Relation between spin-coherent states and boson-coherent states in the theory of magnetism

D. V. Kapor, M. J. Škrinjar, and S. D. Stojanović
Institute of Physics, Faculty of Sciences, 21000 Novi Sad, Yugoslavia
(Received 5 June 1990; revised manuscript received 21 February 1991)

It is shown that the application of both the spin-coherent states and the Holstein-Primakoff representation with boson-coherent states leads to the same classical limit for the spin-operator average values, if the proper procedure for normal ordering is applied. A general expression for the boson expansion is derived, and the first quantum correction is obtained. Consequences of the study of nonlinear excitations in magnetic systems are presented.

I. INTRODUCTION

Nonlinear phenomena in magnetic systems have been a subject of interest for nearly 20 years.[1] Nonlinear equations of motion were first studied in detail for classical systems;[2] so the transition from quantum to classical spins is a subject of continued interest to theoreticians. The usual approach is the application of spin-coherent states[3–6] (SCS) (or generalized coherent states), while the other possibility is the "bosonization" of spin operators. Combined with the application of both of these approaches is the problem of time dependence due to the fact that coherent states do not evolve, in time, into coherent states, in the general case, except for some very particular forms of the Hamiltonian.[7–9] This problem is a very delicate one and it is still the subject of intensive study.

Our aim here is to study another problem, which is quite general and not related to any particular form of the Hamiltonian. The problem concerns the relation of the boson representation to spin-coherent states; more precisely, to its degree of accuracy. In the general case, when studying low-lying excitations, one has well-developed approaches, but it turns out that the results should not be used for the classical limit of boson representation results.

In this paper we review the relevant standard approaches [SCS and Holstein-Primakoff (HP) representations] in Sec. II, then study the classical limit of the bosonization procedure in Sec. III, and finally discuss the consequences of our results on the results of other studies of nonlinear excitations in magnetic systems.

II. HOLSTEIN-PRIMAKOFF REPRESENTATION AND SPIN-COHERENT STATES

The procedure of bosonization in the general case corresponds to the determination of functions of the Bose operators which, acting in the occupation number space, lead to the same results as spin operators acting in the standard basis. More precisely, let B_{n} and $B_{\mathrm{n}}^{\dagger}$ be operators that satisfy the Bose commutation relations. We wish to determine some operator functions[10] $S_{\mathrm{n}}^{\alpha}(B_{\mathrm{n}},B_{\mathrm{n}}^{\dagger})$ $(\alpha=x,y,z)$ which satisfy the angular momentum commutation relations.

We shall suppose that the ground state of the system of N spins corresponds to the state with all spins pointing "up" in the positive direction of the z axis: $M=\langle S_{\mathrm{n}}^{z}\rangle=NS$. In this case, the Holstein-Primakoff[11] representation is defined as

$$S_{\mathrm{n}}^{z}=S-B_{\mathrm{n}}^{\dagger}B_{\mathrm{n}}\ , \tag{1a}$$

$$S_{\mathrm{n}}^{+}=\sqrt{2S}\left[1-\frac{1}{2S}B_{\mathrm{n}}^{\dagger}B_{\mathrm{n}}\right]^{1/2}B_{\mathrm{n}}\ , \tag{1b}$$

$$S_{\mathrm{n}}^{-}=\sqrt{2S}B_{\mathrm{n}}^{\dagger}\left[1-\frac{1}{2S}B_{\mathrm{n}}^{\dagger}B_{\mathrm{n}}\right]^{1/2}\ . \tag{1c}$$

This expression is valid in the system of units $\hbar=1$. For the sake of convenience, we shall use it as long as possible and restore $\hbar$ only when the classical limit is studied.

Problems related to this representation concerning the appearance of nonphysical states ($\langle B^{\dagger}B\rangle>S$) and handling the square root of an operator expression are well known to anyone working with it, especially if one wishes to apply it to the study of higher excited states.

Our interest is in the low-lying states $\langle B^{\dagger}B\rangle<S$, where one can expand the square root to obtain

$$\left[1-\frac{1}{2S}B_{\mathrm{n}}^{\dagger}B_{\mathrm{n}}\right]^{1/2}\approx 1-\frac{1}{4S}B_{\mathrm{n}}^{\dagger}B_{\mathrm{n}}+O(1/S^{2})\ . \tag{2}$$

It is important to stress once more that this equation is meaningful for low-lying states only, and we shall have to reexamine it when the classical limit is considered.

The classical limit means $\hbar\to 0$, $S\to\infty$ with $S\hbar\to S_{\mathrm{cl}}$ (classical angular momentum). Let us first look at the results obtained by using SCS.[3,6] We shall use one of the possible definitions:

$$|a_{\mathrm{n}}\rangle=\frac{1}{(1+|a_{n}|^{2})^{s}}e^{a_{\mathrm{n}}S_{\mathrm{n}}^{-}}|0\rangle,\quad |a\rangle=\prod_{\mathrm{n}}|a_{\mathrm{n}}\rangle\ , \tag{3}$$

where $|0\rangle=|\uparrow\uparrow\cdots\uparrow\rangle$ denotes the above-mentioned ground state of the system:

$$S_{\mathrm{n}}^{z}|0\rangle_{\mathrm{n}}=S|0\rangle_{\mathrm{n}},\quad |0\rangle=\prod_{\mathrm{n}}|0\rangle_{\mathrm{n}}\ . \tag{4}$$

Using this definition, we obtain

$$(S_n^\alpha)_{cl} = \lim_{\hbar\to 0} \langle a_n | \hbar S_n^\alpha | a_n \rangle , \tag{5}$$

giving

$$(S_n^z)_{cl} = S_{cl} \frac{1-|a_n|^2}{1+|a_n|^2} , \tag{6a}$$

$$(S_n^-)_{cl} = 2S_{cl} \frac{a_n^*}{1+|a_n|^2} , \tag{6b}$$

$$(S_n^+)_{cl} = 2S_{cl} \frac{a_n}{1+|a_n|^2} , \tag{6c}$$

and a possible parametrization is

$$a_n = \tan\frac{\theta_n}{2} e^{i\phi_n} . \tag{7}$$

This particular parametrization leads to the representation of $\langle a_n | \hbar S_n^\alpha | a_n \rangle$ as the components of the classical vector $\mathbf{S}_n$ of the length S making polar angles θ_n and ϕ_n.

Now we can construct the Lagrangian density of the system in the form

$$\mathcal{L} = \mathcal{L}_t - \langle a | H | a \rangle . \tag{8}$$

Here H denotes the Hamiltonian of the system and

$$\mathcal{L}_t = \frac{i\hbar}{2} \left\langle a \left| \frac{\overleftrightarrow{\partial}}{\partial t} \right| a \right\rangle = i\hbar S \sum_n \frac{\dot{a}_n a_n^* - a_n \dot{a}_n^*}{1+|a_n|^2} , \tag{9}$$

which gives the classical limit

$$\mathcal{L}_t = i S_{cl} \sum_n \frac{\dot{a}_n a_n^* - \dot{a}_n^* a_n}{1+|a_n|^2} . \tag{10}$$

The phase space in this case is a "nonlinear" one, meaning that the generalized momenta are not linear functions of generalized coordinates. For example,

$$\pi_n^* = \frac{\partial \mathcal{L}}{\partial \dot{a}_n} = i S_{cl} \frac{a_n^*}{1+|a_n|^2} . \tag{11}$$

In order to linearize the phase space, we shall follow Mead and Papanicolaou,[12] and use the sterographic projection

$$\alpha_n = \sqrt{2S} \frac{a_n}{(1+|a_n|^2)^{1/2}} . \tag{12}$$

This leads to the following expression:

$$\mathcal{L}_t(a_n) = \mathcal{L}_t(\alpha_n) = \frac{i\hbar}{2} \sum_n (\dot{\alpha}_n \alpha_n^* - \alpha_n \dot{\alpha}_n^*) . \tag{13}$$

The diagonal matrix elements of the spin operator become

$$\langle a_n | S_n^- | a_n \rangle = \sqrt{2S}\, \alpha_n^* \left[1 - \frac{1}{2S} |\alpha_n|^2 \right]^{1/2} , \tag{14a}$$

$$\langle a_n | S_n^+ | a_n \rangle = \sqrt{2S}\, \alpha_n \left[- \frac{1}{2S} |\alpha_n|^2 \right]^{1/2} , \tag{14b}$$

$$\langle a_n | S_n^z | a_n \rangle = S - |\alpha_n|^2 . \tag{14c}$$

Using the parametrization (7), we obtain

$$\alpha_n = \sqrt{2S} \sin\frac{\theta_n}{2} e^{i\phi_n} \tag{15}$$

and we arrive once again at the classical expressions for the average values of the spinors.

Classical equations of motion can be obtained using $\mathcal{L}$ expressed either in terms of a_n or α_n. We really mean that, although equations for a_n and α_n are different after the parametrization, in both cases the equations for θ_n and ϕ_n are the same.

III. BOSONIZATION AND THE CLASSICAL LIMIT

The other possible approach is the use of the HP representation (3) from the beginning and combining it with boson[13,14] coherent states

$$|\alpha\rangle = \prod_n |\alpha_n\rangle, \quad |\alpha_n\rangle = e^{\alpha_n B_n^\dagger - \alpha_n^* B_n} |0\rangle , \tag{16}$$

where $|0\rangle$ is boson vacuum state. Radcliffe[3] has shown that $\lim_{S\to\infty} |a_n\rangle = |\alpha_n\rangle$ and one has

$$\mathcal{L}_t = \frac{i\hbar}{2} \left\langle \alpha \left| \frac{\overleftrightarrow{\partial}}{\partial t} \right| \alpha \right\rangle = i\hbar \sum_n (\dot{\alpha}_n \alpha_n^* - \alpha_n \dot{\alpha}_n^*) . \tag{17}$$

In order to obtain $\langle \alpha | H | \alpha \rangle$ in terms of matrix elements of spin components, it is necessary to expand the square root in a normal ordered series:

$$\left[1 - \frac{1}{2S} B_n^\dagger B_n \right]^{1/2} = 1 + \sum_{\mu=1}^{\infty} C_\mu(S) B_n^{\dagger\mu} B_n^\mu , \tag{18}$$

where

$$C_\mu(S) \approx \begin{bmatrix} \frac{1}{2} \\ \mu \end{bmatrix} \left[-\frac{1}{2S} \right]^\mu + \begin{bmatrix} \frac{1}{2} \\ \mu+1 \end{bmatrix} \left[-\frac{1}{2S} \right]^{\mu+1} l_\mu^{\mu+1} + O(1/S^{\mu+2}) . \tag{19}$$

Here l_μ^ν is the coefficient of the following expression:

$$(B_n^\dagger B_n)^\nu = \sum_{\mu=1}^{\nu} l_\mu^\nu B_n^{\dagger\mu} B_n^\mu . \tag{20}$$

We are now going to derive the above result, but it is important to notice that the whole derivation is valid only for the Hamiltonian bilinear in spin components [no $(S^z)^2$ terms, for example].

Looking at (20), it is clear that $l_\nu^\nu = 1$; $l_1^\nu = 1$ and one can derive an important recurrent relation

$$l_\mu^{\nu+1} = \mu l_\mu^\nu + l_{\mu-1}^\nu . \tag{21}$$

Using mathematical induction, we obtain

$$l_\mu^\nu = \sum_{p=0}^{p_{max}=\nu-\mu} \mu^p l_{\mu-1}^{\nu-p-1} , \quad \mu \geq 2 \tag{22}$$

and this finally leads to

$$l_\mu^\nu = \sum_{n=0}^{\mu-1} (-1)^n \frac{(\mu-n)^\nu}{n!(\mu-n)!} = \frac{1}{\mu!} \sum_{n=0}^{\mu} (-1)^{\mu-n} \begin{bmatrix} \mu \\ n \end{bmatrix} n^\nu . \tag{23}$$

Now, we can use

$$\begin{aligned} \left[1-\frac{1}{2S}B_n^\dagger B_n\right]^{1/2} &= 1+\sum_{\nu=1}^{\infty} \begin{bmatrix} \frac{1}{2} \\ \nu \end{bmatrix} \left[-\frac{1}{2S}\right]^\nu (B_n^\dagger B_n)^\nu \\ &= 1+\sum_{\nu=1}^{\infty} \begin{bmatrix} \frac{1}{2} \\ \nu \end{bmatrix} \left[-\frac{1}{2S}\right]^\nu \sum_{\mu=1}^{\nu} l_\mu^\nu B_n^{\dagger\mu} B_n^\mu \\ &= 1+\sum_{\mu=1}^{\infty} C_\mu(S) B_n^{\dagger\mu} B_n^\mu \end{aligned} \tag{24}$$

with

$$\begin{aligned} C_\mu(S) &= \sum_{\nu=\mu}^{\infty} l_\mu^\nu \begin{bmatrix} \frac{1}{2} \\ \nu \end{bmatrix} \left[-\frac{1}{2S}\right]^\nu \\ &= \sum_{n=0}^{\mu-1} (-1)^n \frac{1}{n!(\mu-n)!} \sum_{\nu=\mu}^{\infty} (-1)^\nu \begin{bmatrix} \frac{1}{2} \\ \nu \end{bmatrix} \left[\frac{\mu-n}{2S}\right]^\nu \\ &= \sum_{\nu=0}^{\infty} \begin{bmatrix} \frac{1}{2} \\ \mu+\nu \end{bmatrix} \left[-\frac{1}{2S}\right]^{\mu+\nu} \alpha_\nu(\mu) , \end{aligned} \tag{25a}$$

$$\alpha_\nu(\mu) = \sum_{n=0}^{\mu-1} \frac{(-1)^n}{n!(\mu-n)!} (\mu-n)^{\mu+\nu} . \tag{25b}$$

Let us note that some of these results were derived previously (in a more qualitative way) by Goldhirsch,[15] who developed another boson representation, but only studied the asymptotic limit for the first $2S+1$ terms that coincide with the HP representation.

If one uses the asymptotic expansion (19), one only needs to know

$$l_\mu^{\mu+1} = \frac{\mu(\mu+1)}{2} . \tag{26}$$

First of all,

$$\left\langle \alpha_n \left| \frac{B_n^{\dagger\mu} B_n^\mu}{(2S)^\mu} \right| \alpha_n \right\rangle = \frac{|\alpha_n|^{2\mu}}{(2S)^\mu} \tag{27}$$

and introducing

$$\tilde{\alpha}_n = \frac{\alpha_n}{\sqrt{2S}} = \frac{a_n}{\sqrt{1+|a_n|^2}} = \sin\frac{\theta_n}{2} e^{i\phi_n} , \tag{28}$$

which is S independent, we obtain

$$\lim_{S\to\infty} \left\langle \alpha_n \left| \frac{B_n^{\dagger\mu} B_n^\mu}{(2S)^\mu} \right| \alpha_n \right\rangle = |\tilde{\alpha}_n|^{2\mu} , \tag{29}$$

while the next term is

$$\left\langle \alpha_n \left| \frac{B_n^{\dagger\mu} B_n^\mu}{(2S)^{\mu+1}} \right| \alpha_n \right\rangle = \frac{1}{2S} |\tilde{\alpha}_n|^{2\mu} = O(1/S) . \tag{30}$$

Let us use the results to calculate the average value

$$\begin{aligned} \langle \alpha_n | \hbar S_n^- | \alpha_n \rangle &= \sqrt{2S}\, \hbar \left\langle \alpha_n \left| B_n^\dagger \left[1+\sum_{\mu=1}^{\infty} C_\mu(S) B_n^{\dagger\mu} B_n^\mu \right] \right| \alpha_n \right\rangle \\ &\approx \sqrt{2S}\, \hbar \alpha_n^* \left[1-\frac{1}{2S}|\alpha_n|^2\right]^{1/2} + \sqrt{2S}\, \hbar \frac{\alpha_n^*}{2} \sum_{k=2}^{\infty} \begin{bmatrix} \frac{1}{2} \\ k \end{bmatrix} k(k-1)(-1)^k \frac{|\alpha_n|^{2(k-1)}}{(2S)^k} \\ &= 2S\hbar\tilde{\alpha}_n^* (1-|\tilde{\alpha}_n|^2)^{1/2} + \frac{\hbar\tilde{\alpha}_n^*}{2} \sum_{k=2}^{\infty} \begin{bmatrix} \frac{1}{2} \\ k \end{bmatrix} k(k-1)(-1)^k |\tilde{\alpha}_n|^{2(k-1)} . \end{aligned} \tag{31}$$

The classical limit gives

$$\lim_{\substack{\hbar\to\infty \\ S\to\infty}} \langle \alpha_n | \hbar S_n^- | \alpha_n \rangle = (S_n^-)_{\text{cl}} + \frac{\Delta_1}{S} S_n^- + O(1/S^2) , \tag{32}$$

$$\begin{aligned} \Delta_1 S_n^- &= S_{\text{cl}} \frac{\tilde{\alpha}_n^* |\tilde{\alpha}_n|^2}{2} \sum_{k=2}^{\infty} \begin{bmatrix} \frac{1}{2} \\ k \end{bmatrix} k(k-1)(-1)^k |\tilde{\alpha}_n|^{2(k-2)} \\ &= \tfrac{1}{2} S_{\text{cl}} \tilde{\alpha}_n^* |\tilde{\alpha}_n|^2 \tfrac{1}{2}(\tfrac{1}{2}-1)(1-|\tilde{\alpha}_n|^2)^{-3/2} \\ &= -\frac{S_{\text{cl}} \tilde{\alpha}_n^* |\tilde{\alpha}_n|^2}{8(1-|\tilde{\alpha}_n|^2)^{3/2}} = -\frac{S_{\text{cl}} \tilde{\alpha}_n^* |\tilde{\alpha}_n|^2 (1-|\tilde{\alpha}_n|^2)^{1/2}}{8(1-|\tilde{\alpha}|^2)^2} . \end{aligned} \tag{33}$$

Using the parametrization

$$|\tilde{\alpha}_n|^2 = \sin^2\frac{\theta_n}{2} = \frac{1-\cos\theta_n}{2} , \tag{34}$$

we obtain

$$\Delta_1 S_n^- = \frac{S_{\text{cl}}}{8} \sin\theta_n e^{-i\phi_n} \frac{1-\cos\theta_n}{(1+\cos\theta_n)^2} . \tag{35}$$

This is the first quantum correction to the classical value and it is of order $1/S$.

IV. DISCUSSION

Our aim was to show that classical results can be obtained by two ways: either by using SCS, which leads to the nonlinear phase space, or by combining the HP boson representation with Glauber's coherent states, where the phase space is a linear one. In the soliton theory of one-dimensional (1D) magnetic systems,[16] these two representations lead to the same results. On the other hand, the calculations presented here are not limited to one dimension.

Now we must clearly state the advantage of the application of the HP boson representation. It enables one to derive not only classical equations of motion (classical limit) but also the quantum corrections of order $1/S$ and higher, which cannot be obtained with the application of SCS only. (Of course, nothing prevents us from expressing these results in terms of angles defining SCS.)

Finally, we must return to a property of the HP representation which has caused many misunderstandings and produced many misleading results. Our result is an approximate one and it is the way the approximation is performed that really matters.

If one starts from the approximate expression for the operator square root (2), introduces it into (1), and averages the expression over boson-coherent states, one obtains

$$\langle \alpha_n|\hbar S_n^-|\alpha_n\rangle \approx \hbar\sqrt{2S}\,\alpha_n^* - \frac{\hbar\sqrt{2S}}{4S}\alpha_n^*|\alpha_n|^2 = 2S\hbar\tilde{\alpha}_n^* - \hbar S\tilde{\alpha}_n^*|\tilde{\alpha}_n|^2 , \quad (36)$$

whose classical limit is

$$\lim_{\substack{\hbar\to 0\\ S\to\infty}} \langle \alpha_n|S_n^-|\alpha_n\rangle = 2S_{cl}\tilde{\alpha}_n^*[1-\tfrac{1}{2}|\tilde{\alpha}_n|^2+O(|\tilde{\alpha}_n|^4)] . \quad (37)$$

In fact, this result contains no quantum corrections because they follow from the procedure of normal ordering. What we have here is just the expansion of the classical value $(S_n^-)_{cl}$ for $|\tilde{\alpha}_n|^2 \ll 1$. It can contribute only to the linear magnon spectrum. In the soliton theory, it leads to the nonlinear Schrödinger equation[17,18] and not to the classical Landau-Lifshitz equation.

This is an extremely important point for the practical applications, but we shall not insist on listing various results which can be obtained with different approaches which do not respect the procedure proposed here. In subsequent papers we shall establish the relation between our approach and the application of Villain's representation,[19] and we shall also show how to use our approach for the case of a spin chain with a biquadratic exchange interaction.[20]

[1] See, for example, articles by T. Schneider, E. Stoll, A. M. Kosevich, A. V. Mikhailov, M. Steiner, and A. R. Bishop, in *Solitons,* edited by S. E. Trullinger, V. E. Zakharov, and V. L. Pokrovsky (North-Holland, Amsterdam, 1986).

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