Persistent currents in mesoscopic rings, conductance, and boundary conditions

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The persistent current of a mesoscopic ring pierced by a magnetic flux and the conductance of the same sample in an open geometry are two quantities that measure the sensitivity of the spectrum to the boundary conditions. We study the content in harmonics of the variation with the flux of the energy of a *single* level. This content is different from the harmonic content of the *total* energy. We find that there is a well-defined relation between the harmonic content of the persistent current and the correlation function of these currents on an energy range equal to the Thouless correlation energy $E_c=\hbar D/L^2$. These results provide a self-consistency check between the various analytical and numerical results relating the conductance to the single level and total flux dependence of the persistent currents.

I. INTRODUCTION

The recent experimental discovery of persistent currents in an array of mesoscopic disconnected copper rings has increased the interest for a theoretical study of these currents.[1] The existence of a persistent current in a ring pierced by a magnetic flux has been predicted by Büttiker, Imry, and Landauer.[2] This effect, reminiscent of the diamagnetism of aromatic molecules, is an equilibrium property of the ring: the current is given by $I=-\partial E_T/\partial\phi$, where E_T is the total energy and ϕ is the flux inside the ring. It is well known that, using a gauge transformation, the spectrum of the electrons in such a ring is identical to the spectrum of electrons in zero flux, with a change in the boundary conditions. Instead of having a periodic boundary condition, the wave function obeys $\Psi(x+L)=\Psi(x)e^{2i\pi\phi/\phi_0}$, where L is the perimeter of the ring, ϕ the magnetic flux, and ϕ_0 the flux quantum h/e. As a consequence, the persistent current directly measures the sensitivity of the spectrum to the boundary conditions.[23]

Theoretical investigations of the persistent current started three years ago, first in a one-dimensional (1D) ring and then in a multichannel ring.[4–19] The second case is not a trivial generalization of the first one and has a different physical behavior. In particular, it is characterized by a metallic regime which is absent in 1D. The flux dependence of the persistent current is a complicated function which depends on the details of the microscopic realization of the disorder. The probability distribution of the current is characterized by its mean value $\langle I(\varphi)\rangle$ and its root mean square $I_{\mathrm{typ}}(\varphi)$, where $\langle\ \rangle$ is an average over disorder and number of electrons; $\varphi=2\pi\phi/\phi_0$.

It has been shown, first in 1D,[4] and then in multichannel systems,[11–16] that $\langle I(\phi)\rangle$ is finite and periodic in $\phi_0/2$, provided that the number of electrons is kept fixed in each ring. A grand canonical average does not yield this half-flux quantum periodicity and is exponentially small. The experimental results of Levy *et al.* agree with this prediction.[1] More recently, it has also been shown that *interacting* electrons contribute also to a nonzero average, even in the grand canonical ensemble.[15,17,20] At the moment, experiments cannot decide yet between the two mechanisms.

The typical amplitude of the current has been estimated, with a Green-function method, and has been found to vary as $I_t(\varphi=\pi/2)\sim I_0 l_e/L\sim e/\tau_D$.[8] l_e is the elastic mean free path. I_0 is the 1D zero disorder current: $I_0=ev_F/L$ where v_F is the Fermi velocity. τ_D is the diffusion time in the ring. The typical current can also be written as $I_t\sim I_0(M_{\mathrm{eff}}/M)$, where M is the number of channels and the quantity M_{eff} introduced by Imry is the effective number of conducting channels.[21] M_{eff} is identical to the dimensionless conductivity of a strip having the same dimensions and amplitude of disorder than the ring,

$$M_{\mathrm{eff}}=g=\frac{Gh}{2e^2}. \tag{1.1}$$

On the other hand, it has been shown by the same authors that the typical value i_t of the single-level current (also calculated at $\varphi=\pi/2$) scales as $\sqrt{g}$.[8] There is, of course, a great interest in understanding these relations between the persistent current which is an equilibrium property of the ring and the conductance which is a transport quantity. A crucial step in this direction has been made by Thouless when he showed that the conductance is a measure of the sensitivity to the boundary conditions:[22]

$$g=\frac{E_c}{\eta}. \tag{1.2}$$

η is the mean interlevel spacing and the "Thouless energy" E_c is the typical curvature of the levels

$$E_c = \left[\left\langle \left[\frac{\partial^2 e_n}{\partial \varphi^2} \right]^2 \right\rangle \right]^{1/2} \Bigg|_{\varphi \to 0} . \tag{1.3}$$

This quantity also measures the correlation between levels[23] and is the characteristic energy for finite-temperature effects. (Except for the Thouless energy E_c, our convention is to use small letters for single-level quantities and capital letters for total quantities.)

To characterize the sensitivity of the spectrum and calculate the conductance, other quantities have been introduced such as the change in the energy of a level from periodic to antiperiodic boundary conditions $\delta e = |e(0) - e(\pi)|$. The typical value of this quantity has often been used, mainly for computational purpose, as a measure of E_c.

The Thouless argument relates the conductivity to a typical *single-level* quantity.[22] The connection with the analytical results of Ref. 8 where the *total* current is found to scale as g (instead of the *single-level* current which scales as $\sqrt{g}$) is thus not completely straightforward. The aim of this paper is indeed to show that these three relations between the conductance and the persistent current are consistent within one another when taking into account the harmonic contents of the flux dependence of the energy levels and the correlations between the level currents in an energy range equal to E_c

The dependence of $\delta e_t = (\langle \delta e^2 \rangle)^{1/2}$ vs g is still controversial. It has been investigated numerically on a wide range of sizes and values of the disorder. These results, already published, agree with $\delta e_t \propto g^{1/4}/M$.[12] Riedel showed that this behavior can be deduced with a correspondence argument from the weak-disorder regime which is well understood.[18] On the other hand, it has been shown recently that δe_t is related to the fluctuation of the number of levels in an energy range equal to E_c.[15,16] This last quantity is related to the so-called universal conductance fluctuations and depends only *logarithmically* on g. Since there is not yet agreement on the variation $\delta e_t(g)$, we will assume in the following a variation $\delta e_t \propto g^{\beta}/M$ with $0 < \beta < \frac{1}{2}$ ($\delta e_t \propto \ln g/M$ for $\beta = 0$. We show in Appendix A that β necessarily obeys these inequalities) and we will study the consequences of such a behavior on the harmonic content of several quantities. Our main results in the metallic regime are then written in terms of this unique parameter and are the following.

(i) For a *single* level, the typical harmonics of the energy scale as $g^{2\beta}/p^{2\beta+1/2}$ up to a value $p_m \propto \sqrt{g}$, independent of β. This result means that the $\sqrt{g}$ first harmonics of the single-level current scale as $p^{1/2-2\beta}$.

(ii) There is a relationship between the harmonics $\langle I_p \rangle$ of the average *total* current and the typical harmonics of the typical *single-level* current $\langle i_p^2 \rangle$,

$$\langle I_p \rangle = \frac{M}{2\pi I_0} \frac{\langle i_p^2 \rangle}{p} .$$

(iii) As a result, the $\langle I_p \rangle$'s scale as $g^{2\beta}/p^{4\beta}$ up to $p_m \propto \sqrt{g}$.

(iv) At low flux, the average current varies as $\langle I \rangle \propto g\varphi$, independently of β.

(v) We also discuss the correlation functions of the level currents first $\sqrt{g}$ harmonics $\Gamma_p(n) = \langle i_p(n'+n) i_p(n') \rangle_{n'}$. Its Fourier transform presents a sharp low-frequency cutoff below p^2/g. This last result, together with the preceding ones, yields indeed the first $\sqrt{g}$ harmonics of the total current proportional to $g^{\beta+1}/p^{2\beta+3/2}$ and provides a strong argument for $\beta = 0$ in order to be consistent with previous analytical results.[18,24,25]

II. HARMONICS CONTENT OF THE ENERGY LEVELS

In the following, we characterize the flux dependence of the energy levels by their content in harmonics λ_p:

$$e(\phi) = \sum_p \lambda_p \cos(p\varphi) \quad \text{with} \varphi = 2\pi \frac{\phi}{\phi_0} . \tag{2.1}$$

Depending on the way the λ_p vary with p, the flux dependence of the levels as well as the single-level current $i(\varphi) = -\partial e/\partial \phi$ and the curvature $c(\varphi) = \partial^2 e/\partial \varphi^2$ will be dominated by the first harmonics or, on the other hand, will contain contributions of higher harmonics. Our first physical ingredient is that the harmonics for a given level are *uncorrelated.* This is exact in the vanishing disorder limit (when averaging on the number of electrons) and there is no physical reason why disorder would introduce correlations between them. We have also checked this absence of correlations numerically. (The ratio $\langle \lambda_p \lambda_q \rangle^2 / \langle \lambda_p^2 \rangle \langle \lambda_q^2 \rangle$, with $p \neq q$ is found to be of the order of $1/N$ when the averages are taken over N levels and N sufficiently large, typically of the order 1000.) The typical values of $\delta e = |e(0) - e(\pi)|$, $i(\varphi)$ and $c(\varphi)$ are then obtained from the following summations:

$$\langle \delta e^2 \rangle = \sum_{p=1}^{\infty} \langle \lambda_p^2 \rangle [\cos(p\pi) - 1]^2 , \tag{2.2a}$$

$$\langle i^2(\varphi) \rangle = \frac{4\pi^2}{\phi_0^2} \sum_{p=1}^{\infty} p^2 \langle \lambda_p^2 \rangle \sin^2(p\varphi) , \tag{2.2b}$$

$$\langle c^2(\varphi) \rangle = \sum_{p=1}^{\infty} p^4 \langle \lambda_p^2 \rangle \cos^2(p\varphi) . \tag{2.2c}$$

A reasonable ansatz for the p dependence of the typical values of the λ_p is that they decay as a power law: $\langle \lambda_p^2 \rangle = \langle \lambda_1^2 \rangle p^{-2\alpha}$. Moreover, the fact that these three quantities have different dependences versus g implies the existence of a cutoff p_m, a function of g, above which they decay faster (in order to ensure the convergence of the rest of the series). The above quantities are thus functions of this cutoff (except when the series converge) and when $p_m \gg 1$ and $\frac{1}{2} \le \alpha < \frac{3}{2}$, they scale as

$$\delta e_t \propto \lambda_{1,t} \quad (\times \ln p_m \ \text{ if } \ \alpha = \tfrac{1}{2}) , \tag{2.3a}$$

$$i_t(\pi/2) \propto \frac{2\pi}{\phi_0} \lambda_{1,t} p_m^{3/2-\alpha} , \tag{2.3b}$$

$$c_t(0) \propto \lambda_{1,t} p_m^{5/2-\alpha} , \tag{2.3c}$$

where the index t means a typical value. For example, $\lambda_{1,t} = (\langle \lambda_1^2 \rangle)^{1/2}$. Table I and Appendix A present a logi-

TABLE I. Dependence of the three quantities of interest the cutoff p_m, for various values of the exponent α. "Const" stands for a constant which has no dependence on g. See Appendix A.

α		$\frac{1}{2}$		$\frac{3}{2}$		$\frac{5}{2}$	
$\delta e_t/\lambda_{1,t}$	$p_m^{1/2-\alpha}$	$\ln p_m$	const		const		const
$i_t(\pi/2)/\lambda_{1,t}$	$p_m^{3/2-\alpha}$		$p_m^{3/2-\alpha}$	$\ln p_m$	const		const
$c_t(0)/\lambda_{1,t}$	$p_m^{5/2-\alpha}$		$p_m^{5/2-\alpha}$		$p_m^{5/2-\alpha}$	$\ln p_m$	const

cal way to show that α must be contained in these limits, since from Refs. 22 and 8, $c_t(0)=E_c \propto g$ and $i_t(\pi/2) \propto \sqrt{g}/M$ we deduce the cutoff dependence

$$p_m \propto \sqrt{g} \tag{2.4}$$

independent of the exponent α. This result has to be opposed to the one obtained in the ballistic regime, where p_m scales linearly with the number of channels (cf. Appendix B). We believe that this is related to the intrinsic difference between the ballistic and diffusive motions where the characteristic time scales vary respectively, linearly and quadratically with the length scales. The important difference between the typical value of the band current at $\pi/2$, $i_t(\pi/2) \propto \sqrt{g}/M$ and its low flux value: $i_t(\varphi) \propto g\varphi/M$ is thus directly related to the harmonics content of the energy levels. The crossover between this linear behavior at low flux and the saturation $\propto \sqrt{g}$ at larger flux occurs when φ is of the order of the inverse of the cutoff p_m, i.e., $\varphi_{\rm co} \approx 1/\sqrt{g}$. At this flux, the typical excursion of a level is of the order of the interlevel distance.[14] On the other hand, in the ballistic regime, $\varphi_{\rm co}$ is of the order of $1/M$ because of the linear dependence of the levels at low flux.

We can also deduce from these formulas the relation between α and the exponent β of the dependence of δe_t,

$$\alpha = 2\beta + \tfrac{1}{2}. \tag{2.5}$$

Let us now compare these result with our numerical simulations on the Anderson model. The transfer term is taken as a constant t between first neighbors. The field effect is simply to change the boundary condition along the ring so that the transfer term gets a phase factor $\exp(2i\pi\phi/\phi_0)$ after one loop around the ring. Open boundary conditions are taken in the two other directions. The disorder is given by a random choice of the on-site energy between $-W/2$ and $W/2$. We have studied the flux dependence of the energy levels on rings of various sizes and disorder, and measured its content in harmonics. The typical values of the first ten harmonics of the energy levels are shown in Fig. 1 for several values of disorder W/t. We notice a first behavior $\lambda_{p,t} \propto 1/p^{\alpha}$ with $\alpha = 0.75 \pm 0.25$ and then $\lambda_{p,t} \propto 1/p^3$ for larger p.[26] The position p_m of the crossover is proportional to $\sqrt{M}/W$ where W is the amplitude of the disorder. Since, according to elementary scattering theory, l_e varies like $1/W^2$ and $g = Ml_e/L$, this result agrees with $p_m \propto \sqrt{g}$ expected from our previous argument. More precisely, since $\lambda_{p,t} \propto g^{\beta}/Mp^{\alpha}$ and using the relation (2.5), we expect that $M\lambda_{p,t}g^{1/4}$ vs $p/\sqrt{g}$ is a universal curve. This universal behavior is found numerically and shown in Fig. 1. From this curve, the numerical estimate of α cannot allow us to conclude between $\beta=0$ and $\beta=\frac{1}{4}$.

We have also measured directly δe_t versus disorder. This has been done extensively on a wide range of sizes and values of the disorder. Our results are depicted in Fig. 2 where the quantity $M\delta e_t$ is plotted versus $1/g$ [Fig. 2(a)] and ln(1/g) [Fig. 2(b)], for rings of different sizes. The data corresponding to a number of channels $M \leq 100$ can be well fitted by a g^{β} behavior where $\beta = 0.25 \pm 0.05$, as already published.[12] However, one can see that the same data can also be described by a logarithmic dependence of g, but the range of validity of this fit is different from the previous one. More recently, we have performed numerical calculations on rings of size $64\times14\times14$ (this represents 3 h of computing time for each value of disorder, on a Cray-2 computer). When described by a power law g^{β}, these results lead to a very small exponent $\beta \sim 0.1$. On the other hand, a logarithmic variation still describes the data quite well. These results thus seem to indicate that δe_t varies logarithmically with g, in agreement with the recent analytical results of Refs. 16 and 20. However, we do not understand why the behavior $\delta e_t(g)$ is so much size dependent, since the variation $I_t(g)$ of the typical current versus g is already found correctly in small sizes. Moreover, we are still puzzled by the following point: since the harmonics of the single-level current vary as $p^{1/2-2\beta}$ (up to $\sqrt{g}$), the value $\beta=0$

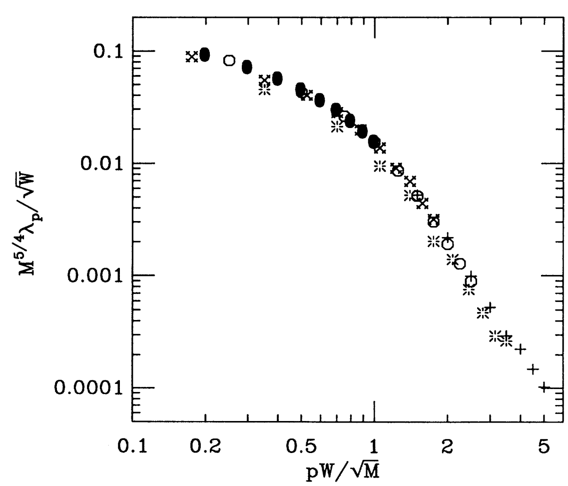

FIG. 1. Typical values of the first ten harmonics of the energy levels for several values of disorder and number of channels in a universal plot. Stars: $64\times4\times4$, $W/t=1.4$ Plusses: $64\times4\times4$, $W/t=2$. Crosses: $64\times8\times8$, $W/t=1.4$. Circles: $64\times8\times8$, $W/t=2$. Black circles: $64\times14\times14$, $W/t=1.4$.

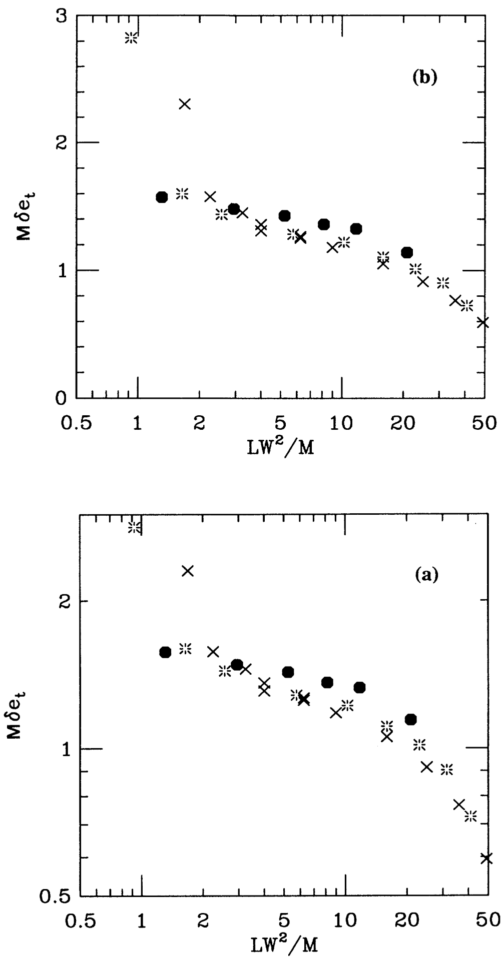

FIG. 2. $m\delta e_t$ vs $LW^2/M \propto g$ for different ring sizes. Crosses: $64\times 8\times 8$, stars: $64\times 10\times 10$; black circles: $64\times 14\times 14$. (a) On a double logarithmic scale. (b) On a semilogarithmic scale.

implies that these harmonics *increase* with their order. We have shown analytically (Appendix B) that the typical harmonics of the single-level current $i_{p,t}$ are *independent* of p in the zero-disorder limit. Such an *increase* in the diffusive regime seems to us very unphysical.

III. AVERAGE CURRENT

The quantities we have discussed so far are the *single-level* currents which *a priori* are not measurable. However, it has been shown numerically that, when the averages are performed at fixed number of particles, there is a relationship

$$\langle I_2 \rangle \propto M \langle i_1^2 \rangle$$

between the second harmonics of the average current and the typical first harmonic of the single-level current.[11–13] By that time, this relation could be justified only in the localized regime.[11] Imry[14] has demonstrated its validity also in the diffusive regime (see also Refs. 15, 16, and 19). The argument connects the flux derivative of the average of the *total* energy to the typical variation of the Fermi level

$$\langle \Delta E(\varphi) \rangle = \tfrac{1}{2} n_0 \langle \Delta e_F^2(\varphi) \rangle , \tag{3.1}$$

where n_0 is the average density of states. This formula can be rewritten in terms of the total and single-level currents. Assuming again no correlation between harmonics, one gets immediately

$$\langle I(\varphi) \rangle = \frac{n_0 \pi}{\phi_0} \sum_{p=1}^{\infty} p \langle \lambda_p^2 \rangle \sin(2p\varphi) \tag{3.2}$$

or, using the expression of the unit current $I_0 = ev_F/L = 2M/n_0\phi_0$,

$$\langle I(\varphi) \rangle = \sum_{p=1}^{\infty} \langle I_p \rangle \sin(2p\varphi) \tag{3.3}$$

with

$$\langle I_p \rangle = \frac{M}{2\pi I_0} \frac{\langle i_p^2 \rangle}{p} , \tag{3.4}$$

where i_p is the p^{th} harmonic of the level current, $i_p = 2\pi p \lambda_p / \phi_0$. Using the above results, one gets

$$\langle I(\varphi) \rangle = \frac{n_0 \pi}{\phi_0} \langle \lambda_1^2 \rangle \sum_{p=1}^{\sqrt{g}} p^{1-2\alpha} \sin(2p\varphi) \tag{3.5}$$

or using Eq. (2.5) and the dependence $\lambda_{1,t}(g)$

$$\frac{\langle I(\varphi) \rangle}{I_0} \propto \frac{\pi}{2M} g^{2\beta} \sum_{p=1}^{\sqrt{g}} p^{-4\beta} \sin(2p\varphi). \tag{3.6}$$

This current has the $\phi_0/2$ periodicity. The coefficient $\beta = \frac{1}{4}$ implies that the harmonics of the average total current decrease as $1/p$. On the other hand, $\beta = 0$ (Refs. 15 and 16) implies that the $\sqrt{g}$ first harmonics have the same order of magnitude. It is interesting to notice that at low flux both results give the same behavior $\langle I(\varphi) \rangle \propto g\varphi$ because the different harmonics content is compensated by a different prefactor. This behavior is valid until $\varphi_{co} \approx 1/\sqrt{g}$, where $\langle I(\varphi) \rangle$ attains its maximum value $\sqrt{g}$.

In numerical simulations,[12] the average current has been estimated from the quantity

$$\langle I \rangle_{\text{calc}} = \frac{\pi}{\phi_0} \langle E_N(\pi) + E_N(0) - 2E_N(\pi/2) \rangle_N . \tag{3.7}$$

$\langle I \rangle_{\text{calc}}$ has the following harmonics content:

$$\langle I \rangle_{\text{calc}} = \sum_{p=1}^{\infty} \frac{\langle I_p \rangle}{p} [1 - \cos(p\pi)]$$

$$\propto \frac{\pi}{2M} g^{2\beta} \sum_{p=1}^{\sqrt{g}} p^{-4\beta-1} [1 - \cos(p\pi)]. \tag{3.8}$$

This calculated quantity provides a good estimation of the second harmonics of average current $\langle I_2 \rangle$ if one assumes that the contribution of high harmonics decays faster than $1/p$ (i.e., $\beta > 0$). However, when $\beta = 0$, all the even harmonics up to $\sqrt{g}$ contribute to $\langle I \rangle_{\text{calc}}$

$$\langle I \rangle_{\mathrm{calc}} \propto \frac{I_0}{M} \sum_{p=1}^{\sqrt{g}} \frac{1}{2p} \propto \frac{I_0}{M} \ln g . \tag{3.9}$$

Our numerical results concerning $\langle I \rangle_{\mathrm{calc}}$ are shown in Fig. 3 for rings of various sizes. On Fig. 3(a), the ratio $r = MW\langle I \rangle_{\mathrm{calc}}/\sqrt{L}$ is plotted versus W. The data, already published,[12] obtained with a number of channels $M \le 100$ are consistent with r=const in the diffusive regime, i.e.,

$$\langle I \rangle_{\mathrm{calc}} \propto \frac{I_t}{\sqrt{M_{\mathrm{eff}}}} . \tag{3.10}$$

However, as for δe_t, our recent numerical results with bigger size show a different behavior. They cannot be described with relation (3.10), because the ratio r is found to increase with W in the whole diffusive regime. On the other hand, Fig. 3(b) shows that a logarithmic behavior of $\langle I \rangle_{\mathrm{calc}}$ vs g: $\langle I \rangle_{\mathrm{calc}} = (I_0/M)(A + B\,\mathrm{Lng})$ well describes all our data. The dimension less coefficient B is found of the order of 0.1 (within plus or minus 10%) in agreement with analytical results which yield $B = 1/4\pi$.[15,16]

The previous discussion of the harmonics content also lead us to reconsider the numerical estimate of the ratio:

$$\frac{\langle I \rangle_{\mathrm{calc}}}{\langle i^2 \rangle_{\mathrm{calc}}} , \tag{3.11}$$

where $\langle i^2 \rangle_{\mathrm{calc}}$ was defined in Ref. 12 as

$$\langle i^2 \rangle_{\mathrm{calc}} = \left[\frac{\pi}{2\phi_0} \right]^2 \langle [e_n(0) - e_n(\pi)]^2 \rangle_n = \left[\frac{\pi}{2\phi_0} \delta e_t \right]^2 . \tag{3.12}$$

This quantity has the following harmonics content:

$$\langle i^2 \rangle_{\mathrm{calc}} = \sum_{p=1}^{\infty} \frac{\langle i_p^2 \rangle}{p^2} [1 - \cos(p\pi)]^2 . \tag{3.13}$$

The ratio (3.11) was considered as an estimation of $\langle I_2 \rangle / \langle i_1^2 \rangle$. This is correct only if $\beta > 0$. However, given the harmonics content of $\langle I \rangle_{\mathrm{calc}}$ and $\langle i^2 \rangle_{\mathrm{calc}}$ [see Eqs. (3.18) and (3.13)], the analytical result (3.4), $\langle I_p \rangle = (M/2\pi I_0)(\langle i_p^2 \rangle / p)$, valid for each p, implies that

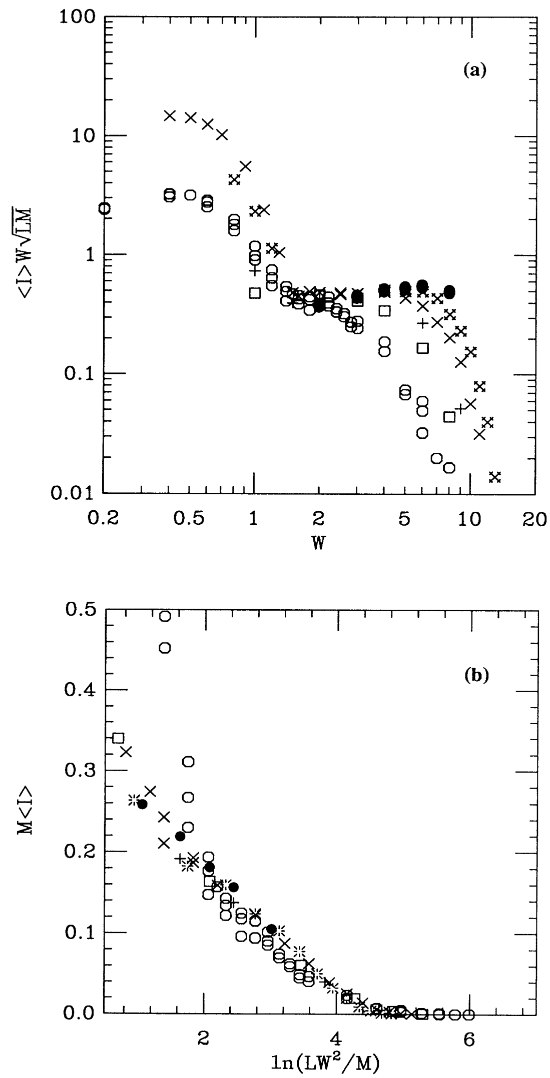

FIG. 3. (a) Plot of the quantity $\langle I \rangle_{\mathrm{calc}} \times \sqrt{MLW}$ vs disorder W (in t units) for various sizes of the ring. Circles: 64×4×4; crosses: 64×8×8; squares: 128×8×8; stars: 64×10×10; plusses: 128×10×10; black circles: 64×14×14. Note that $\langle I \rangle_{\mathrm{calc}} \sqrt{MLW}$ is nearly independent of W only in the diffusive regime whose extension in W corresponds to $l_e < L < Ml_e$. (b) $M\langle I \rangle_{\mathrm{calc}}$ vs lng. Circles: 64×4×4; crosses: 64×8×8; squares: 128×8×8; stars: 64×10×10; plusses: 128×10×10; black circles: 64×14×14.

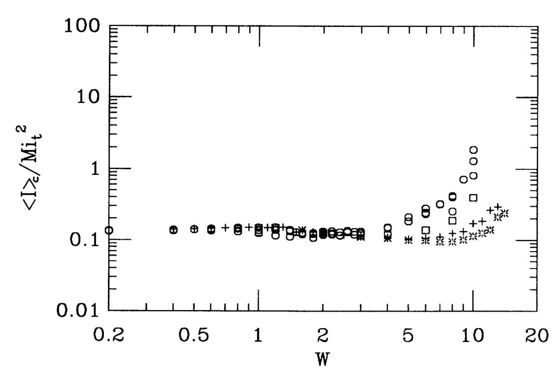

FIG. 4. $M(\langle I \rangle_{\mathrm{calc}} / \langle i^2 \rangle_{\mathrm{calc}})$ vs disorder for different sizes of the ring. Circles: 64×4×4; plusses: 64×8×8; squares: 128×8×8; stars: 64×10×10; black circles; 64×14×14. In the localized regime, the averages have to be done differently since the distribution of currents is not normal.

$$\frac{\langle I\rangle_{\text{calc}}}{\langle i^2\rangle_{\text{calc}}}=\text{const}=\frac{M}{2\pi I_0}, \tag{3.14}$$

which agrees very well with the numerical results (Fig. 4).

IV. CORRELATIONS BETWEEN LEVELS

We are now interested in the correlations between the currents of different levels in the diffusive regime. The first harmonics $\lambda_1(n)$ of successive energy levels are depicted in Fig. 5 as a function of their label n. They are obtained from simulations on a ring of dimensions $128\times10\times10$, for two different values of the disorder. In the first case, $W/t=1$, the electronic motion is nearly ballistic. One can see that $\lambda_1(n)$ presents oscillations whose periodicity is nearly equal to M, the number of channels in the system [Fig. 5(a)]. The function $\lambda_1(n)$ is, however, not completely smooth but presents a certain roughness on a scale shorter than M. The noise spectrum $B_1(f)$ of the quantity $\lambda_1(n)$, i.e., the average square of its Fourier transform, is shown in Fig. 6(a). One can see a peek at a frequency $1/M$, followed by a power-law decay at higher frequency $B_1(f)\propto 1/f$. With increasing disorder, for $W/t=2$, the quantity $\lambda_1(n)$ does not present any clear oscillations any more [Fig. 5(b)]. However, from its spectral analysis, it can be seen that it is not a white uncorrelated noise but it presents strong correlations on the scale of the order of M_{eff}. At high frequency, $B_1(f)$ decays as a power law but there is a sharp cutoff at low frequency below which $B_1(f)$ is very small. It is easy to verify that this cutoff scales as W^2, i.e., as the inverse of the conductance g (Figs. 6 and 7). Since in the diffusive regime, the correlation energy scales as the inverse square of the system length, we expect the correlation of the higher harmonics to be correlated on an energy scale varying like E_c/p^2. These results suggest that it is possible to write

$$\lambda_p(n)=\int_{p^2/g}^{1} v_p(f)\cos(2\pi f n)df \tag{4.1}$$

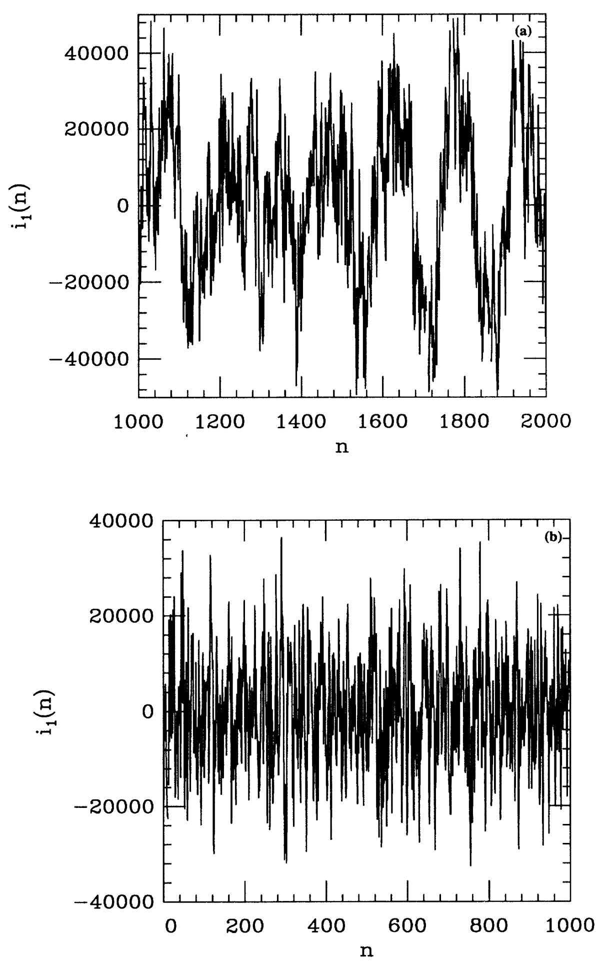

FIG. 5. The first harmonics $\lambda_1(n)$ of successive energy levels as a function of their label n, for a ring of dimensions $128\times10\times10$. (a) $W/t=1$, the system is nearly ballistic; quasi-periodic oscillations are clearly seen. (b) $W/t=2$, the oscillations are buried into the noise.

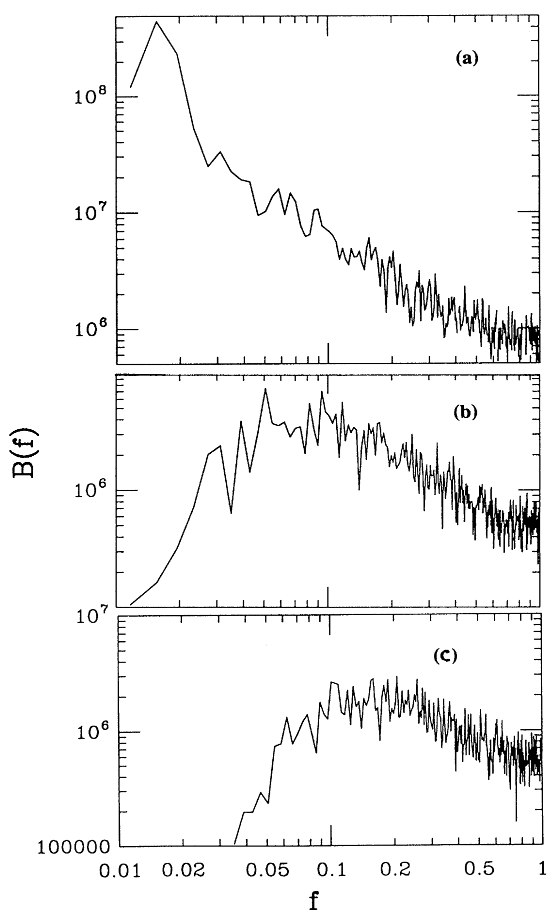

FIG. 6. Noise spectrum $B_1(f)$ of the quantity $\lambda_1(n)$ for several values of the disorder. (a) $W/t=1$; (b) $W/t=1.5$; (c) $W/t=2$; (d) $W/t=3$. The size of the ring is $128\times10\times10$.

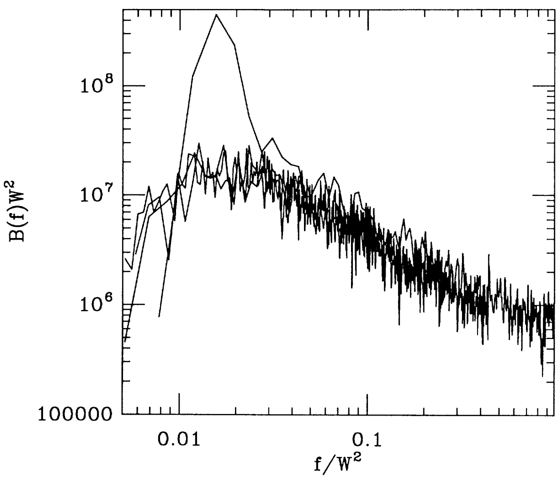

FIG. 7. Same data as in Fig. 6 after renormalization of the frequency and the noise spectrum by, respectively, the inverse of the Thouless energy and the conductance.

with a power-law frequency decay:

$$\langle v_p(f)\rangle^2 = B_p(f) = \left[\frac{p^2}{g}\right]^{\gamma-1} \frac{\langle \lambda_p^2\rangle}{f^\gamma} \propto \frac{g^{2\beta}}{p^{2\alpha} f^\gamma}. \tag{4.2}$$

We show in a forthcoming paper[28] that such a form can also be justified analytically. The exponent γ also describes the decay of the current correlations for $n \le M_{\text{eff}}$,

$$\Gamma_p(n) = \langle i_p(n'+n) i_p(n')\rangle_{n'} \propto p^2 \frac{\lambda_{p,t}^2}{n^{1-\gamma}}$$

$$\propto \frac{g^{2\beta}}{p^{2\alpha-2} n^{1-\gamma}}. \tag{4.3}$$

$\Gamma_1(n)$ is depicted in Fig. 8 at weak disorder. It presents oscillations for $n > M_{\text{eff}}$. These oscillations are not visible anymore at higher disorder.

The harmonics $I_p(N)$ of the total current $I(N,\varphi) = \sum_{i=1}^{N} i(n,\varphi)$ can thus be simply expressed in terms of this correlation function

$$I_p(N) = p \int_{p^2/g}^{1} v_p \frac{(f)}{2\pi f} \sin(2\pi f n)\, df, \tag{4.4}$$

$$\langle I_p(N)^2\rangle = p^2 \int_{p^2/g}^{1} \frac{\langle v_p(f)^2\rangle}{4\pi^2 f^2} df = \frac{\lambda_{p,t}^2 g^2}{p^2}$$

$$\propto \frac{g^{2\beta+2}}{p^{2\alpha+2}}$$

$$\propto \frac{g^{2\beta+2}}{p^{4\beta+3}}. \tag{4.5}$$

Since we know that $\langle I_p(N)^2\rangle$ scales as g^2, this implies $\beta=0$. As an important result we find that the harmonics

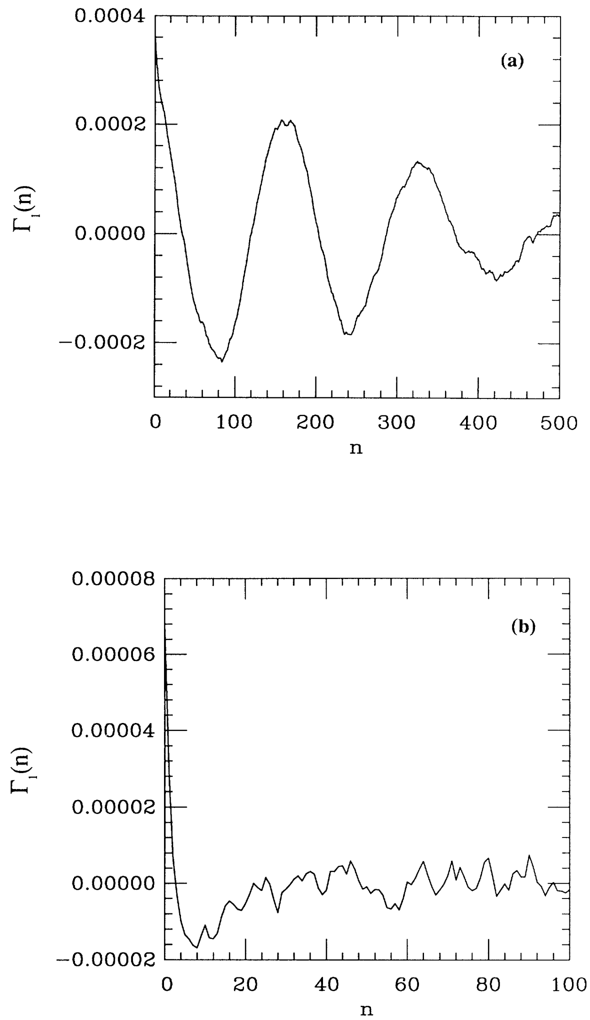

FIG. 8. Correlation function $\Gamma_1(n)$ of the first harmonics of the energy levels in a ring of dimensions $128\times10\times10$. (a) At weak disorder ($W/t=1$), it presents oscillations for $n > M_{\text{eff}}$. (b) These oscillations are not visible anymore at higher disorder ($W=2$).

of the typical total current ($\langle I_p(N)^2\rangle)^{1/2}$ scale as $1/p^{3/2}$. One recovers a well-established result.[18,24,25]

The content in harmonics of the total current is much weaker than for the single-level current. As a result, the *total* quantities $\mathcal{E}_c = (\partial^2 E_n/\partial\varphi^2)_t$, $I_t(\phi=\pi/2)$, and δE_t have the same behavior versus disorder and scale as g ($\mathcal{E}_c$ has actually a $g \ln g$ behavior).

V. CONCLUSION

We have shown that, in a mesoscopic ring, there are precise relations between the sensivity of the energy levels to the boundary conditions, their content in harmonics, and the correlations on an energy range equal to the Thouless energy $E_c = \hbar D/L^2$. These relations can be expressed in terms of a single exponent β which describes the dependence versus the conductance g of the typical energy shift δe_t of the levels between periodic and antiperiodic boundary conditions

$$\delta e_t = \langle [e(0)-e(\pi)]^2\rangle^{1/2} \propto \frac{g^\beta}{M} \quad \left[\propto \frac{\ln g}{M} \text{ if } \beta=0\right].$$

The first $\sqrt{g}$ typical harmonics of the energy levels $\lambda_{p,t}$ decay as

$$\lambda_{p,t} \propto \frac{g^\beta}{p^{2\beta+1/2}}.$$

These relations are well confirmed by numerical simulations. The exponent β is, however, difficult to determine numerically since it appears to depend on the size of the system studied, even in a range where the linear variation of the typical current versus Le is well obeyed. From the previous results performed on sizes $L\times\sqrt{M}\times\sqrt{M}$ with $L=64,128$ and $M<100$, it was found that the data could be well described by an exponent $\beta=0.25$. Computations with larger number of channels $M>100$ are rather consistent with $\beta=0.$, i.e., a logarithmic dependence of δe_t vs g. A possible reason for this difference could be that, for too small M, the motion is not always diffusive in transverse directions.[27] The logarithmic behavior obtained for larger M is in agreement with the analytical calculations which describe a diffusive motion in the three directions.[15,16] The study of the correlation functions of the $\lambda_p(n)$ also confirm the value $\beta=0$.

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APPENDIX A

Since the three quantities shown in Table I have a different dependence on g, this implies that $\alpha<\frac{3}{2}$. In this case, the fact that $c_t(0)/i_t(\pi/2)$ scales as $\sqrt{g}$ immediately implies that $p_m=\sqrt{g}$. If α was smaller or equal to $\frac{1}{2}$, the ratio $i_t(\pi/2)/\delta e_t$ should be proportional to $p_m=\sqrt{g}$. Since $i_t(\pi/2)$ is also proportional to $\sqrt{g}$, this would imply that $\delta e_t=\text{const}$. In such a case, $\lambda_{1,t}$ would scale as $g^{\alpha/2-1/4}$. The condition $\alpha<1/2$ would then imply a decrease of $\lambda_{1,t}$ when g increases, which is physically impossible. As a result, only values of $\frac{1}{2}\le\alpha<\frac{3}{2}$ are compatible with our physical results. From Eq. (2.5), this implies $0\le\beta<\frac{1}{2}$.

APPENDIX B: WEAK DISORDER

In the following, we discuss the content in harmonics of the single-level current in the limit of vanishing disorder where an analytical derivation is possible. The current $i(\phi)$ is the derivative of a function $e(\phi)$ which contains $2M+2$ sharp alternating minima and maxima between $-\phi_0/2$ and $\phi_0/2$. The current can thus be written as

$$i(\varphi) = -\frac{e^2}{mL^2}\varphi + \varepsilon\sum_{i=0}^{M+1}(-1)^i\frac{\Delta_i}{4}[\text{sgn}(\varphi-\varphi_i)+\text{sgn}(\varphi-\varphi_i)], \tag{B1}$$

where $\varepsilon(-1)^i\Delta_i$ is the amplitude of the i^{th} discontinuity located at $\varphi=\pm\varphi_i$. $\Delta_i>0$ and $\varepsilon=\pm1$. Since $i(\pi)=0$, there is a sum rule

$$\frac{e^2}{mL^2}\pi = \varepsilon\sum_{i=0}^{M+1}(-1)^i\frac{\Delta_i}{2}. \tag{B2}$$

The Fourier transform of the current is then

$$i_p = \frac{2\pi p\lambda_p}{\phi_0} = \frac{\varepsilon}{p\phi_0}\sum_{i=0}^{M+1}(-1)^i\Delta_i\cos(p\varphi_i) \tag{B3}$$

or

$$i_p = \frac{\varepsilon}{p\phi_0}\left\{\frac{\Delta_1}{2} + \sum_{i=0}^{M}\frac{(-1)^i}{2}[\Delta_i\cos(p\varphi_i)-\Delta_{i+1}\cos(p\varphi_{i+1})] + \frac{(-1)^{M+1}}{2}\Delta_{M+1}\cos(p\pi)\right\}, \tag{B4}$$

or, by adding and subtracting $\Delta_i\cos(p\varphi_{i+1})$ in the square brackets

$$i_p = \frac{\varepsilon}{p\phi_0}\left\{\frac{\Delta_1+(-1)^{M+1}\Delta_{M+1}\cos(p\pi)}{2} + \sum_{i=0}^{M}(-1)^i\left[\Delta_i\sin\left[\frac{p(\varphi_{i+1}-\varphi_i)}{2}\right]\sin\left[\frac{p(\varphi_{i+1}+\varphi_i)}{2}\right]+(\Delta_i-\Delta_{i+1})\cos(p\varphi_{i+1})\right]\right\}. \tag{B5}$$

The three terms of this sum are, respectively, of order Δ, $M\Delta\sin(p\pi/M)$, and Δ/N, where N is the index of the level, so that the first and third ones are negligible (for $N \gg 1$). Since the average value of $(\varphi_{i+1}-\varphi_i)$ is π/M, the remaining term is written as

$$i_p \approx \frac{\pi\varepsilon}{2M\phi_0}\sum_{i=0}^{M}(-1)^i\Delta_i \sin\left[\frac{p\varphi_i}{2}\right]\frac{\sin(p\pi/M)}{p\pi/M}. \quad \text{(B6)}$$

This very good approximation describes the single-level current as a succession of square functions. If we now approximate the $\sin(x)/x$ function by a constant for $x \le 1$ and assume that the φ_i are sufficiently randomly distributed in order that

$$\left\langle \left[\sum_{i=0}^{M}\Delta_i \sin(p\varphi_i)\right]^2\right\rangle \approx M\langle\Delta_i^2\rangle \approx MI_0^2/2. \quad \text{(B7)}$$

We obtain that the typical values of the M first harmonics have all the same order of magnitude and decay like $1/p$ for $p > M$:

$$p \le M, \quad i_{p,t} \propto \frac{I_0}{\sqrt{M}},$$
$$p \ge M, \quad i_{p,t} \propto \sqrt{M}\frac{I_0}{p}. \quad \text{(B8)}$$

As a result, the typical band current i_t is independent of M and of order I_0.

The same kind of calculation can be done for the total current. In this case, the discontinuities have all positive sign. As a result the main contribution to I_p is

$$I_p \approx \frac{\pi\varepsilon}{2M\phi_0}\sum_{i=0}^{M}\Delta_i \cos\left[\frac{p\varphi_i}{2}\right]\cos\frac{(p\pi/M)}{p\pi/M} \quad \text{(B9)}$$

so that the typical values of the harmonics decay like $1/p$ for all p

$$p \ge M, \quad I_{p,t} \propto \sqrt{M}\frac{I_0}{p} \quad \text{(B10)}$$

This last relation together with Eq. (B8) means that the M first harmonics of the band current are strongly correlated on an energy range containing M levels,

$$p \le M, \quad I_{p,t} \propto M i_{p,t}/p. \quad \text{(B11)}$$

On the other hand, for $p \ge M$ the harmonics are completely uncorrelated

$$I_{p,t} = \sqrt{M}\, i_{p,t}. \quad \text{(B12)}$$

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