Solitary surface transverse waves on an elastic substrate coated with a thin film

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A proof is given of the existence of stable guided solitary surface acoustic waves propagating in the form of envelope solitons on a structure made of a nonlinear substrate and a superimposed linear elastic thermodynamical interface (a very thin film) of mathematically vanishing thickness. A thin gold film on top of a lithium niobate substrate is such a system. The mathematical analysis starting with the theory of material interfaces is carried by using the Whitham-Newell technique of treatment of nonlinear, dispersive, small-amplitude, almost monochromatic waves. In the process, "wave-action" conservation equations and "dispersive" nonlinear dispersion relations are established for this type of surface waves that could also be approached by using Whitham's averaged-Lagrangian technique as modified by Hayes to account for the transverse-modal behavior. It is shown that the whole problem is reduced to studying a single nonlinear Schrödinger equation at the interface, thus providing solutions which are the mechanical analogs of optical solitons known to propagate in nonlinear optical fibers.

I. INTRODUCTION

A recent but already flourishing subject of investigation, the treatment of *waveform evolution for nonlinear, elastic surface acoustic waves* (for short SAW's) of the Rayleigh type has developed through works by various authors (e.g., Kalyanasundaram,[1] Lardner,[2,3] David,[4] Planat,[5] Maugin,[6] and Lardner and Tupholme[7]) with a special interest in anisotropic crystals. These works used variants of the multiple-scale technique in order to study the long-time (or large traveled distance) evolution of SAW signals propagating along a nonlinear elastic substrate while avoiding the appearance of undesirable secular terms in the small-parameter expansion of the solutions. Extensions of these techniques to *nonlinear piezoelectric* SAW's have been presented by Tupholme[8] and Harvey, Craine, and Syngellakis[9] and to *nonlinear magnetoelastic* SAW's propagating along a magnetostrictive substrate by Abd-Alla and Maugin.[10] Kalyanasundaram[11] has also studied the evolution of Bleustein-Gulyaev (piezoelectric) SAW's on an elastically and electrically nonlinear substrate (these waves have the shear horizontal nature; see below). The proceedings edited by Parker and Maugin[12] contain the most relevant contributions to that matter in concise form. The above studies concern the evolution of initially *sinusoidal* signals. Parker[13] and David and Parker,[14] following earlier works by Parker[15] and Parker and Talbot,[16] envisage SAW's of *arbitrary form* propagating on a weakly nonlinear elastic or piezoelectric substrate. All these studies account for elastic or mixed, electroelastic or magnetoelastic, nonlinearities, and they all yield an eventual *distortion* of propagating signals, typically materialized by a steepening and a tendency to wave breaking (but for some exceptional cases in the last quoted studies) since there is no mechanism (such as *dispersion*) present to prevent this effect.

For bulk elastic waves in crystals, we already know that physical dispersion (due to long-range, or nonlocal, interactions) combined to anharmonicity (i.e., nonlinearity) yields the celebrated Boussinesq equation (BE) for anharmonic crystals, and this can be shown to admit solitary-wave solutions either directly or through the application of a reductive perturbation method yielding a Korteweg–de Vries (KdV) equation as a secularity condition (see Maugin, Ref. 6, pp. 89–96 for a nonmathematical approach to this). As is now well known, solitary waves are those localized nonlinear waves of permanent profile which travel without distortion as a result of exact compensation, or balance, between nonlinearity and dispersion. In addition, these strange waves are called solitons if they actually behave like particles during collision, i.e., they come out unaltered from a collision, but for a phase shift. Together with the BE, the KdV equation and the sine-Gordon equation, we note for further use that the nonlinear Schrödinger (NLS)—cubic—equation exhibits such dynamical solutions.[17]

An inevitable question then arises: it is possible to exhibit *surface acoustic solitary waves* propagating along an *elastic* structure? These "elementary" waves would fulfill the dream of many signal-processing engineers since, if they can be generated at all to start with, they could propagate over very long distances (compared to a typical wavelength of elastic waves) without any appreciable distortion and, on the way, they could, for instance, meet an oppositely moving wave without loosing their individuality (while nonlinearity without noticeable dispersion for SAW's is nowadays used for producing convolution and correlation; see Ref. 6 for these aspects). As emphasized before, nonlinearity and dispersion are two prerequisites for the existence of such signals. Another one, of a more technical aspect, is that amplitude and velocity of such signals are strongly correlated and one must enter in the system an initial signal which is exactly of, or at least very close to (then we expect a rapid adaptation), the ideal profile allowed by the theory.

For SAW's, if we do not introduce long-range interactions causing physical dispersion, an easy way to produce geometrical dispersion is, following Love,[18–20] to superimpose a layer of "slower" material, with complete adhesion, on the substrate. Not only does this produce dispersion since a characteristic length (the thickness of the superimposed layer) is involved, but (i) the solution becomes multimode since the superimposed layer acts as a wave guided and (ii) it allows, on an isotropic substrate, for the (otherwise prohibited) propagation of so-called Love waves, i.e., SAW's carrying an elastic displacement in shear which is polarized parallel to the limiting surface, hence is "horizontal," whence the naming of SH (shear horizontal) SAW's. The combination of this superimposed layer (and its associated dispersion) and nonlinearity has, indeed, attracted the attention of several researchers, among them Bataille and Lund,[21] Kalyanasundaram,[22,23] Teymur,[24] and Maradudin.[25] Of these, however, only the last quoted indeed foresaw the propagation of a solitary wave on clearly physically based equations for crystals. The second author obtains coupled-amplitude equations in the manner of nonlinear optics, from which he obtains bounded periodic solutions involving Jacobian elliptic functions. The third author indeed arrives at a possible solitary-wave solution by studying the nonlinear modulation of waves by means of a singular perturbation technique, but for materials which are not crystals. The first authors in fact use an *ad hoc* wave equation [their equation (24) in Ref. 21] which is inspired by the above-mentioned Boussinesq model. Their governing equation really is a modified Boussinesq equation (MBE; with higher-order nonlinearity than the usual BE) which is also obtained in other situations such as for waves propagating down nonlinear elastic rods of finite cross section.[26] We shall, therefore, follow Maradudin. However, we present an alternative to his starting point and this allows us to precisely specify the conditions in which SH-SAW solitary wave will propagate along an anisotropic elastic substrate covered by a very thin layer of soft material. Numerical values are given for such a possible structure showing the reality of the phenomenon.

As a starting point we consider that the superimposed layer is so thin that mathematically it can be viewed as an elastic interface (membrane) of vanishing thickness. The corresponding formalism has recently been developed in several works, particularly noteworthy are those of Bedeaux, Albano, and Mazur[27] and Daher and Maugin,[28] and Murdoch[29] who has more particularly studied the case of such an interface superimposed on a linear elastic substrate. Interestingly enough, such an elastic structure that one may think of as produced by epitaxial growth[30] (for our forthcoming study of solitons, this could use a nonlinear elastic semiconductor substrate such as GaAs on which would grow a soft layer of about 1 μm in thickness) becomes a monomode guide while keeping the requested dispersion, and it still allows for the existence of SH saws which are identical to the lowest Love mode in the linear approximation. This is examined in Secs. II and III.

The nonlinear analysis is performed in three steps in Sec. IV. First an evaluation is made of the nonlinear contributions in terms of harmonic production. This allows one to look for a nonlinear generalization of dispersion relations in a sensible form. Then an asymptotic expansion is performed which provides the missing perturbation terms in the nonlinear dispersion relations. These, in fact, are dispersive nonlinear dispersion relations or, more appropriately, equations determining the wave amplitude, and they are obtained simultaneously with "conservation of wave action" equations (in the manner of Whitham and Hayes) both in the substrate and at the interface. Finally, in a small-amplitude, almost monochromatic limit, it is shown that the whole problem is governed by a single nonlinear Schrödinger (NLS) equation at the interface. A brief study of the stability of the envelope-soliton solutions of this equation in terms of the working point of the carrier wave (frequency, wave number), the nonlinearity parameter, and the linear dispersion characteristics, then allows us to specify the stable solutions ("dark" and "bright" solitons) which propagate depending on the sign of the nonlinearity parameter (effective fourth-order elasticity coefficient) and the ratio of linear shear-wave velocities in the substrate and the superimposed film. This, in turn, provides a criterion for selecting the material of the thin soft film once the nonlinear substrate has been chosen (Sec. V). Appendixes A and B give details of the obtention of basic equations and of the NLS equation through an alternate method, respectively. The paper is self-contained.

II. EQUATIONS OF ELASTIC INTERFACES

A. General equations

We assume that, within a continuous material region D of Euclidean physical space $\mathbb{E}^3$, there exists a moving regular surface Σ which splits D into two subregions D^+ and D^-, the unit oriented normal to Σ pointing toward the D^+ region. There are material fields A attached to, or defined at, all regular points of D^+ and D^-; such fields may suffer a finite jump at Σ, defined by $[A]=A^+-A^-$, where $A^{\mp}$ are uniform limits of A in approaching Σ on its two faces along its normal. In addition, the surface Σ itself is considered as a material body, of vanishingly small thickness, which is endowed with its own mass density and thermodynamical properties. One may think of such a "thermodynamical interface" as an extremely thin layer of which the material differs from that on both sides D^+ and D^-, or a region of D where the material suffers a drastic change in its properties on a very small length scale (e.g., its characteristic parameters may have much steeper gradients than in the remainder of D). Quantities attached and properties pertaining to Σ are distinguished from others by a superimposed caret.

At time t, in the so-called current configuration K_t of continuum mechanics,[31,32] only a *three*-dimensional Cartesian tensor notation is used (even for tensor-valued quantities defined on Σ, which are essentially two-dimensional geometric objects). Thus x_i, $i=1,2,3$, denotes Cartesian coordinates at time t, $\nabla_i A = A_{,i}$ is the three-dimensional gradient. Let n_i be the components of the unit oriented normal to Σ. Then the local projector onto Σ, P_{ij}, is the symmetric idempotent tensor defined by

$$P_{ij} = \delta_{ij} - n_i n_j \ , \tag{2.1}$$

and the tangential gradient on Σ is noted

$$\hat{\nabla}_i \hat{A} = P_{ij} \nabla_j \hat{A}, \quad n_i \hat{\nabla}_i = 0 \ . \tag{2.2}$$

The Einstein summation convention on repeated indices is understood.

It can be shown by various means (one of which is an elegant application of the principle of virtual power[28]—see Ref. 27 for another point of view providing identical equations) that the fundamental laws of motion in continuum mechanics (local balance laws of linear and angular momenta) take on the following component form: at points $\mathbf{x} \in D^+$ and D^-

$$\rho \frac{dv_i}{dt} = \nabla_j t_{ij} + f_i, \quad t_{ij} = t_{ji} \tag{2.3}$$

and at points $\mathbf{x} \in \Sigma$

$$\hat{\rho} \frac{\hat{d}\hat{v}_i}{dt} = \hat{\nabla}_j \hat{t}_{ij} + [\rho(\hat{v}_i - v_i)(v_j - \nu_j) + t_{ij}] n_j + 2\Omega \hat{R}_i + \hat{f}_i \tag{2.4}$$

with

$$\hat{t}_{ij} = \hat{t}_{ji}, \quad \hat{t}_{ij} n_j = 0, \quad \hat{R}_i n_i = 0 \ . \tag{2.5}$$

In the first equation, (2.3), which is standard in continuum mechanics, ρ is the mass density, v_i are the components of the velocity field, t_{ij} those of the symmetric Cauchy stress tensor, and f_i those of a body force per unit volume. d/dt is the usual material time derivative such as

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \ . \tag{2.6}$$

In Eq. (2.4) $\hat{v}_i$ is the velocity of material particles that belong to Σ (these cannot leave Σ so that $\hat{v}_i n_i = 0$), ν_j is the absolute velocity of Σ itself in $\mathbb{E}^3$, $\hat{\rho}$ is the mass density (per unit area) of the matter of which Σ is built, t_{ij} is the essentially two-dimensional surface stress tensor on Σ, $\hat{f}_i$ is a force per unit surface acting on points on Σ (this is a traction in usual mechanics), $\hat{R}_i$ are the components of a so-called double normal force (this occurs in theories of membranes[28]), $\Omega = -(\frac{1}{2})\nabla_i n_i$ is the mean curvature of Σ, and $\hat{d}/dt$ is the material derivative in following the motion defined by $\hat{v}_i$, thus [compare to Eq. (2.6)]

$$\frac{\hat{d}}{dt} = \frac{\partial}{\partial t} + \hat{\mathbf{v}} \cdot \hat{\nabla} \ . \tag{2.7}$$

If $\hat{R}_i = \hat{f}_i = 0$, Σ possesses neither inertia ($\hat{\rho} = 0$) nor stress properties ($\hat{t}_{ij} = 0$), and it reduces to a so-called material surface for which $v_j = \nu_j$ at Σ, then Eq. (2.4) reduces to

$$t_{ij}^{*} n_j = t_{ij}^{-} n_j =: T_i \ , \tag{2.8}$$

which is the usual stress boundary condition at the material surface $\Sigma = \partial D^-$ bounding D^-. That is, the general equation (2.4) is the balance of linear momentum for Σ and the jump of fields, defined in D^+ and D^-, across Σ provides a source term in that equation (of which Laplace's equation for capillarity is none but a special case). Typically, equations such as (2.3) and (2.4) are used for treating liquid-vapor interfaces and, in the case of elastic solids, after inclusion of the relevant electromagnetic terms, for studying the acoustoelectronics of semiconductor junctions.[33,34]

We shall apply Eqs. (2.3) and (2.4) in the rather simple context where Σ is flat ($\Omega = 0$), and it is considered a material surface ($v_j = \nu_j$), thus reducing the set (2.3) and (2.4) to

$$\rho \frac{dv_i}{dt} = \nabla_j t_{ij} + f_i \ , \tag{2.9}$$

$$\hat{\rho} \frac{\hat{d}\hat{v}_i}{dt} = \hat{\nabla}_j \hat{t}_{ij} + [t_{ij}] n_j + \hat{f}_i \ . \tag{2.10}$$

It remains to specify the mechanical behavior. The region D^+ is considered to be a vacuum ($t_{ij}^{+} = 0$) while D^- is made of a nonlinear elastic solid (such as lithium niobate $LiNbO_3$), and Σ is elastic. A linear elastic behavior is sufficient for the latter. Accounting for some nonlinear elastic properties in D^- and having in view nonlinear dynamical effects during which Σ will deform, we should, as in all nonlinear elasticity problems, reformulate Eqs. (2.9) and (2.10) so that the problem is prescribed on a fixed domain. In other words Eqs. (2.9) and (2.10) need to be rewritten in a so-called Lagrangian representation in a reference configuration K_R. This representation is obtained as

$$\rho_0 \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial T_{ij}}{\partial X_j} + J f_i \tag{2.11}$$

at points in D_0^- in K_R, and

$$\hat{\rho}_0 \frac{\partial^2 \hat{u}_i}{\partial t^2} = \left[\frac{\hat{J}}{J} \right] P_{ik} \frac{\partial \hat{T}_{kj}}{\partial \hat{X}_j} + [T_{ij}] N_j + \hat{J} \hat{f}_i \tag{2.12}$$

at Σ_0, the fixed image of Σ back in K_R, with the unit oriented outward normal of components N_j. The objects T_{ij} and $\hat{T}_{ij}$ are so-called (nonsymmetric) Piola-Kirchoff stress tensors defined by

$$T_{ij} = J \frac{\partial X_j}{\partial x_k} t_{ik}, \quad \hat{T}_{ij} = \hat{J} \hat{X}_{jk} \hat{t}_{ik} \ , \tag{2.13}$$

where

$$J = \det \left| \frac{\partial x_i}{\partial X_k} \right| \ , \tag{2.14}$$

$$\hat{J} = J (N_K C_{Kl}^{-1} N_l)^{1/2} \ , \tag{2.15}$$

$$C_{kl}^{-1} = \frac{\partial X_k}{\partial x_i} \frac{\partial X_l}{\partial x_i} \ , \tag{2.16}$$

$$\hat{X}_{jk} = \frac{\partial X_j}{\partial x_i} P_{ik} \ , \tag{2.17}$$

and

$$\rho_0=\rho J,\quad \hat{\rho}_0=\hat{\rho}\hat{J}\ . \tag{2.18}$$

The last two equations are the equations of mass conservation, in integrated form, between the reference configuration K_R (densities ρ_0 and $\hat{\rho}_0$) and the current configuration K_t, in the bulk and on Σ, respectively. J is the Jacobian of the motion for points in the bulk (change in volume) and $\hat{J}$ is the corresponding quantity for points on Σ (change in area); $x_i(X_j,t)$ is the direct motion and $X_j(x_i,t)$ is the inverse motion, if X_j are the so-called Lagrangian coordinates. The abbreviation $\partial/\partial\hat{X}_j$ means

$$\frac{\partial\hat{T}_{ij}}{\partial\hat{X}_j}=\frac{\partial x_k}{\partial X_p}P_{kj}\frac{\partial\hat{T}_{ip}}{\partial x_j}\ . \tag{2.19}$$

Equations (2.11) and (2.12) can be deduced from (2.9) and (2.10) because the following identities hold true (see Daher and Maugin[34] for these transformations)

$$\frac{\partial}{\partial x_j}\left[J^{-1}\frac{\partial x_j}{\partial X_p}\right]=0,\quad \left[\!\left[J^{-1}\frac{\partial x_j}{\partial X_p}n_j\right]\!\right]=0 \tag{2.20}$$

and we have

$$\frac{dv_i}{dt}=\frac{\partial}{\partial t}v_i(X_j,t)=\frac{\partial^2}{\partial t^2}u_i(X_j,t)\ , \tag{2.21}$$

where $u_i=x_i-X_i$ is the displacement, a similar formula holding for dv_i/dt.

For a volume nonlinear elastic behavior we have

$$T_{ij}=\frac{\partial W}{\partial E_{pj}}\frac{\partial x_i}{\partial X_p}=\frac{\partial W}{\partial E_{pj}}\left[\delta_{ip}+\frac{\partial u_i}{\partial X_p}\right]\ , \tag{2.22}$$

while for the surface linear elastic behavior we shall have

$$\hat{T}_{ij}=\frac{\partial\hat{W}}{\partial\hat{E}_{ij}}\ , \tag{2.23}$$

with

$$E_{pj}=\frac{1}{2}\left[\frac{\partial u_p}{\partial X_j}+\frac{\partial u_j}{\partial X_p}+\frac{\partial u_m}{\partial X_j}\frac{\partial u_m}{\partial X_p}\right]\ , \tag{2.24}$$

$$\hat{E}_{ij}\simeq\tfrac{1}{2}P_{ik}P_{jl}\left[\frac{\partial u_k}{\partial X_l}+\frac{\partial u_l}{\partial X_k}\right]\ , \tag{2.25}$$

where E_{pj} are the components of the Lagrange finite-strain tensor and $\hat{E}_{ij}$ are the components of the two-dimensional small-strain tensor on Σ. W and $\hat{W}$ are densities of elastic energy per unit volume and area, in D_0 and on Σ_0, respectively, in the reference configuration. For isotropic behaviors W and $\hat{W}$ depend only on the elementary invariants of E_{pj} and $\hat{E}_{ij}$, respectively. Let $f=O(|\partial u_i/\partial X_j|)$ and $e=O(|E_{pj}|)$. For $\hat{W}$ a quadratic expression [i.e., $O(e^2)$ or $O(f^2)$] is sufficient (tr = trace)

$$\begin{aligned}\hat{W}&=\frac{\hat{\lambda}}{2}\hat{I}_1^2+\hat{\mu}\hat{I}_2\ ,\\ \hat{I}_1&=\mathrm{tr}\hat{\mathbf{E}}\ ,\\ I_2&=\mathrm{tr}\hat{\mathbf{E}}^2\ ,\end{aligned} \tag{2.26}$$

while for W an expansion up to, and including, fourth order in $|E_{pj}|$ is envisaged. Generalizing thus the expression of Murnaghan[35] and following Kalyanasundaram[22] [see also Bland,[36] Eq. (3.59), p. 49] we write thus

$$\begin{aligned}W=\frac{\lambda}{2}I_1^2+\mu I_2+\alpha I_1^3+\bar{\beta}I_1I_2+\gamma I_3+\xi I_1^4+\eta I_1^2I_2\\ +\nu I_1I_3+\delta I_2^2\ ,\end{aligned} \tag{2.27}$$

where

$$I_1=\mathrm{tr}\mathbf{E},\quad I_2=\mathrm{tr}\mathbf{E}^2,\quad I_3=\mathrm{tr}\mathbf{E}^3\ , \tag{2.28}$$

where λ and μ and $\hat{\lambda}$ and $\hat{\mu}$ are elasticity coefficients of the second order (SOEC's), respectively, in the bulk and on Σ; α, $\bar{\beta}$, and γ are elasticity coefficients of the third order (TOEC's) in the bulk, and ξ, η, ν, and δ are elasticity coefficients of the fourth order (FOEC's) in the bulk.

We have at hand all ingredients to study dynamical problems in D_0 and on Σ_0. We shall consider the case where $f_i=\hat{f}_i=0$ and $T_{ij}^+=0$, and Eqs. (2.11) and (2.12) are reduced to

$$\begin{aligned}\rho_0\frac{\partial^2u_i}{\partial t^2}&=\frac{\partial T_{ij}}{\partial X_j}\ ,\\ \hat{\rho}_0\frac{\partial^2u_i}{\partial t^2}&=(\hat{J}/J)P_{ik}\frac{\partial\hat{T}_{kj}}{\partial\hat{X}_j}-T_{ij}^-N_j\ .\end{aligned} \tag{2.29}$$

B. Wave equations

We now specialize the problem to the SH-SAW configuration. D_0^- is the half-space $X_2>0$ while $X_2<0$ is a vacuum. Σ is none other than the plane-limiting surface $X_2=0$ in the reference configuration K_R. Thus $\mathbf{N}=(0,-1,0)$. We consider dynamic solutions such as (Fig. 1)

$$\mathbf{u}=(0,0,u_3(X_1,X_2,t)\quad\text{for } X_2>0\ , \tag{2.30a}$$

$$\hat{\mathbf{u}}=(0,0,\hat{u}_3(X_1,t))\quad\text{for } X_2=0\ . \tag{2.30b}$$

Obviously, we must check the matching condition

$$\hat{u}_3(X_1,t)=u_3(X_1,X_2=0,t)\ . \tag{2.31}$$

We shall consider a propagation along X_1. A dynamical solution such as (2.30) is then said to be transversely horizontally polarized. It is also called a SH (for shear horizontal) wave motion,[19,20] as often contrasted to the Rayleigh wave motion whose elastic displacement is polarized orthogonally to the X_3 direction, hence parallel to the sagittal plane. With the special choice (2.30), the only surviving components of Eqs. (2.29) read ($\hat{J}/J\simeq1$)

$$\begin{aligned}\rho_0\frac{\partial^2u_3}{\partial t^2}&=\frac{\partial}{\partial X_1}T_{31}+\frac{\partial}{\partial X_2}T_{32},\quad X_2>0\\ \hat{\rho}_0\frac{\partial^2\hat{u}_3}{\partial t^2}&=\frac{\partial}{\partial X_1}\hat{T}_{31}+T_{32},\quad X_2=0\ .\end{aligned} \tag{2.32}$$

Accounting for Eqs. (2.22)–(2.28) and for the assumptions (2.30) and (2.31), we show with some algebraic manipulations (see Appendix A) that Eqs. (2.32) provide the fol-

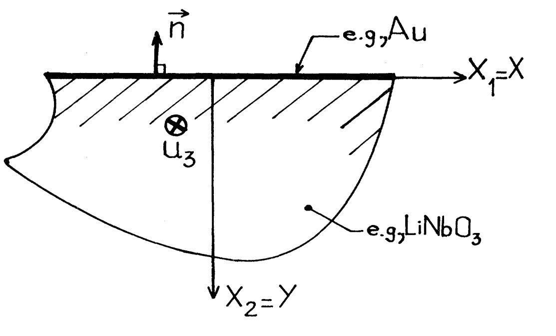

FIG. 1. Surface-wave problem.

lowing two equations [keeping terms which are $O(f^3)$ at most]: for $X_2>0$

$$(\rho_0/\mu)_{3,tt}=\frac{\partial}{\partial X_1}\{u_{3,1}[1+\Delta_{\text{eff}}(u_{3,1}^2+u_{3,2}^2)]\}$$
$$+\frac{\partial}{\partial X_2}\{u_{3,2}[1+\Delta_{\text{eff}}(u_{3,1}^2+u_{3,2}^2)]\}\ , \qquad (2.33)$$

for $X_2=0$

$$(\hat{\rho}_0/\hat{\mu})\hat{u}_{3,tt}=\frac{\partial^2\hat{u}_3}{\partial X_1^2}+(\mu/\hat{\mu})u_{3,2}[1+\Delta_{\text{eff}}(u_{3,1}^2+u_{3,2}^2)]\ , \qquad (2.34)$$

wherein

$$A_{,t}=\frac{\partial A}{\partial t},\quad A_{,\alpha}=\frac{\partial A}{\partial X_\alpha},\quad \alpha=1,2$$

and we have set

$$\Delta_{\text{eff}}=\delta_{\text{eff}}/\mu \qquad (2.35)$$

and

$$\delta_{\text{eff}}=\delta+(\bar{\beta}+\tfrac{3}{2}\gamma)+\left[\frac{\lambda}{2}+\mu\right]. \qquad (2.36)$$

Equations (2.33) through (2.36) are the basic equations in the further developments. They call for the following comments.

(i) There are no nonlinear terms which are $O(f^2)$ in Eqs. (2.33) and (2.34). This results from the peculiar setting (2.30) and the isotropy assumed for the material. Then one understands now why the expansion (2.27) had to be carried up to terms of order four in $e=O(|E_{ij}|)$. The same situation occurs in other problems in nonlinear electroacoustics, for instance when studying the so-called frequency-amplitude effect (also called anisochronism[6,37]) for certain crystal cuts in elastic resonators, or when examining intermodulation through nonlinear thickness-shear vibrations in electroelastic resonators.[6,38] As a matter of fact, these two situations afford one possible means for determining the effective fourth-order elasticity coefficient $\delta_{\text{eff}}=\mu\Delta_{\text{eff}}$ of the order of 10^{10}–10^{12} N/m^2 in materials such as quartz or lithium tantalate.[38] In passing, but this may be quite relevant to the difficulty of placing in evidence the effects to be studied in forthcoming sections, we note the difficulty of determining experimentally FOEC's (one usual technique for this being the shock-compression technique where FOE's are determined through the second variation of the velocity of propagation of shock waves, in an initially stressed material, with respect to the initial pressure).

Still another phenomenon where FOEC's show up is that of dynamical (phonon) echoes in powders of piezoelectric grains.[39] All this to emphasize that, while fourth-order elasticity *a priori* seems to be of too high a degree to be relevant because of the smallness of strains appearing during many experiments $e=O(10^{-4})$ in crystals, it does, in practice, yield many interesting nonlinear effects. The developments below still add another such effect.

(ii) The FOEC defined in Eq. (2.36) is an effective one which includes contributions from SOEC's μ and λ (the classical Lamé coefficients in linear isotropic elasticity) and from TOEC's $\bar{\beta}$ and γ. Notice that all contributions in (2.36) are more or less of the same order, and this, obviously, increases the difficulty in the determination of the "thermodynamic" FOEC δ.

(iii) Equations (2.33) and (2.34) present themselves as two wave equations with highly nonlinear contributions, but, in truth, Eq. (2.34) is none other than a generalized boundary condition associated with the field equation (2.33). The presence of a wave operator in this boundary condition makes the whole system [(2.33) and (2.34)] dispersive and it is the possible compensation between this dispersion and the already noted nonlinearity which favors the existence of interesting phenomena. Such systems have been obtained in linear frameworks by various authors using different methods. For instance, Murdoch[29] obtained such equations by a theory very much like that of thermodynamical interfaces. More interestingly perhaps is the approach of Tiersten,[40,41] following along the line of previous works by Mindlin, to deduce the "boundary condition" (2.34)—in the fully linear case—by applying a limit argument where the three-dimensional region D_0^- is covered with a layer of infinitesimally small thickness in which a true equation of motion is verified; but this is brought back at, or squeezed to, $X_2=0$ by a limit procedure to first order in the thickness, then producing an equation of the type of (2.34) at $X_2=0$ (this is also dealt with by Maradudin[42]).

(iv) A final remark concerns the validity of the system (2.33) and (2.34) in the nonlinear framework for a pure SH mode. It is known that in a nonlinear elastic body, whether anisotropic or isotropic, a pure bulk transverse mode cannot exist without being accompanied by a longitudinal elastic vibration. However, the latter is at least second order with respect to the former (cf. Maugin,[6] pp. 36 and 37). Translated to the framework of SAW's, this means that a pure SH mode, although predominantly excited by some generating device, will necessarily be accompanied by a Rayleigh (sagitally polarized) mode, the later being second order in magnitude according to the above-recalled result. In other words, a "small" sagittal component will be driven nonresonantly via third-order elasticity (second harmonics) in both bulk and surface equations of motion (compare Maradudin and Mayer[43]). In turn, the next order approximation (that is studied in this paper) for SH waves will be slightly altered by these couplings (an alteration to be found at the price of very lengthy algebra). In the present first analytic approach, we content ourselves with the simplified system (2.33) and (2.34), bearing in mind the eventual coupling with the sagittal mode and the resulting complications for further analytic and numerical works [a phenomenological modeling of this coupling directly on Eq. (4.38) will be developed elsewhere]. The main effect, however, remains the one reproduced in this paper as the nonlinear wave obtained remains predominantly shear-horizontally polarized.

C. Nondimensional problem

Let k_a and ω_a be the characteristic wave number and frequency. We set

$$T=\omega_a t,\quad X=k_a X_1,\quad Y=k_a X_2,\quad U=k_a U_3\ ,$$
$$c_T^2=\mu/\rho_0, c_s^2=\hat{\mu}/\hat{\rho}_0,\quad c=c_T/c_s=\beta^{-1}\ , \tag{2.37}$$
$$k_a=(\mu/\hat{\mu})=2\pi/\lambda_a,\quad \omega_a^2=c_s^2 k_a^2,\quad \Delta=\beta^{-2}\Delta_{\text{eff}}\ .$$

That is, c, or β^{-1}, is the ratio of linear shear-wave velocities in the bulk and on the interface. On account of Eqs. (2.37) our problem takes on the following nondimensional form:

$$\Box_B U-\Delta\beta^2 T_B^{\text{NL}}(U)=0,\quad Y>0\ , \tag{2.38}$$
$$\Box_s \hat{U}-U_Y-\Delta\beta^2 T_s^{\text{NL}}(U)=0,\quad Y=0\ , \tag{2.39}$$
$$U(X,Y=0,T)=\hat{U}(X,T),\quad Y=0\ , \tag{2.40}$$
$$U(X,Y\to\infty,T)=0\ , \tag{2.41}$$

where we have set

$$\Box_B\equiv\beta^2\frac{\partial^2}{\partial T^2}-\left[\frac{\partial^2}{\partial X^2}+\frac{\partial^2}{\partial Y^2}\right]\ ,$$
$$\Box_s\equiv\frac{\partial^2}{\partial T^2}-\frac{\partial^2}{\partial X^2}\ , \tag{2.42}$$

$$T_B^{\text{NL}}(U)=(U_X\Phi)_X+(U_Y\Phi)_Y=0(U^3)\ ,$$
$$T_S^{\text{NL}}(U)=U_Y\Phi=0(U^3)\ , \tag{2.43}$$
$$\Phi=U_X^2+U_Y^2\geq 0\ ,$$

where subscripts T, X, and Y denote partial derivatives; Φ in fact, is an invariant of strains for isotropic nonlinear elasticity for SH motion.

Equations (2.38) through (2.41) are the relevant equations for our surface-wave problems. They deserve the following comments.

(i) Equation (2.40) is the condition of perfect bonding of the thin film and the substrate while (2.41) is the radiation condition typical of surface wave motion.

(ii) There remain two parameters in the nondimensionalization. One is the nonlinearity parameter Δ, which is related to the FOEC δ_{eff}. As only third-order terms in the gradient of U are involved in T_B^{NL} and T_S^{NL}, Eqs. (2.38) and (2.39) are third-harmonic generators insofar as nonlinearities are concerned. However, the second parameter β^2 accounts for dispersion as it is, in fact, the ratio of two lengths [or the ratio of the two (squared) characteristic velocities of the system]. Indeed,

$$\beta^2=\frac{1}{c^2}=\frac{\hat{\mu}}{\mu}\frac{\rho_0}{\hat{\rho}_0}=\frac{1}{2\pi}\left[\frac{\rho_0}{\rho_L}\right]\left[\frac{\lambda_a}{H}\right] \tag{2.44}$$

if we remember that the thermodynamical interface $\Sigma_0=\{Y=X_2=0\}$ of surface mass density $\hat{\rho}_0$ and surface rigidity $\hat{\mu}$ in fact represents a thin three-dimensional layer of thickness H and mass density ρ_L. Then the system (2.38)–(2.41) contains, potentially, the possibility to allow for solitary-wave propagation.

(iii) If the thin film carries neither inertia nor surface tension (or there is no superimposed thin film), then (2.38)–(2.41) reduces to

$$\left[\frac{\partial^2}{\partial T^2}-\left[\frac{\partial^2}{\partial X^2}+\frac{\partial^2}{\partial Y^2}\right]\right]U$$
$$-\Delta[(U_X\Phi)_X+(U_Y\Phi)_Y]=0,\quad Y>0 \tag{2.45a}$$
$$U_Y[1+\Delta\Phi]=0,\quad Y=0\ , \tag{2.45b}$$
$$U(X,Y\to\infty,T)=0\ , \tag{2.45c}$$

where β has been set equal to 1 without loss in generality. The system (2.45) was considered by Mozhaev.[44] As this system does not present any dispersion [Eq. (2.45b) is a true boundary condition for (2.45a)], it *cannot* allow for the propagation of true solitons. However, because (2.45b) can be satisfied in two ways, $U_Y=0$ (a condition identical to that prevailing in linear isotropic elasticity, for which we know that SH surface waves are *not* possible) or $\Phi(Y=0)=-\Delta^{-1}$, Δ necessarily negative (as $\Phi\geq 0$), it may present strange nonlinear SAW solutions (SH waves existing because of the nonlinearity). We refer the reader to Mozhaev[44] and Kosevich[45] for these.

(iv) The system (2.38)–(2.41) is derivable from a Lagrangian. We let the reader check this, the Lagrangian density per unit X length being given, in nondimensional variables, by

$$\mathcal{L}=\int_0^\infty[\tfrac{1}{2}(\beta^2U_T^2-\Phi-\tfrac{1}{2}\Delta\beta^2\Phi^2)]dY+\tfrac{1}{2}(\hat{U}_T^2-\hat{U}_X^2)\ , \tag{2.46}$$

where kinetic and potential energies are easily identifiable.

III. LINEAR PROBLEM

A. Murdoch waves

Before attacking the full problem (2.38)–(2.41) and for further use, it is advisable to consider the linear problem. In the absence of nonlinearity in the substrate ($\Delta=0$) the system (2.38)–(2.41) reduces to

$$\begin{aligned} &\Box_B U=0, \quad Y>0 , \\ &\Box_S \hat{U}-U_Y=0, \quad Y=0 , \\ &U(X,Y=0,T)=\hat{U}(X,T), \quad Y=0 \\ &U(X,Y\to\infty,T)=0 , \end{aligned} \tag{3.1}$$

which is none other than the system considered by Murdoch.[29] Trying harmonic SAW solutions of the type

$$U(X,Y,T)=A_0\exp(-\chi Y)\exp[i(kX-\omega T)], \quad \chi>0 \tag{3.2}$$

for $A_0 \neq 0$ one is led to the following linear "bulk" and "surface" conditions

$$\begin{aligned} &\mathcal{D}_B(k,\omega;\chi)\equiv\beta^2\omega^2-k^2+\chi^2=0 , \\ &\mathcal{D}_S(k,\omega;\chi)\equiv\omega^2-k^2-\chi=0 , \end{aligned} \tag{3.3}$$

which, upon elimination of the penetration parameter χ, yield the "linear" dispersion relation

$$\mathcal{D}_L(\omega,k)=\mathcal{D}_M(\omega,k)\equiv\beta^2\omega^2-k^2+(\omega^2-k^2)^2=0 . \tag{3.4}$$

An alternate form of this is Murdoch's one[29]—also Kosinski,[46] pp. 198–200,

$$\mathcal{D}_M(s,K)\equiv K^2-\frac{1-s^2}{(s^2-s_0^2)^2}=0 , \tag{3.5}$$

where

$$s=v/c_T, \quad s_0=\beta, \quad v=\omega/k, \quad K=lk . \tag{3.6}$$

Here v is the phase velocity and $l=\hat{\rho}_0/\rho_0$ is a characteristic length [compare to Eq. (2.44)]. The linear harmonic SH SAW solution (3.2) with dispersion relation (3.4) or (3.5), which we call a "Murdoch SAW," exists only for $s_0<s<1$ or $c_s<v<c_T$, which means that the superimposed thin film is "slow" as compared to the substrate (a condition also prevailing for Love waves). Typical dispersion relations $s(K)$ or $\omega(k)$ are plotted in Figs. 2 and 3.

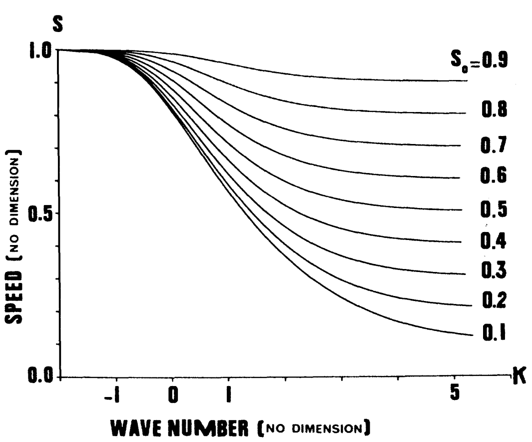

FIG. 2. Linear dispersion relation $s(K)$ for various s_0.

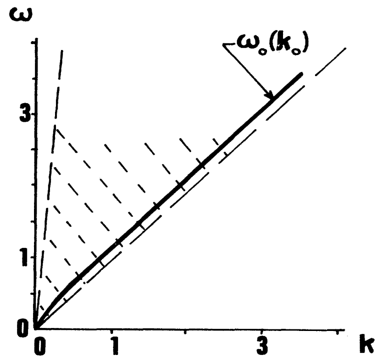

FIG. 3. Linear dispersion relation $\omega(k)$ for a fixed s_0.

B. Comparison with Love waves

The representations in Fig. 2 are reminiscent of the lowest mode of Love waves (cf. Ref. 18, p. 210). Indeed, the (multimode) dispersion relation of Love waves, using the same notation as in (3.5), reads

$$\tan\left[(s^2-s_0^2)^{1/2}\frac{kH}{s_0}\right]-\frac{p}{s_0}\frac{(1-s^2)^{1/2}}{(s^2-s_0^2)^{1/2}}=0 ,$$

$$p\equiv\rho_0/\rho_L . \tag{3.7}$$

To obtain from this the case of an elastic interface of vanishing thickness we should notice that ρ_L goes to $\hat{\rho}_0/H=(l/H)\rho_0$ so that p goes to H/l, and H is of order ε. Expanding then the tangent function in (3.7), we readily obtain the expression (3.5). Thus we can say that our approach using the zero-thickness elastic interface has *isolated* the lowest mode of Love waves as the slow-layer thickness goes to zero while keeping a mass and an elasticity.

The systems (2.45) and (3.1) represent two extreme limit cases—respectively, no dispersion and no nonlinearities—of the general system (2.38)–(2.41) that we shall now consider.

IV. NONLINEAR PROBLEM

A. Evaluation of nonlinear contributions

In the nonlinear regime, there exists a strong relationship between velocity of propagation, amplitude of the signal, frequency, and wave number, and not only between the latter two quantities. To clarify the matter for a system as complicated as (2.38)–(2.41) we proceed in three steps by using elements of the kinematic wave theory (e.g., Whitham[47]), asymptotic expansions and the small-amplitude, almost monochromatic limit (see, e.g., Benney and Newell[48] and Newell[49]). In a first step, in order to generalize Eqs. (3.3) to the nonlinear case we first try to evaluate the alteration brought in by the nonlinear contributions to Eqs. (3.3), hence a dependence of the dispersion relation on the amplitude. To that effect we consider for small amplitudes a solution of the form

$$U=f(\theta,Y)=A\exp(-\chi Y)\cos\theta + C\Delta\beta^2 A^3\exp(-3\chi Y)\cos3\theta+\cdots\ , \qquad (4.1)$$

$$\theta=kX-\omega T\ ,$$

where odd-order harmonics only contribute because of the very form of Eqs. (2.38) and (2.39).. Substituting from (4.1) in $T_B^{\rm NL}$ and $T_S^{\rm NL}$, the first term in (4.1) will yield contributions such as

$$T_B^{\rm NL}=\tfrac{1}{4}[(9\chi^3-3k^4+2\chi^2k^2)\cos\theta +(3k^4+3\chi^4-6k^2\chi^2)\cos(3\theta)] \times A^3\exp(-3\chi Y)+\cdots \qquad (4.2)$$

and

$$T_S^{\rm NL}=-\tfrac{1}{4}[\chi(k^2+3\chi^2)\cos\theta +\chi(\chi^2-k^2)\cos(3\theta)]A^3+\cdots\ . \qquad (4.3)$$

The contribution in $\cos\theta$ in these two expressions will resonate with the linear parts of Eqs. (2.38) and (2.39) so that these equations provide

$$(-\beta^2\omega^2+k^2-\chi^2)Ae^{-\chi Y}\cos\theta =\Delta\beta^2\left[(9\chi^4-3k^4+2k^2\chi^2)\frac{\cos\theta}{4}A^3e^{-3\chi Y}\right]+B\ , \qquad (4.4)$$

$$(-\omega^2+k^2+\chi)A\cos\theta=-\tfrac{1}{4}\Delta\beta^2\chi(k^2+3\chi^2)A^3\cos\theta+S\ , \qquad (4.5)$$

where B and S gather higher-harmonic contributions in $\cos(3\theta)$, $\cos(9\theta)$, etc.

On comparing (4.4) and (4.5) with (3.3) and accounting for this last remark, we can look for nonlinear generalizations of (3.3) in the following perturbed form:

$$\mathcal{D}_B^{\rm NL}(k,\omega,\chi,A)=\varepsilon l^{(1)}+\varepsilon^2 l^{(2)}+\cdots\ , \qquad (4.6a)$$

$$\mathcal{D}_S^{\rm NL}(k,\omega,\chi,A)=\varepsilon m^{(1)}+\varepsilon^2 m^{(2)}+\cdots\ , \qquad (4.6b)$$

where, accounting for (4.4) and (4.5), we have set

$$\mathcal{D}_B^{\rm NL}(k,\omega,\chi,A)\equiv\mathcal{D}_B(k,\omega;\chi) +\frac{\Delta\beta^2}{4}(9\chi^4-3k^4+2k^2\chi^2)A^2e^{-2\chi Y}\ , \qquad (4.7a)$$

$$\mathcal{D}_S^{\rm NL}(k,\omega,\chi,A)\equiv\mathcal{D}_S(k,\omega;\chi)-\frac{\Delta\beta^2}{4}\chi(k^2+3\chi^2)A^2\ , \qquad (4.7b)$$

in which $\mathcal{D}_B$ and $\mathcal{D}_S$ are none other than the expressions defined in (3.3). In (4.6) the perturbations have to be found through an asymptotic expansion.

B. Asymptotic expansion

Now we assume that Eqs. (4.6) hold well and consider solutions of (2.38) and (2.39) in the form

$$U(X,Y,T)=f(\theta,A)+\varepsilon U_1+\varepsilon^2 U_2+\cdots \qquad (4.8)$$

with the scaling

$$x=\varepsilon X,\quad y=\varepsilon Y,\quad t=\varepsilon T\ , \qquad (4.9)$$

where $\varepsilon=0(A)$ and θ is a generalized phase such that

$$\theta_X=k,\quad \theta_T=-\omega,\quad \theta_Y=i\chi\ , \qquad (4.10)$$

the imaginary "i" in the last expression visualizing a nonpropagative behavior along the depth coordinate Y. Discarding intermediate calculations, from Eq. (2.38) we obtain

$$\left[\mathcal{D}_B(\omega,k;\chi)\frac{\partial^2}{\partial\theta^2}-\varepsilon\left[2\omega\beta^2\frac{\partial^2}{\partial t\partial\theta}+\omega_t\beta^2\frac{\partial}{\partial\theta}+2k\frac{\partial^2}{\partial x\partial\theta}+k_x\frac{\partial}{\partial\theta}+i\chi_y\frac{\partial}{\partial\theta}+2i\chi\frac{\partial^2}{\partial\theta\partial y}\right]+\varepsilon^2\Box_B -\frac{\Delta\beta^2}{4}(9\chi^4-3k^4+2k^2\chi^2)A^2e^{-2\chi Y}-\beta^2\Delta\varepsilon B_1-\beta^2\Delta\varepsilon^2B_2\right](u_0+\varepsilon u_1+\cdots+\varepsilon^n u_n+\cdots)=0\ , \qquad (4.11)$$

where there is no need to give explicitly B_1 and B_2, and $\Box_B$ is computed with the variables (x,y,t). Accounting now for (4.6a) and (4.7b) we can rewrite (4.12) as

$$\left[-\tfrac{1}{4}\Delta\beta^2(9\chi^4-3k^4+2k^2\chi^2)A^2\exp(-2\chi Y)\left[1+\frac{\partial^2}{\partial\theta^2}\right] -\varepsilon\left[2\omega\beta^2\frac{\partial^2}{\partial t\partial\theta}+\omega_t\beta^2\frac{\partial}{\partial\theta}+k_x\frac{\partial}{\partial\theta}+i\chi_y\frac{\partial}{\partial\theta}+2k\frac{\partial^2}{\partial x\partial\theta}+2i\chi\frac{\partial^2}{\partial y\partial\theta}\right] +\varepsilon^2\Box_B+(\varepsilon l^{(1)}+\varepsilon^2 l^{(2)}+\cdots)\frac{\partial^2}{\partial\theta^2}-\beta^2\Delta\varepsilon B_1-\beta^2\Delta\varepsilon^2B_2\right](u_0+\varepsilon u_1+\varepsilon^2u_2+\cdots)=0\ . \qquad (4.12)$$

As $u_0=O(\varepsilon)$, at order one, this yields the "linear" solution (A can still be a function of X, Y, and T)

$$u_0 = A\exp(i\theta), \quad \theta = kX - \omega T + i\chi Y, \tag{4.13}$$

while at the next order (4.12) gives

$$-\tfrac{1}{4}\Delta\beta^2(9\chi^4 - 3k^4 + 2k^2\chi^2)A^2\exp(-2\chi Y)\left[1 + \frac{\partial^2}{\partial\theta^2}\right]u_1 - \beta^2\Delta B_1 u_0 = \frac{i}{A}\exp(i\theta)((\beta^2\omega A^2)_t + (kA^2)_x + (i\chi A^2)_y) + l^{(1)}A\exp(i\theta), \tag{4.14}$$

from which we deduce that $u_1 = 0$ and $l^{(1)} = 0$ and there holds the following "bulk" conservation law of wave action

$$\mathcal{CA}_B \equiv \frac{\partial}{\partial t}(\beta^2\omega A^2) + \vec{\nabla}\cdot\vec{K}_B = 0 \tag{4.15}$$

where $\vec{K}_B$ is a two-dimensional vector of components

$$\vec{K}_B = (kA^2, i\chi A^2). \tag{4.16}$$

This seems to be the first case where the notion of conservation of wave action is treated in two spatial dimensions where one, x, is of the propagative type and the other, the transverse coordinate y, is of the vanishing (in some sense modal) type although Hayes[50] envisaged such a generalization (but this is not all as there is also another such conservation law at the interface, see below).

At the next order ε [ε^2 in (4.11) is really ε^3 as $u_0 = O(\varepsilon)$], (4.11) gives

$$-\tfrac{1}{4}\Delta\beta^2(9\chi^4 - 3k^4 + 2k^2\chi^2)A^2\exp(-2\chi Y) \times \left[1 + \frac{\partial^2}{\partial\theta^2}\right]u_2 - \Delta\beta^2 B_2 u_0 = -\Box_B(A)\exp(i\theta)l^{(2)}A. \tag{4.17}$$

As u_2 solutions of the form $\exp(i\theta)$ make the first term in the left-hand side of Eq. (4.17) vanish, the right-hand side is selected in order to avoid the appearance of secular terms. This is the spirit of multiple-scale techniques. As a matter of fact, it is sufficient to make it zero, yielding thus

$$l^{(2)} = A^{-1}\Box_B A. \tag{4.18}$$

At the third order we have thus found (4.6a) in the form

$$\mathcal{D}_B^{\rm NL}(k,\omega,\chi,A) - \frac{\varepsilon^2}{A}\Box_B A = 0, \quad Y > 0. \tag{4.19}$$

This, in effect, may be referred to as the "bulk" nonlinear dispersion relation which results from a double expansion in which ε and A are of the same order. We can also say that the dispersion relation not only depends on the amplitude, but it has itself become dispersive as it involves a wave operator. As the notion of true harmonic motion is lost, we should rather refer to (4.19) as a wave-amplitude equation.

Now we should proceed in a similar manner for Eq. (4.6b). We simply state the results. $\mathcal{D}_S = 0$ holds at the first order; at the second order $m^{(1)} = 0$ and there holds the *surface* conservation of wave action:

$$\mathcal{CA}_S \equiv \frac{\partial}{\partial t}(\omega A^2) + \vec{\nabla}\cdot\vec{K}_S = 0, \quad Y = 0 \tag{4.20}$$

$$\vec{K}_S = \left[kA^2, -\frac{i}{2}A^2\right]. \tag{4.21}$$

At the next order we can select $m^{(2)}$ in such a way that the following "surface" nonlinear dispersion relation holds

$$\mathcal{D}_S^{\rm NL}(\omega,k,\chi,A) - \frac{\varepsilon^2}{A}\Box_S A = 0, \quad Y = 0. \tag{4.22}$$

To the above results should be added the relations of kinematic wave theory which follow from the fact that θ is a potential for the triplet $(k,\omega,i\chi)$:

$$\frac{\partial k}{\partial t} + \frac{\partial\omega}{\partial x} = 0, \quad \frac{\partial\chi}{\partial t} - i\frac{\partial\omega}{\partial y} = 0, \tag{4.23}$$

where the first is the conservation of wave number and the second gives the time evolution (if any) of the penetration parameter χ once the solution $\omega(X,Y,T)$ is known.

In all, Eqs. (4.15), (4.20), (4.19), (4.22), and (4.23) provide a system for evaluating A, ω, and k once χ has been eliminated. This system clearly is unexploitable in this state of generality, but we shall take profit of it in the so-called quasimonochromatic, small-amplitude approximation.

Note. As the system (2.38)–(2.41) is derivable from a Lagrangian variational principle via the Lagrangian density (2.46), it is theoretically possible to use Whitham's method of the averaged Lagrangian (see Refs. 51 and 52 and p. 223 of Ref. 47)—applied to (2.46) in the form generalized by Hayes[50] for several spatial dimensions, and deduce Euler-Lagrange equations for the amplitude and the phase in the form of dispersive nonlinear dispersion relations and the conservation of wave action. This, however, requires some ingenuity while carrying special solutions in (2.46) and effecting the time average, as shown in a general manner by the very few existing applications of Whitham's method to cases simpler than the present one (see Luke[53] and Yuen and Lake[54] as exceptional cases in fluid mechanics). The authors intend to give this application to the present case in a separate work.

C. Small-amplitude, quasimonochromatic limit

Let us assume that we are close to a harmonic regime characterized by a frequency ω_0, a wave number k_0, and penetration parameter χ_0, which altogether satisfy the linear dispersion relations (3.3) that, therefore, we rewrite as

$$\mathcal{D}_B(k_0,\omega_0;\chi_0)\equiv\beta^2\omega_0^2-k_0^2+\chi_0^2=0\ , \tag{4.24a}$$

$$\mathcal{D}_S(k_0,\omega_0;\chi_0)\equiv\omega_0^2-k_0^2-\chi_0=0\ . \tag{4.24b}$$

Accordingly, we note that the actual (nonlinear solution) k, ω, and χ have the perturbation form

$$k=k_0+\varepsilon\Phi_x,\quad \omega=\omega_0-\varepsilon\Phi_t,\quad \chi=\chi_0-i\varepsilon\Phi_y\ , \tag{4.25}$$

where Φ is a phase perturbation and $(\omega_0,k_0)\in\mathcal{D}_M(\omega_0,k_0)=0$. We need to evaluate the slowly varying quantities Φ and A when (4.25) holds well. We introduce the new variables and function (compare Benney and Newell[48])

$$\xi=x-\omega_0't+i\chi_0'y,\quad \tau=\varepsilon t,\quad A\rightarrow\varepsilon A\ , \tag{4.26}$$

where a prime denotes the derivative with respect to k_0 of the solutions $\omega_0(k_0)$ and $\chi_0(k_0)$ of (4.24). Working as if the triplet (k_0,ω_0,χ_0) were variables, we deduce from (4.24a) and (4.24b) the following useful relations:

$$\omega_0\omega_0'\beta^2-k_0+\chi_0\chi_0'=0\ , \tag{4.27a}$$

$$\omega_0'\beta^2+\omega_0\omega_0''\beta^2-1+\chi_0'^2+\chi_0\chi_0''=0 \tag{4.27b}$$

for $Y>0$, and $(Y=0)$

$$2\omega_0\omega_0'-2k_0-\chi_0'=0\ , \tag{4.28a}$$

$$2\omega_0'^2+2\omega_0\omega_0''-2-\chi_0''=0\ . \tag{4.28b}$$

We should consider the bulk and surface conditions simultaneously. We give the detail only for the former. Substituting (4.25) into (4.19) and accounting for (4.24a), we obtain

$$\varepsilon(-2\omega_0\beta^2\Phi_t-2k_0\Phi_x)+\varepsilon^2(\beta^2\Phi_t^2-\Phi_x^2)+(-2\varepsilon i\chi_0\Phi_y-\varepsilon^2\Phi_y^2)$$
$$+\tfrac{1}{4}\Delta\beta^2A^2\exp(-2\chi Y)(9\chi_0^4-3k_0^4+2k_0^2\chi_0^2)=\frac{\varepsilon^2}{A}(\beta^2A_{tt}-A_{xx}-A_{yy})\ . \tag{4.29}$$

Accounting now for (4.26) we get

$$-2\omega_0\beta^2\Phi_\tau\varepsilon^2+2\omega_0\beta^2\varepsilon\Phi_\xi\omega_0'-2\varepsilon k_0\Phi_\xi+2\varepsilon\chi_0\chi_0'\Phi_\xi+\varepsilon^2\beta^2\omega_0'^2\Phi_\xi^2-\varepsilon^2\Phi_\xi^2+\varepsilon^2\chi_0'\Phi_\xi^2$$
$$+\frac{\varepsilon^2}{4}\Delta\beta^2A^2\exp(-2\chi_0Y)(9\chi_0^4-3k_0^4+2k_0^2\chi_0^2)=\frac{\varepsilon^2}{A}(\beta^2\omega_0'^2-1+\chi_0'^2)A_{\xi\xi}\ , \tag{4.30}$$

where only terms of $O(\varepsilon^2)$ at most have been kept. Equation (4.30) is automatically satisfied at $O(\varepsilon)$ because of the first of Eqs. (4.27). At order ε^2, Eq. (4.30) gives, on account of the second of (4.27),

$$\Phi_\tau=\left[\frac{\omega_0''}{2}+\frac{\chi_0\chi_0''}{2\omega_0\beta^2}\right]\left[\frac{A_{\xi\xi}}{A}-\Phi_\xi^2\right]+\frac{\Delta A^2}{8\omega_0}\exp(-2\chi_0Y)(9\chi_0^4-3k_0^4+2k_0^2\chi_0^2)\ . \tag{4.31}$$

We proceed in the like manner to examine Eq. (4.19). The first order in ε is automatically satisfied because of (4.27a), while at the next order we obtain

$$A_\tau=\frac{1}{2\beta^2\omega_0}(\beta^2\omega_0'^2+\chi_0'^2-1)(2\Phi_\xi A_\xi+A\Phi_{\xi\xi})\ . \tag{4.32}$$

Finally, introducing the complex quantity

$$a=A\exp(i\Phi)\ , \tag{4.33}$$

we note that

$$a_\tau\exp(-i\Phi)=A_\tau+iA\Phi_\tau\ . \tag{4.34}$$

This allows us to gather Eqs. (4.31) and (4.32) in a single complex equation for a $(Y>0)$ as

$$a_\tau=\frac{i}{2}\left[\omega_0''+\frac{\chi_0\chi_0''}{\beta^2\omega_0}\right]a_{\xi\xi}$$
$$+i\frac{\Delta}{8\omega_0}\exp(-2\chi_0Y)(9\chi_0^4-3k_0^4+2k_0^2\chi_0^2)a^2a^* \tag{4.35}$$

on account of (4.27b). Here the asterisk denotes complex conjugation.

The same method is applied to the surface conditions (4.22) and (4.20) on account of Eqs. (4.28). It is found that Φ and A satisfy the following coupled evolution equations (at $Y=0$):

$$\Phi_\tau=\frac{(\omega_0'^2-1)}{2\omega_0}\left[\Phi_\xi^2-\frac{A_{\xi\xi}}{A}\right]-\frac{1}{8\omega_0}\beta^2\Delta\chi_0(k_0^2+3\chi_0^2)\ ,$$
$$A_\tau=\frac{1}{2\omega_0}(\omega_0'^2-1)(A\Phi_{\xi\xi}+2\Phi_\xi A_\xi)\ , \tag{4.36}$$

and, introducing (4.33) reduces (4.36) to a single complex equation

$$a_\tau=\frac{i}{2}\left[\omega_0''-\frac{\chi_0''}{2\omega_0}\right]a_{\xi\xi}-i\frac{\beta^2\Delta}{8\omega_0}\chi_0(k_0^2+3\chi_0^2)a^2a^*\ , \tag{4.37}$$

valid at $Y=0$.

D. The governing nonlinear Schrödinger (NLS) equation

In the same way that the true linear dispersion relation of Murdoch waves is obtained, by eliminating χ_0 between the two equations (4.24), to obtain a single equation for our nonlinear problem we need to combine Eqs. (4.35) and (4.37) at $Y=0$ and eliminate χ_0''. This procedure yields the single nonlinear Schrödinger (NLS) cubic equation for the complex field a at $Y=0$,

$$ia_\tau+p(\omega_0,k_0)a_{\xi\xi}+q(\omega_0,k_0)|a|^2a=0\ , \tag{4.38}$$

where the coefficients depend on the working regime $(\omega_0,k_0)\in\mathcal{D}_M(\omega_0,k_0)=0$, and, in particular, on the curvature ω_0'' of the linear dispersion curve $\omega_0(k_0)$ as

$$p(\omega_0,k_0)=\tfrac{1}{2}\omega_0''\ , \tag{4.39a}$$

$$q(\omega_0,k_0)=\tfrac{3}{8}\Delta\beta^4\omega_0\frac{(\beta^2\omega_0^2-2k_0^2)}{\beta^2+2(\omega_0^2-k_0^2)}\ , \tag{4.39b}$$

and we recall that in Eq. (4.38)

$$\xi=\xi(Y=0)=x-\omega_0't,\quad \tau=\varepsilon t \tag{4.40}$$

so that ξ has recovered its usual meaning of a characteristic variable in the propagation direction; $\omega_0'=d\omega_0/dk_0$ is the group velocity of the linear, harmonic wave process. It is easily shown from (3.4) written for (ω_0,k_0) that

$$\omega_0'=\frac{k_0[1+2(\omega_0^2-k_0^2)]}{\omega_0[\beta^2+2(\omega_0^2-k_0^2)]}\ , \tag{4.41a}$$

$$\omega_0''=\frac{(\omega_0^2-k_0^2)[2(\beta^4\omega_0^2-k_0^2)-\beta^2(\omega_0^2-k_0^2)]}{\omega_0^3[\beta^2+2(\omega_0^2-k_0^2)]^3}\ . \tag{4.41b}$$

This will allow us to discuss the possible solutions of Eq. (4.38).

Note. The NLS equation (4.38) can be obtained in a different way, which is sketched out in Appendix B.

E. Bright and dark surface acoustic solitons

Equation (4.38) is known to exhibit soliton solutions,[55,56] especially in the context of the nonlinear optics of optical fibers,[57,58] and also in plasma physics.[59,60] Two types of solutions, with different stability characteristics, are possible, depending on the sign of the product pq of the remaining two coefficients. This was established by Zakharov and Shabat[59,60] by the method of inverse scattering. In our case the sign of pq depends on the pair $(\omega_0,k_0)\in\mathcal{D}_M(\omega_0,k_0)=0$, and this requires a discussion when the expressions (4.39) and (4.41) are taken into account.

1. Bright solitons

According to Refs. 59 and 60, for $pq>0$ one obtains stable "bright" soliton solutions in the form

$$\hat{a}(\xi,\tau)=\left[2\left|\frac{p}{q}\right|\right]^{1/2}\eta\exp[-4i(\alpha^2-\eta^2)p\tau-2i\alpha\xi+i\varphi_0]\operatorname{sech}[2\eta(\xi-\xi_0)+8\eta\alpha p\tau] \tag{4.42}$$

or, at $Y=0$,

$$\hat{u}(X,T)=\varepsilon\eta\left[\left|\frac{\omega_0''}{q}\right|\right]^{1/2}\exp\{i[-2\varepsilon^2\omega_0''T(\alpha^2-\eta^2)-2\alpha\varepsilon(X-\omega_0'T)+\varphi_0+k_0X-\omega_0T]\}$$
$$\times\operatorname{sech}[2\varepsilon\eta(X-\omega_0'T-X_0)+4\alpha\eta\varepsilon^2\omega_0''T]\ , \tag{4.43}$$

which contains four parameters η (the amplitude), α, X_0, and φ_0. Taking $\alpha=0$, without loss in generality, reduces (4.43) to

$$\hat{u}(X,T,\alpha=0)=\varepsilon\eta\left[\left|\frac{\omega_0''}{q}\right|\right]^{1/2}\exp\{i[2\varepsilon^2\omega_0''T\eta^2+\varphi_0+k_0X-\omega_0T]\}\operatorname{sech}[2\varepsilon\eta(X-\omega_0'T-X_0)]\ . \tag{4.44}$$

Figure 4 gives a sketch of $|a|^2$ vs $(\xi-\xi_0)$, which explains the naming of bright soliton in the optical context as $|a|^2$ would then represent the intensity of light.

Assume that $\Delta>0$. The condition $pq>0$, on account of Eqs. (4.39) and (4.41), takes on the form

$$2(\beta^4\omega_0^2-k_0^2)-\beta^2(\omega_0^2-k_0^2)<0 \tag{4.45}$$

as $\omega_0>k_0$ for pairs (ω_0,k_0) belonging to the linear dispersion relation (see Fig. 3). This can be rewritten as

$$\beta^2\omega_0^2(2\beta^2-1)<k_0^2(2-\beta^2)\ . \tag{4.46}$$

If $\beta^2<\frac{1}{2}$ we have then

$$\frac{\omega_0^2}{k_0^2}>\frac{2-\beta^2}{\beta^2(1-2\beta^2)}>1 \tag{4.47}$$

so that pairs $(\omega_0,k_0)\in\mathcal{D}_M(\omega_0,k_0)=0$ *cannot* be working

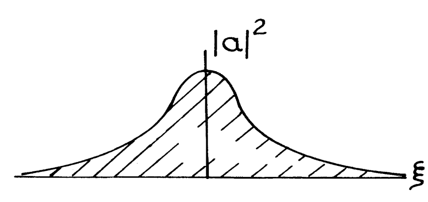

FIG. 4. Bright soliton.

points for the carrier wave. On the contrary, if $\beta^2 > \frac{1}{2}$ (but $\beta^2 < 1$ according to the linear analysis), then

$$\frac{\omega_0^2}{k_0^2} < \frac{2-\beta^2}{\beta^2(2\beta^2-1)} , \tag{4.48}$$

and this is allowed.

Conclusion. $\{\Delta > 0, \frac{1}{2} < \beta^2 < 1\}$ provides the stable bright soliton solution of the soliton-envelope type sketched in Fig. 4.

2. *Dark solitons*

According to Refs. 59 and 60 for $pq < 0$, one obtains stable "dark" soliton solutions in the form

$$\left[\left|\frac{q}{2p}\right|\right]^{1/2} \hat{a}(\xi,\tau) = \frac{(\lambda+i\nu)^2+\exp[2\nu(\xi-\xi_0-2\lambda p\tau)]}{1+\exp[2\nu(\xi-\xi_0-2\lambda p\tau)]} , \tag{4.49}$$

where there are three parameters, λ, ν, and ξ_0, but only two of these are independent as (4.49) holds for $\nu=(1-\lambda^2)^{1/2}$. If we take $\lambda=0$, hence $\nu=1$, we are left with a one-parameter (X_0 or ξ_0) solution (the constant amplitude at $\xi\to\pm\infty$ is also a parameter),

$$\left|\frac{q}{2p}\right| |\hat{a}(\xi,\tau)|^2=2[1-\text{sech}^2(\xi-\xi_0)] , \tag{4.50}$$

of which a graph is given in Fig. 5. It explains the naming of "dark" soliton in the optical context. The displacement $\hat{u}(X,T)$ at $Y=0$ is obtained as

$$\hat{u}(X,T)=\varepsilon\left[\left|\frac{\omega_0''}{q}\right|\right]^{1/2} \tanh[\varepsilon(X-X_0-\omega_0'T)] \times\exp[i(k_0X-\omega_0T)] . \tag{4.51}$$

The corresponding stability condition is discussed as in the previous case.

Conclusion. For $\{\Delta > 0, \beta^2 < \frac{1}{2}\}$ we have the stable dark soliton solution in the form (4.50) or (4.51). That is, we have a mechanical analog, in the form of an elastic SH surface wave propagating on a nonlinear substrate coated with a very thin "slow" film, of optical dark solitons in optical fibers. This result was announced in Ref. 61.

If the nonlinearity coefficient Δ is negative, then the above conclusions are interchanged so that stable bright solitons correspond to $\beta^2 < \frac{1}{2}$ and stable dark solitons to $\frac{1}{2} < \beta^2 < 1$. Remarkably enough, once the sign of the nonlinearity parameter is known, the stability is decided only by the value of the ratio of linear shear elastic waves in the substrate and the thin film (see below). Finally, it is important that the amplitude of the signal be as large as possible. This is obtained for the maximum of the ratio $|p/q|(\omega_0,k_0)$ along the curve $\omega_0(k_0)$ in Fig. 3. This provides the optimal working point for the carrier wave.

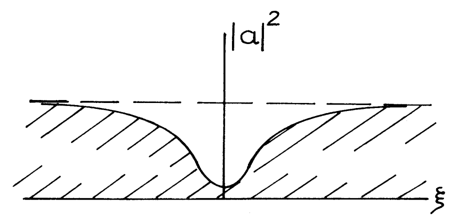

FIG. 5. Dark soliton.

Finally we note that the penetration depth of the soliton SH SAW, Y_0, is still given by the linear solution at our order of approximation, i.e., it also depends on the working regime by

$$\begin{aligned} Y_0 &= \chi_0^{-1} \\ &= (k_0^2-\beta^2\omega_0^2)^{-1/2} \\ &= \frac{\lambda_0}{2\pi}(1-\beta^2 v_0^2)^{-1/2} \\ &= \frac{\lambda_0}{2\pi}[1-(V^2/c_T^2)]^{-1/2} , \end{aligned} \tag{4.52}$$

from Eq. (4.24a) with $v_0=\omega_0/k_0$, $\lambda_0=2\pi/k_0$, and V is the phase velocity in dimensional units.

V. CHOICE OF MATERIALS

We may consider a substrate of lithium niobate $LiNbO_3$ (obviously, $LiNbO_3$ is not isotropic but this approximation is sufficient to give an idea of the looked-for effect), a reputedly nonlinear crystal which also presents a high electromechanical coupling coefficient, which may be useful in experimental investigations and signal-processing applications. Thus[6] $\rho_0=4.7\times10^3$ kg/m^3 and $\mu=c_{44}=6\times10^{10}$ N/m^2, and $\Delta > 0$.[38] Hence $c_T=3.735\times10^3$ m/sec.

The remaining critical parameter is

TABLE I. Film materials for which $\beta^2 < \frac{1}{2}$ ($LiNbO_3$ substrate).

Material	Gold	Cadmium	Copper	Silver	Zinc	Platinum	Tantalum	Bismuth
μ_1 (10^{10} N/m^2)	2.7	1.92	4.83	3.03	4.34	6.10	6.92	1.2
ρ_L (kg/m^3)	19.3	8.65	8.96	10.5	7.14	21.4	16.6	9.8
β^2	0.109	0.173	0.422	0.226	0.476	0.223	0.326	0.095

TABLE II. Film materials for which $\frac{1}{2}<\beta^2<1$ ($LiNbO_3$ substrate).

Material	Aluminum	Tungsten	Vanadium	Titanium	Nickel
μ_L (10^{10} N/m^2)	2.61	16.06	4.67	4.38	8.39
ρ_L kg/m^3)	2.7	19.3	6.1	4.51	8.9
β^2	0.757	0.651	0.599	0.760	0.738

$$\beta^2=\frac{\hat{\mu}}{\mu}\frac{\rho_0}{\hat{\rho}_0}=\frac{\mu_L H}{\mu}\frac{\rho_0}{\rho_L H}=\frac{\mu_L}{\mu}\frac{\rho_0}{\rho_L}\ ,$$

which does *not* depend on the thickness H of the film (we may have $H\simeq 1\ \mu$m).

Tables I and II (values taken from Ref. 62) give the estimated β^2. Accordingly, with a substrate of $LiNbO_3$ a thin film of gold, cadmium, copper, silver, zinc, platinum, tantalum, or bismuth would allow for the stable propagation of dark soliton SH SAW's. As a rule the superimposed film, in addition to being "slower" than the substrate, must be very "slow" (for Au, $\hat{c}_T=\hat{\mu}/\hat{\rho}_0=1182$ m/sec).

VI. CONCLUSION

We have shown analytically that an elastic structure made of a nonlinear elastic substrate and a thin linear elastic film coated on it can allow (with a good choice of the two materials) for the propagation of stable and unstable envelope solitons which are mechanical analogs of envelope light solitons observed in nonlinear optical fibers. From the point of view of engineering practice, with the availability of electronic generators of hyperbolic functions it should be possible to enter the required signal (or one which can rapidly adapt to the required shape) via a piezoelectric transducer in the appropriate structure and thus to realize a real elastic-soliton SAW line. From the point of view of applied mathematics, the Whitham-Newell type of approach has been preferred over others (e.g., the derivative expansion method[63] as used, for instance, by Teymur[64]). Its elegance in treating simultaneously bulk and interface conditions must be emphasized. A direct numerical approach of the basic nonlinear dispersive system (2.38), (2.41) and the direct variational approach through (2.46) will be given in other works.

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APPENDIX A: BASIC WAVE EQUATIONS

For solutions (2.30) we symbolically have $\partial/\partial X_3=0$. The relevant stress components in Eqs. (2.32) read, on account of Eqs. (2.22) and (2.23)

$$T_{3\alpha}=\frac{\partial W}{\partial E_{p\alpha}}(\delta_{3p}+u_{3,p}),\quad \alpha=1,2,\quad p=1,2,3\ , \tag{A1}$$

$$\hat{T}_{32}=\frac{\partial \hat{W}}{\partial \hat{E}_{32}}\ . \tag{A2}$$

Thus (A2) gives trivially

$$\hat{T}_{32}=\hat{\mu}\hat{u}_{3,2}\quad \text{at } X_2=0\ . \tag{A3}$$

As to (A1), it yields

$$\begin{aligned} T_{31}&=\frac{\partial W}{\partial E_{11}}u_{3,1}+\frac{\partial W}{\partial E_{21}}u_{3,2}+\frac{\partial W}{\partial E_{31}}\ ,\\ T_{32}&=\frac{\partial W}{\partial E_{12}}u_{3,1}+\frac{\partial W}{\partial E_{22}}u_{3,2}+\frac{\partial W}{\partial E_{32}}\ . \end{aligned} \tag{A4}$$

In general, for isotropic solids we have

$$\frac{\partial W}{\partial E_{pq}}=\frac{\partial W}{\partial I_1}\delta_{pq}+2\frac{\partial W}{\partial I_2}E_{pq}+3\frac{\partial W}{\partial I_3}E_{pk}E_{kq}\ , \tag{A5}$$

where $\mathbf{E}$, in components, is defined by (2.24). In the present case, on account of Eq. (2.30a), we have for the E_{ij}'s

$$\begin{aligned} E_{11}&=\tfrac{1}{2}u_{3,1}^2=O(f^2)\ ,\\ E_{22}&=\tfrac{1}{2}u_{3,2}^2=O(f^2)\ ,\\ E_{33}&=0\ ,\\ E_{12}&=\tfrac{1}{2}u_{3,1}u_{3,2}=O(f^2)\ ,\\ E_{13}&=\tfrac{1}{2}u_{3,1}=O(f)\ ,\\ E_{32}&=\tfrac{1}{2}u_{3,2}=O(f)\ , \end{aligned} \tag{A6}$$

while

$$\begin{aligned} I_1&=\tfrac{1}{2}(u_{3,1}^2+u_{3,2}^2)=O(f^2)\ ,\\ I_2&=O(f^2)\ ,\\ I_3&=O(f^2)\ . \end{aligned} \tag{A7}$$

For the special case of the truncated expansion (2.27), we note that

$$\frac{\partial W}{\partial I_1}=\lambda I_1+\bar{\beta} I_2+4\xi I_1^3+2\eta I_1 I_2+\nu I_3=O(f^2)\ ,$$
$$\frac{\partial W}{\partial I_2}=\mu+\bar{\beta} I_1+\eta I_1^2+2\delta I_2=O(1)\ , \tag{A8}$$
$$\frac{\partial W}{\partial I_3}=\gamma+\nu I_1=O(1)\ .$$

Substituting from (A8) and (A7) into (A5), accounting then for (A6) and carrying the results into (A4) while keeping terms $O(f^3)$ at most, we are led to

$$T_{31}=\mu u_{3,1}+\delta_{\text{eff}}(u_{3,1}^3+u_{3,1}u_{3,2}^2)\ ,$$
$$T_{32}=\mu u_{3,2}+\delta_{\text{eff}}(u_{3,2}^3+u_{3,2}u_{3,1}^2)\ , \tag{A9}$$

from which there follow Eqs. (2.33) and (2.34) via Eqs. (2.32) while accounting for (A3).

APPENDIX B: ANOTHER METHOD YIELDING THE NLS EQUATION

Here we show that Eq. (4.38) can also be obtained by working practically from the start at the interface. To do this, the following trick is necessary. We study the modulation $u(\theta,\bar{Y},x,y,t)$, $\theta=kX-\omega T$, $\bar{Y}=-\chi Y$, $x=\varepsilon X$, $y=\varepsilon Y$, $t=\varepsilon T$ about the monochromatic surface-wave solution $u=A\exp(i\theta+\bar{Y})$. We account for these in Eqs. (2.38) and (2.39) and consider asymptotic expansions for $u(Y>0)$ and $\hat{u}(Y=0)$. Accounting for the fact that $\omega^2-k^2=\chi$ at the first order of approximation, noting that $\partial/\partial\bar{Y}=1$, integrating the equation resulting from (2.39) with respect to χ, and combining this at $Y=0$ with the equation issued from (2.38), we obtain the single equation

$$\Bigg\{[\beta^2\omega^2-k^2+(\omega^2-k^2)^2]\frac{\partial^2}{\partial\theta^2}-\varepsilon\Bigg[\Bigg[2\omega\beta^2\frac{\partial^2}{\partial t\partial\theta}+\omega_t\beta^2\frac{\partial}{\partial\theta}+2k\frac{\partial^2}{\partial x\partial\theta}++k_x\frac{\partial}{\partial\theta}-\chi_y\Bigg]$$
$$+2\chi\Bigg[2\omega\frac{\partial^2}{\partial t\partial\theta}+\omega_t\frac{\partial}{\partial\theta}+2k\frac{\partial^2}{\partial x\partial\theta}+k_x\frac{\partial}{\partial\theta}\Bigg]\Bigg]+\varepsilon^2\Bigg[\Bigg[\beta^2\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\Bigg]+2(\omega^2-k^2)\Bigg[\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}\Bigg]\Bigg]$$
$$-\beta^2\Delta\varepsilon S_1-\tfrac{3}{4}\Delta\beta^2A^2(\chi_0^4-k_0^4)-\beta^2\Delta\varepsilon^2S_2\Bigg\}(\hat{u}_0+\varepsilon\hat{u}_1+\varepsilon^2\hat{u}_2+\cdots)=0\ . \tag{B1}$$

Accounting for the nonlinear contribution present in this equation and noting that

$$\chi^2\simeq-\beta^2\omega^2+k^2\ , \tag{B2}$$

we look for a perturbed nonlinear Murdoch dispersion relation in the form [compare to Eq. (3.4)]

$$\mathcal{D}_M^{\text{NL}}(\omega,k,A)\equiv[\beta^2\omega^2-k^2+(\omega^2-k^2)^2]$$
$$-\tfrac{3}{4}\Delta\beta^4A^2\omega^2(2k^2-\beta^2\omega^2)$$
$$=\varepsilon g^{(1)}+\varepsilon^2 g^{(2)}+\cdots\ . \tag{B3}$$

Substituting from this in Eq. (B1), at orders ε and ε^2 we obtain the "dispersive" nonlinear dispersion relation and the "conservation of wave action" as

$$\mathcal{D}_M^{\text{NL}}=\frac{\varepsilon^2}{A}[\beta^2A_{tt}-A_{xx}-A_{yy}+2(\omega^2-k^2)(A_{tt}-A_{xx})] \tag{B4}$$

and

$$2(\omega^2-k^2)[(\omega A^2)_t+(kA^2)_x]$$
$$+[(\beta^2\omega A^2)_t+(kA^2)_x+iA^2\chi_y]=0\ , \tag{B5}$$

respectively. The right-hand side in Eq. (B4) is of importance only when $A=O(\varepsilon)$ and $\mathcal{D}_M(\omega,k)=O(\varepsilon^2)$. Let us consider $A\to\varepsilon A$, and $\omega\simeq\omega_0$, $k\simeq k_0$ such that $\mathcal{D}_M(\omega_0,k_0)=0$. We set (small-amplitude, quasimonochromatic limit)

$$\theta=k_0X-\omega_0T+\Phi(x,y,t),\quad \bar{Y}=-\chi_0Y\ , \tag{B6a}$$

so that

$$\omega=\omega_0-\varepsilon\Phi_t,\quad k=k_0+\varepsilon\Phi_x,\quad \chi=\chi_0-i\varepsilon\Phi_y\ . \tag{B6b}$$

Then Eq. (B5) at the first order yields

$$2(\omega_0^2-k_0^2)(AA_t\omega_0+AA_xk_0)+\beta^2AA_t\omega_0+k_0AA_x=0\ . \tag{B7}$$

Introducing $\xi=x-\omega_0't+i\chi_0'y$, $\tau=\varepsilon t$, we transform this into

$$2(\omega_0^2-k_0^2)(k_0-\omega_0\omega_0')-\beta^2\omega_0\omega_0'+k_0=0\ . \tag{B8}$$

But this is nothing else but the first differential with respect to k_0 of the Murdoch dispersion relation (4.4a) so that it is identically satisfied. Then at the next order, the equation obtained after rearranging terms and accounting for (4.28a) and the following expression of ω_0'':

$$\omega_0''=\frac{1-\beta^2\omega_0'^2-4(\omega_0\omega_0'-k_0)^2-2(\omega_0^2-k_0^2)(\omega_0'^2-1)}{\omega_0[\beta^2+2(\omega_0^2-k_0^2)]}\ , \tag{B9}$$

yields

$$A_\tau = -\frac{\omega_0''}{2}(A\Phi_{\xi\xi}+2\Phi_\xi A_\xi) \ . \qquad (B10)$$

Exploiting now Eq. (B3) on account of (B6), the first order yields the Murdoch dispersion relation $\mathcal{D}_M(\omega_0,k_0)=0$. The second order yields again the first differential of Murdoch's dispersion relation, Eq. (B8), and the next order, on account of (B9) yields

$$\Phi_\tau = \frac{\omega_0''}{2}\left[\frac{A_{\xi\xi}}{A}-\Phi_\xi^2\right]-\frac{3}{8}\frac{\Delta\beta^4 A^2\omega_0(2k_0^2-\beta^2\omega_0^2)}{[\beta^2+2(\omega_0^2-k_0^2)]} \ . \qquad (B11)$$

Introducing now the complex function $a=A\exp(i\Phi)$ and combining Eqs. (B10) and (B11) by noting (4.34) produces the NLS equation (4.38) at $Y=0$ with $\xi=x-\omega_0't$. Q.E.D.

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