorem 1)}\\ &= 2M v_1 \ . \end{aligned} \tag{5.1}$$

Next, choosing to consider a highly localized triplet correlation, one obtains

$$\begin{aligned}\langle \mu_1\mu_2\mu_4 \rangle_{k,Q} &= \langle \mu_1\mu_2\mu_4 \rangle_{d,L} \quad \text{(by theorem 3)}\\ &= M^3 \langle (\sigma_0+\sigma_1)(\sigma_0+\sigma_2)(\sigma_3+\sigma_4) \rangle_{d,L} \quad \text{(by theorem 4)}\\ &= M^3 \langle (\sigma_0+\sigma_1)(\sigma_0+\sigma_2)(\sigma_3+\sigma_4) \rangle_{h,K} \quad \text{(by theorem 5)}\\ &= M^3 [2(\langle \sigma_0 \rangle_{h,K} + \langle \sigma_0\sigma_1\sigma_3 \rangle_{h,K} + \langle \sigma_0\sigma_1\sigma_4 \rangle_{h,K}) + \langle \sigma_1\sigma_2\sigma_3 \rangle_{h,K} + \langle \sigma_1\sigma_2\sigma_4 \rangle_{h,K}]\\ &= M^3 [2(y_1+y_2+y_4)+y_3+y_6] \ , \end{aligned} \tag{5.2}$$

where the definitions (2.24) have been used and, for convenience, the degeneracy relation (2.25) has also been substituted. Using the superposition theorem 2 together with Figs. 3 and 4, one writes

$$y_1 = \langle \sigma_0 \rangle_{h,K} = v_1 \ , \tag{5.3a}$$

$$y_2 = \langle \sigma_0\sigma_1\sigma_2 \rangle_{h,K} = A(2v_1+v_2)+Bv_1 \ , \tag{5.3b}$$

$$y_3 = \langle \sigma_0\sigma_4\sigma_5 \rangle_{h,K} = \langle \sigma_1\sigma_2\sigma_3 \rangle_{h,K} = v_2 \ , \tag{5.3c}$$

$$\begin{aligned} y_4 = y_5 &= \langle \sigma_0\sigma_1\sigma_4 \rangle_{h,K}\\ &= \langle \sigma_0\sigma_3\sigma_6 \rangle_{h,K} = A(v_1+v_2+v_3)+Bv_3 \ , \end{aligned} \tag{5.3d}$$

$$y_6 = \langle \sigma_1\sigma_2\sigma_4 \rangle_{h,K} = \langle \sigma_4\sigma_1\sigma_2 \rangle_{h,K} = A(v_2+v_3+v_8)+Bv_5 \ . \tag{5.3e}$$

For calculational convenience, one notes above that, prior to elimination of all "circled" ○ sites within a thermal average symbol by the superposition theorem 2, it was initially desirable to make minimal the number of such sites by using symmetry arguments, e.g.,

$$\langle \sigma_0\sigma_4\sigma_5 \rangle_{h,K} = \langle \sigma_1\sigma_2\sigma_3 \rangle_{h,K}$$

and

$$\langle \sigma_0\sigma_1\sigma_4 \rangle_{h,K} = \langle \sigma_0\sigma_3\sigma_6 \rangle_{h,K} \ .$$

In developing expressions (5.3), the reader is also aided by reviewing the illustrative example (4.1) of the superposition theorem 2. Substituting (5.3) into (5.2), one has succeeded in writing a three-site kagomé Ising correlation $\langle \mu_1\mu_2\mu_4 \rangle_{k,Q}$ in terms of the known spanning correlations $v_1, \ldots, v_{11}$ of the triangular Ising model. These systematic procedures for finding exact solutions of the spontaneous magnetization $y_{1k} = \langle \mu_0 \rangle_{k,Q}$ and the triplet correlation $y_{5k} = \langle \mu_1\mu_2\mu_4 \rangle_{k,Q}$ as linear superpositions of the known spanning correlations can be similarly repeated to obtain exact solutions for the other (approximately 50) kagomé localized correlations spanned by $v_1, \ldots, v_{11}$.

The resulting exact solution curves for the kagomé spontaneous magnetization $y_{1k} = \langle \mu_0 \rangle_{k,Q}$ and triplet

$$y_{5k} = \langle \mu_1\mu_2\mu_4 \rangle_{k,Q}$$

correlation as well as the quintet correlation

$$y_{6k} = \langle \mu_0\mu_1\mu_2\mu_3\mu_4 \rangle_{k,Q}$$

are displayed in Fig. 8 (also recall that exact solutions for kagomé highly localized correlations such as y_{5k} and y_{6k} are immediately offered by multiplying the ratio solutions (2.9) with y_{1k} from Table I). As anticipated, in the ordered region, exact solution curves for these odd-number correlations are continuous, monotonically decreasing functions of temperature which vanish at the same bulk critical temperature T_c and with the same critical exponent $\frac{1}{8}$ (branch point singularity) as the upper-envelope spontaneous magnetization but with differing critical amplitudes. Using the critical amplitudes $V_1, V_2, \ldots, V_{11}$ of the triangular Ising model recorded in (3.14) together with the previously stated transformation relations

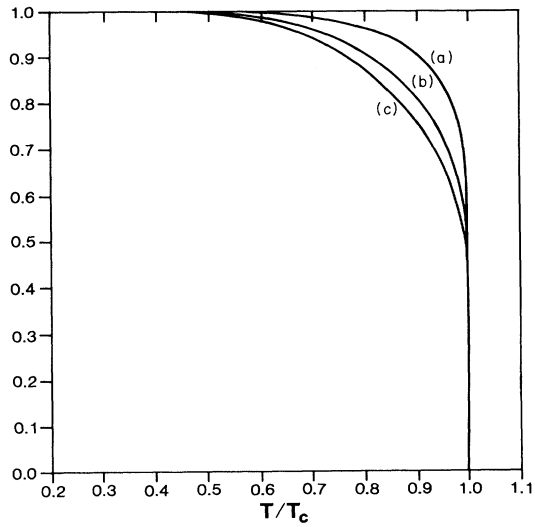

FIG. 8. Exact solution curves for kagomé Ising-model (a) spontaneous magnetization $\langle \mu_0 \rangle_{k,Q}$ and correlations (b) $\langle \mu_1\mu_2\mu_4 \rangle_{k,Q}$, (c) $\langle \mu_0\mu_1\mu_2\mu_3\mu_4 \rangle_{k,Q}$ vs (reduced) temperature $Q_c/Q(=T/T_c)$, where $Q_c=\frac{1}{4}\ln(3+2\sqrt{3})=0.466\,56\ldots$. (Note the differing restricted ranges of the scales.)

among the (dimensionless) interaction parameters Q, L, K, and R, the critical amplitudes of all Ising correlations spanned by $v_1, v_2, \ldots, v_{11}$ can be determined exactly and systematically as illustrated in the Appendix.

VI. SUMMARY AND CONCLUSIONS

After first employing relatively simple procedures for obtaining exact solutions of highly localized odd-number Ising correlations upon the kagomé, square, and honeycomb lattices, the present investigations proceeded to establish a systematic and unifying method for finding many additional exact solutions of Ising-model odd-number correlations on various planar lattices. In the latter theoretical formulation, the triangular Ising model satisfied an enveloping strategy. In particular, knowledge of all 11 odd-number correlations upon a select 7-site cluster of the triangular Ising model was shown sufficient to secure *all* honeycomb, decorated-honeycomb, and kagomé Ising-model odd-number correlations upon their correspondingly select 10-site, 19-site, and 9-site clusters, respectively, and convenient prescriptions were developed for extracting their critical amplitudes. The numbers of such multisite correlations are very considerable, e.g., approximately 80 and 50 for the honeycomb and kagomé Ising models, respectively, and for such n-site (n odd integer) correlations, $n_{\max}=9$, 9, 19 for the honeycomb, kagomé, and decorated-honeycomb Ising models, respectively, where the latter $n_{\max}$ values are significantly greater than existing literature values. With foreknowledge of the spontaneous magnetization and three triplet correlations of the triangular Ising model, the present method of catenated transformation theorems in conjunction with linear algebraic correlation identities is exceedingly simpler than other approaches which, e.g., may be based upon a clustering property (asymptotic separability) of appropriately chosen even-number correlations for determining the localized odd-number correlations.

Finally, one notes that, consolidating the present and earlier[3] planar Ising correlations, one is now able to construct exact solutions for the *joint configurational probabilities* of the Ising spins under consideration since each joint configurational probability can be represented as a linear combination of *both* even- and odd-number multisite Ising correlations with simple rational number coefficients. Also, in a lattice-gas transcription of the Ising-model magnet, any thermal average of a multiproduct of idempotent site-occupation numbers (*lattice-gas correlations*) may be similarly represented as a linear combination of both even- and odd-number Ising spin correlations with simple rational number weights. Such recognitions are fertile in both traditional and topical examples of application for exact solutions of planar Ising localized correlations.

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APPENDIX

The critical amplitudes of the odd-number correlations spanned by $v_1, v_2, \ldots, v_{11}$ can be obtained in a direct and systematic manner as follows. Using the transformation relations stated in theorems 1, 3, and 5, the (dimensionless) interaction parameters Q, K, and R of the kagomé, honeycomb, and triangular Ising models, respectively, are connected by

$$e^{4Q}=2e^{2K}-1\ , \tag{A1}$$

$$2\cosh(2K)=e^{4R}+1\ . \tag{A2}$$

For temperatures *slightly below criticality*, one, therefore, may write

$$Q(K)=Q(K_c)+Q'(K_c)(K-K_c)+\cdots$$
$$=Q_c+\frac{\sqrt{3}}{3}(K-K_c)+\cdots \tag{A3}$$

having used the relation (A1) and the critical value $K_c=\frac{1}{2}\ln(2+\sqrt{3})$ stated in Table I. Expression (A3) may be rearranged to read

$$\frac{Q-Q_c}{Q_c}=\frac{\sqrt{3}}{3}\left[\frac{K_c}{Q_c}\right]\left[\frac{K-K_c}{K_c}\right]+\cdots\ . \tag{A4}$$

Substituting the ratio value

$$K_c/Q_c=2\ln(2+\sqrt{3})/\ln(3+2\sqrt{3})$$

from Table I, the last expression (A4) yields, neglecting second-order small quantities, the *scaling relation*

$$\epsilon_k = \frac{2\sqrt{3}}{3}\frac{\ln(2+\sqrt{3})}{\ln(3+2\sqrt{3})}\epsilon_h \ , \tag{A5}$$

having defined the fractional deviations of the (dimensionless) interaction parameters from criticality as

$$\epsilon_k \equiv (Q - Q_c)/Q_c, \quad \epsilon_h \equiv (K - K_c)/K_c \tag{A6}$$

for the kagomé and honeycomb Ising models, respectively. Similarly, one may derive, using (A2) and the critical values R_c, K_c stated in Table I, the *scaling relation*

$$\epsilon_h = \frac{\sqrt{3}}{2}\frac{\ln 3}{\ln(2+\sqrt{3})}\epsilon_t \ , \tag{A7}$$

having defined

$$\epsilon_t \equiv (R - R_c)/R_c \tag{A8}$$

as the fractional deviation of the (dimensionless) interaction parameter R from its critical value R_c for the triangular Ising model.

To illustrate the simple procedures for extracting critical amplitudes, consider as a first example the relation connecting spontaneous magnetizations

$$\langle \mu_0 \rangle_{k,Q} = 2Mv_1 \tag{A9}$$

derived previously in (5.1). For temperatures *slightly below criticality,* the latter relation (A9) gives, in leading order,

$$A_{1k}\epsilon_k^{1/8} = 2M_c V_1 \epsilon_t^{1/8} \ . \tag{A10}$$

Substituting the scaling relations (A5), (A7), and the critical value M_c from Table I, the expression (A10) then yields the critical amplitude A_{1k} of the kagomé spontaneous magnetization to be

$$A_{1k} = \sqrt{2}(2\sqrt{3}-3)^{1/2}[\ln(3+2\sqrt{3})/\ln 3]^{1/8} V_1 = 1.238\,655\ldots \ , \tag{A11}$$

having used the critical amplitude value (3.14a) for V_1 of the triangular Ising-model spontaneous magnetization.

As another example, consider the honeycomb quintet correlation

$$y_{11} = \langle \sigma_0\sigma_1\sigma_2\sigma_3\sigma_6 \rangle_{h,K} = A(v_2 + 2v_3) + Bv_1 \tag{A12}$$

represented in terms of the spanning correlations v_1, v_2, v_3 by use of theorem 2. For temperatures *slightly below criticality,* the latter expression (A12) gives, in leading order,

$$A_{11h}\epsilon_h^{1/8} = [A_c(V_2 + 2V_3) + B_c V_1]\epsilon_t^{1/8} \ , \tag{A13}$$

whereupon substituting the critical values A_c, B_c from Table I and the scaling relation (A7), expression (A13) yields

$$A_{11h} = [2\sqrt{3}\ln(2+\sqrt{3})/3\ln 3]^{1/8} \times (\sqrt{3}/9)[2(V_2 + 2V_3) - V_1] = 0.903\,470\ldots \ , \tag{A14}$$

having used the critical amplitude values (3.14a)–(3.14c) for V_1, V_2, V_3 of the triangular Ising model. The result (A14) for A_{11h} agrees with the result previously obtained by Barry, Múnera, and Tanaka.[4]

Following similar lines of reasoning, the above procedures for extracting exact critical amplitudes can be readily extended to include the decorated-honeycomb Ising model by deriving and employing an additional scaling relation.

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