Low-frequency behavior of the kinetic coefficients in localization regimes in strong magnetic fields

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Using a method of integration over commuting and anticommuting variables, we study the motion of noninteracting electrons in a system with short-range disorder and in a strong magnetic field. An explicit calculation shows that the static conductivities σ_{xx} and σ_{yx} vanish if the Fermi energy is situated in the lower tail of the lowest Landau level. At the same time, the Hall resistivity ρ_{xy} remains finite, although the longitudinal resistivity diverges. These results are valid both for two-dimensional (2D) and for 3D systems. We compare our theoretical findings with experiments on magnetic-field-induced metal-insulator transitions in 3D systems. Furthermore, the result for the Hall conductivity of 2D systems is generalized to higher localization regions in order to study the deformation of the Hall plateaus in microwave experiments on GaAs-$Al_xGa_{1-x}As$ heterostructures.

I. INTRODUCTION

In the present work, we intend to give a complete account of our recent work on the low-frequency asymptotics of the Kubo conductivities of an electron gas subject to disorder and a strong magnetic field. In previous articles[1] we focused our attention on the application of the theoretical results to experiments, whereas, in the following, emphasis will be put on a comprehensive presentation of our method.

The motivation of the present investigation has been threefold. In a recent experiment, Hopkins *et al.*[2] studied the behavior of the resistivities in a three-dimensional (3D) system of uncompensated degenerately doped Ge:Sb slightly above the critical concentration as a function of the magnetic field. Increasing the magnetic field beyond 4 T, they observed an increase of ρ_{xx} by about 3 orders of magnitude whereas the Hall coefficient changed only by a factor of 2–4.

This phenomenon cannot be explained by the expected magnetic freezing out[3] because this would imply a simultaneous reduction of the apparent carrier concentration and thus a drastic increase of the Hall coefficient. Hopkins *et al.* conjectured that some magnetic-field-induced localization mechanism might be involved.

One would be inclined to assume that the observed transition occurs when the lowest Landau level (LLL) crosses the Fermi energy but we are not able to describe the critical behavior near the transition. However, an explicit analytical calculation of kinetic coefficients far from the transition in the lower tail of the disorder-broadened density of states (DOS) is possible. This region can be reached experimentally, provided that the magnetic field is strong enough. In this field range, there are no extended states below the Fermi energy and thus the longitudinal and the Hall dc conductivity vanish simultaneously. However, any conclusion concerning the resistivities is more difficult. Since the dc resistivities are obtained by an inversion of the conductivity tensor, we have to consider finite frequencies ω first—a direct inversion of the dc-conductivity tensor which vanishes in the localization regime is mathematically undefined. Only afterwards may the limit $\omega \rightarrow 0$ be performed in order to obtain the dc resistivities. We will see that the energy scale on which ω is assumed to be small when referring to the low-frequency asymptotics is determined by the disorder-induced level broadening.

It will be shown below that, in the energy region under consideration, the leading term of the longitudinal conductivity σ_{xx} is proportional to $i\omega$, which is just the characteristic of an insulator or dielectric having a finite static polarizability. At the same time, the Hall conductivity in the low-frequency limit is proportional to ω^2. From this proportionality it follows immediately that, although the longitudinal dc resistivity ρ_{xx} diverges, the Hall resistivity ρ_{xy} remains finite as $\omega \rightarrow 0$.

In the metallic region far from the transition where the classical Boltzmann equation is applicable, the finiteness of the Hall resistivity is guaranteed by the well-known formula $\rho_{xy} = -B/en$. In contrast to the Drude formulas for the conductivities, this result is independent of the disorder parameter. One could formally even consider the limit of vanishing relaxation time τ without changing the result for the Hall resistivity. However, here we consider the ultra-quantum-mechanical limit where the Boltzmann equation cannot be used. We indeed obtain a finite ρ_{xy} but our result differs from the above one. Let us emphasize that the finiteness of the Hall resistivity in the case under consideration is not related to the existence of edge states because they exist only if there is at least one Landau level (LL) below the Fermi energy.

The low-frequency behavior of the Hall conductivity in a 2D electron gas (EG) obtained in the present work can be also useful to understand phenomena related to the

quantum Hall effect (QHE). This question leads us to another set of experiments, i.e., to transmission measurements of GaAs-$Al_xGa_{1-x}As$ heterostructures in microwave waveguides[4,5] which indicate that Hall plateaus persist up to frequencies above 30 GHz. Nevertheless, the width of the plateaus decreases with increasing frequency and they seem to develop a finite slope. Until now, the QHE at finite frequencies has only been discussed qualitatively in the percolation limit (cf. Ref. 6). Using our approach, we can study the deformation of the Hall plateaus quantitatively. Of course, in the static limit $\omega \to 0$ we obtain exact quantization of the Hall conductivity.

The method we will use to determine the Kubo conductivities is the calculation of functional integrals in supersymmetric representation near nontrivial saddle points (instantons). This technique has been applied by Affleck[7] in the calculation of the tails of the DOS of a 2D EG in the disorder-broadened LLL. However, this approach did not reveal any new physics since the exact result for the DOS in the one-band model had already been obtained by Wegner[8] (see also, Ref. 9). The advantage of the integration around instantons is that, in contrast to the procedure yielding the exact solution, it may be applied not only to a one-particle Green's function but is especially suitable for the averaged product of n retarded (advanced) Green's functions which will turn out to be important in the subsequent calculations. Our third motivation was thus to apply and to generalize this method to the calculation of kinetic coefficients.

The article is organized as follows. In Sec. II we will introduce the basic notations of our model and define the quantities to be calculated subsequently in terms of supersymmetric functional integrals. The purpose of Sec. III is mainly didactical since we rephrase Affleck's saddle-point method for the calculation of the density of states. In Sec. IV we calculate the leading terms of the low-frequency expansion of the Kubo conductivities. In Sec. V we relate our theoretical results to the above-mentioned experiments.

II. THE MODEL

We consider noninteracting electrons in two and three dimensions under the influence of a perpendicular magnetic field **B**, i.e., a system described by the one-particle Hamiltonian

$$H = H_0 + V(r) ,$$

where

$$H_0 = \frac{1}{2m}(\mathbf{p} - e\,\mathbf{A})^2, \quad \mathbf{A} = \tfrac{1}{2}B(-y,x,0) . \tag{2.1}$$

With a white-noise random potential V, i.e., it has the averages

$$\overline{V(\mathbf{r})} = 0, \quad \overline{V(\mathbf{r}_1)V(\mathbf{r}_2)} = \lambda\delta(\mathbf{r}_1 - \mathbf{r}_2) . \tag{2.2}$$

The magnetic field is assumed to be strong in the sense that the disorder-induced broadening of the LL's is small compared with the Landau-level distance

Whereas the density of states $\rho(E)$ can be expressed in terms of the averaged retarded and advanced Green's function $G^{\pm}$, i.e.,

$$\rho(E) = -\frac{1}{\pi}\mathrm{Im}\overline{G^{+}(\mathbf{r},\mathbf{r};E)} , \tag{2.3}$$

kinetic quantities depend, in general, on higher-order correlation functions. We want to calculate the frequency-dependent Kubo conductivities,[10] i.e.,

$$\sigma_{\mu\nu}(\omega) = \beta\int_0^{\infty} e^{i\omega t - \eta t}\langle j_\nu; j_\mu(t)\rangle dt , \tag{2.4}$$

where we have used the Kubo scalar product

$$\langle A;B\rangle = \beta^{-1}\mathrm{Tr}\int_0^{\beta}\rho\, A(i\hbar\lambda)B^{+}d\lambda$$

$$\text{with } \beta^{-1} = k_B T . \tag{2.5}$$

In terms of retarded and advanced Green's functions $G^{\pm}$, the zero-temperature conductivities can be expressed equivalently by the relation[11]

$$\begin{aligned}
&\sigma_{\mu\nu}(\omega) = \frac{1}{i\omega}[\chi_{\mu\nu}(-\omega) - \chi_{\mu\nu}(0)] , \\
&\chi_{\mu\nu}(\omega) = \frac{e^2}{2\pi i V}\int dE\, f(E)\mathrm{Tr}\{\overline{v_\mu[G^{+}(E) - G^{-}(E)]v_\nu G^{-}(E+\omega)} - \overline{v_\mu G^{+}(E-\omega)v_\nu[G^{+}(E) - G^{-}(E)]}\} ,
\end{aligned} \tag{2.6}$$

which is equivalent to the more compact formulation[12]

$$\begin{aligned}
\sigma_{\mu\nu}(\omega) = \frac{e^2}{2\pi}\frac{1}{V}\int dE\, f(E)\mathrm{Tr}\{&\overline{v_\mu[G^{-}(E) - G^{+}(E)]v_\nu G^{-}(E-\omega)G^{-}(E)} \\
&- \overline{v_\mu G^{+}(E+\omega)G^{+}(E)v_\nu[G^{-}(E) - G^{+}(E)]}\} .
\end{aligned} \tag{2.7}$$

Equation (2.7) is most convenient for the calculation of the conductivities in the metallic region. However, in the localization regime in which we are going to work, the coordinate representation is more convenient. For this purpose we use the equation of motion for the velocities

$$v_\mu = i[H, r_\mu] \tag{2.8}$$

in the form

$$G(z)v_\mu G(z') = i[G(z)r_\mu - r_\mu G(z') + (z - z')G(z)r_\mu G(z')] . \tag{2.9}$$

For any finite frequency ω, a repeated use of Eq. (2.9) in the velocity-velocity correlation function of Eq. (2.7) yields

$$\sigma_{\mu\nu}(\omega)=\frac{e^2\omega}{2\pi V}\int\int\int r_\nu(r_\mu-r'_\mu)\{[f(E)-f(E-\omega)]\overline{G^+(\mathbf{r},\mathbf{r}';E)G^-(\mathbf{r}',\mathbf{r};E-\omega)} \\ +f(E)[\overline{G^+(\mathbf{r},\mathbf{r}';E+\omega)G^+(\mathbf{r}',\mathbf{r};E)}-\overline{G^-(\mathbf{r},\mathbf{r}';\mathrm{E})\mathrm{G}^-(\mathbf{r}',\mathbf{r};\mathrm{E}-\omega)}]\}\mathrm{dr}\,\mathrm{dr}'\,\mathrm{dE}\ . \tag{2.10}$$

Shifting the energy argument of the above integral by $\pm\omega/2$, the frequency-dependent conductivities can be decomposed into two contributions

$$\sigma_{\mu\nu}(\omega)=\sigma^{(-)}_{\mu\nu}(\omega)+\sigma^{(+)}_{\mu\nu}(\omega)\ , \tag{2.11}$$

where

$$\sigma^{(\mp)}_{\mu\nu}(\omega)=\frac{e^2\omega^2}{4\pi V}\int\int\int r_\nu(r_\mu-r'_\mu)\frac{f(E+\omega/2)\mp f(E-\omega/2)}{\omega}K^{(\mp)}(\mathbf{r},\mathbf{r}';E,\omega)dr\,dr'\,dE\ , \tag{2.12}$$

the first of which—in the limit $\omega\to0$—is determined by states near the Fermi surface only, whereas for the calculation of $\sigma^{(-)}$ one has to integrate over all states below the Fermi energy, i.e.,

$$K^{(-)}(\mathbf{r},\mathbf{r}';E,\omega)=\overline{2G^+(\mathbf{r},\mathbf{r}';E+\omega/2)G^-(\mathbf{r}',\mathbf{r};E-\omega/2)}-\overline{G^+(\mathbf{r},\mathbf{r}';E+\omega/2)G^+(\mathbf{r}',\mathbf{r};E-\omega/2)} \\ -\overline{G^-(\mathbf{r},\mathbf{r}';E+\omega/2)G^-(\mathbf{r}',\mathbf{r};E-\omega/2)}\ , \\ K^{(+)}(\mathbf{r},\mathbf{r}';E,\omega)=\overline{G^+(\mathbf{r},\mathbf{r}'E+\omega/2)G^+(\mathbf{r}',\mathbf{r};E-\omega/2)}-\overline{G^-(\mathbf{r},\mathbf{r}';E+\omega/2)G^-(\mathbf{r}',\mathbf{r};E-\omega/2)}\ , \tag{2.13}$$

where we have adopted units in which $\hbar=1$, V denotes the volume, and f is the Fermi function.

In principle, it is possible to give the functional-integral representation of the kernels $K^{(\pm)}$ and to calculate the corresponding conductivities straightforwardly. However, since we are mainly interested in the low-frequency asymptotics, it is more convenient to perform the ω expansion first in those quantities which are analytic near $\omega=0$. This is the case for the kernel $K^{(+)}$ which, up to linear order in ω, reads

$$K^{(+)}(\mathbf{r},\mathbf{r}';E,\omega)=2iI_2(\mathbf{r},\mathbf{r}';E)+\omega\int I_3(\mathbf{r},\mathbf{r}',\tilde{\mathbf{r}};E)d\tilde{r}+O(\omega^2) \tag{2.14}$$

with

$$I_2(\mathbf{r}_1,\mathbf{r}_2;E)=\mathrm{Im}\overline{G^+(\mathbf{r}_1,\mathbf{r}_2;E)G^+(\mathbf{r}_2,\mathbf{r}_1;E)}\ , \\ I_3(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3;E)=\mathrm{Re}[\overline{G^+(\mathbf{r}_1,\mathbf{r}_2;E)G^+(\mathbf{r}_2,\mathbf{r}_3;E)G^+(\mathbf{r}_3,\mathbf{r}_1;E)}-(\mathbf{r}_2\leftrightarrow\mathbf{r}_3)]\ . \tag{2.15}$$

To leading order, the low-frequency expansion of the energy integral contributions to the longitudinal conductivity thus yields

$$\sigma^{(+)}_{xx}(\omega)=\frac{ie^2\omega}{2\pi V}\int\int\int f(E)(x_1-x_2)^2I_2(\mathbf{r}_1,\mathbf{r}_2;E)dr_1\,dr_2\,dE\ . \tag{2.16}$$

Since $I_2(\mathbf{r},\mathbf{r}';E)$ is symmetric with respect to interchanging $\mathbf{r}\leftrightarrow\mathbf{r}'$ and is rotationally invariant in the x,y plane, as will be shown below, it can only contribute to $\sigma^{(+)}_{xx}$ but not to $\sigma^{(+)}_{yx}$. The first nonvanishing contribution to $\sigma^{(+)}_{yx}$ thus comes from $I_3(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3;E)$, which changes its sign when any two of the spacial arguments $\mathbf{r}_i\leftrightarrow\mathbf{r}_j$, $i\neq j$, are interchanged. Using this property we can write

$$\int\int\int x_1(y_1-y_2)I_3(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3;E)dr_1\,dr_2\,dr_3=-\tfrac{1}{3}\int\int\int\Delta(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)I_3(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3;E)dr_1\,dr_2\,dr_3\ , \tag{2.17}$$

where

$$\Delta(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)=\tfrac{1}{2}(\mathbf{r}_1\times\mathbf{r}_2+\mathbf{r}_2\times\mathbf{r}_3+\mathbf{r}_3\times\mathbf{r}_1)_z \tag{2.18}$$

denotes the oriented area of the triangle spanned by $\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3$. Consequently, the leading term of the low-frequency expansion of the Hall-conductivity contribution $\sigma^{(+)}_{yx}$ reads

$$\sigma^{(+)}_{yx}(\omega)=\frac{e^2\omega^2}{6\pi V}\int\int\int f(E)\Delta(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)I_3(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3;E)dr_1\,dr_2\,dr_3\,dE\ . \tag{2.19}$$

In the tails of DOS, i.e., in the region where $\rho(E)$ is exponentially small, one has localization. In this region, the low-frequency asymptotics of $\sigma^{(-)}_{xx}$ for high LL's have already been obtained elsewhere[11] and the calculation for $\sigma^{(-)}_{yx}$ is not essentially more complicated. We will treat the Fermi-surface contributions to the conductivities in Sec. IV B and show that they satisfy, to leading order in ω, the relation

$$\sigma_{\mu\nu}^{(-)}(\omega)=\sigma_{\mu\nu}^{(+)}(\omega) \ . \tag{2.20}$$

The calculations which we will present in the following sections are based on the functional-integral representation of the averaged Green's functions in terms of bosonic and fermionic fields. The commuting and anticommuting fields will be denoted by s and χ, respectively. $\Phi=(s,\chi)$ is the corresponding supervector.

In order to keep a main line of reasoning, we first want to focus our attention on all those quantities which depend on products of either advanced or retarded Green's functions only. We have to calculate the DOS of Eq. (2.3) from the functional integral

$$\overline{G^{+}(\mathbf{r},\mathbf{r};E)}=-i\int[d\bar{\Phi}][d\Phi]\chi\bar{\chi}(\mathbf{r})\exp(-S) \ , \tag{2.21}$$

where S denotes the action

$$S=-i\int\bar{\Phi}(E-H_0+i\eta)\Phi\,dr+\frac{\lambda}{2}\int(\bar{\Phi}\Phi)^2\,dr \ . \tag{2.22}$$

One possible way to calculate $I_{2,3}$ would be to introduce one pair of superfields $\Phi_i\bar{\Phi}_i$ for each Green's function involved. However, the calculation simplifies considerably if the functions $I_{2,3}$ are represented as a functional-integral representation in terms of one pair of fields $\Phi,\bar{\Phi}$ only. This can be done as follows:

$$I_2(\mathbf{r}_1,\mathbf{r}_2;E)=-\mathrm{Im}\int[d\bar{\Phi}][d\Phi]s(\mathbf{r}_1)s^*(\mathbf{r}_2)\chi(\mathbf{r}_2)\bar{\chi}(\mathbf{r}_1)\exp(-S) \tag{2.23}$$

and

$$\begin{aligned} &I_3(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3;E)=\mathrm{Re}[J(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3;E)-J(\mathbf{r}_1,\mathbf{r}_3,\mathbf{r}_2;E)] \ , \\ &J(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3;E)=i\int[d\bar{\Phi}][d\Phi]\chi(\mathbf{r}_1)\bar{\chi}(\mathbf{r}_2)s(\mathbf{r}_2)s^*(\mathbf{r}_3)\chi(\mathbf{r}_3)\bar{\chi}(\mathbf{r}_1)\exp(-S) \ . \end{aligned} \tag{2.24}$$

In order to obtain the energy integral contribution to the conductivities $\sigma^{(+)}$, one has to insert Eqs. (2.23) and (2.24) into Eqs. (2.16) and (2.19). We want to emphasize that the representation of the I_3 given in Eq. (2.24) in terms of one pair of fields only which, at first glance, might not seem to be a natural choice constitutes a major achievement and simplifies the subsequent calculation considerably.

In contrast to the fact, $\sigma_{\mu\nu}^{(-)}$ contains mixed products of Green's functions, i.e., $\overline{G^{+}G^{-}}$, and we will have to introduce two pairs of fields for the proof of Eq. (2.20). However, we want to postpone this discussion and first focus our attention on those quantities which can be obtained with exactly the same techniques as those used by Affleck[7] to determine the DOS in the tails of the LLL of a 2D EG.

III. THE DENSITY OF STATES

For the calculation of the DOS in the region of the tails, one has to expand the action S around its nontrivial extrema (instantons) and to perform the functional integration over the quadratic fluctuations around the instanton solutions. The main contribution in the region under consideration comes from the one-instanton configuration. According to Eqs. (2.3) and (2.21) we have to perform the functional integration

$$\mathrm{Re}\int[d\bar{\Phi}][d\Phi]\chi(\mathbf{r})\bar{\chi}(\mathbf{r})\exp(-S) \tag{3.1}$$

in this approximation. A most transparent way to understand the technique of functional integration over superfields to expand the fields s, χ in terms of an orthonormal set of functions $\varphi^{(m)}(\mathbf{r})$ labeled by the integer m [the set of z and (x,y)-dependent functions h and u are labeled, respectively, by m_1 and m_2], but only precisions about the (x,y) dependence are required in the following:

$$\begin{aligned} &\varphi^{(m)}(\mathbf{r})=h^{(m_1)}(z)u^{(m_2)}(x,y) \ , \\ &u^{(m)}(x,y)=\frac{1}{(2\pi l^2 m!)^{1/2}}\left[\frac{x+iy}{\sqrt{2}l}\right]^m\exp\left[-\frac{x^2+y^2}{4l^2}\right] , \end{aligned} \tag{3.2}$$

where l denotes the magnetic length, i.e.,

$$s(\mathbf{r})=\sum_m a_m\varphi^{(m)}(\mathbf{r}), \quad \chi(\mathbf{r})=\sum_m g_m\varphi^{(m)}(\mathbf{r}) \ . \tag{3.3}$$

The coefficients a_m occurring in the expansion of s are complex and the coefficients g_m in the expansion of χ are Grassmann variables. The functional integration now reduces simply to an integration over the set of variables $a_m,a_m^*,g_m,\bar{g}_m$, together with a spatial average. Denoting by $\varepsilon=E-\omega_c/2$ the energy distance from the center of the LLL, we thus can rewrite Eq. (3.1) in the form

$$\begin{aligned} &\rho(E)=\frac{1}{\pi V}\mathrm{Re}\sum_{m=0}^{\infty}\int\prod_l da^*\,da_l\,d\bar{g}_l\,dg_l\,g_m\bar{g}_m\exp(-S) \ , \\ &S=-i\sum_m(\varepsilon+i\eta)(a_m^*a_m+\bar{g}_mg_m) \\ &\quad+\frac{\lambda}{2}\sum_{n,m,k,l}(a_n^*a_m+2\bar{g}_ng_m)I_{nmkl}a_k^*a_l \ , \\ &I_{nmkl}=\int\varphi^{(m)*}(\mathbf{r})\varphi^{(n)}(\mathbf{r})\varphi^{(k)*}(\mathbf{r})\varphi^{(l)}(\mathbf{r})dr \ . \end{aligned} \tag{3.4}$$

This procedure is equivalent to the one used in Ref. 7 and the following sections. We chose it here to visualize that the essential techniques used throughout the article can be understood on the basis of the integration rules for commuting and anticommuting variables only.

After a transformation to real bosonic integration variables, we rotate the contour of functional integration by $-\pi/4$ and end up with the variables $b_{l_1}=e^{-i\pi/4}\mathrm{Re}a_l$, $b_{l_2}=e^{-i\pi/4}\mathrm{Im}a_l$ (cf. Ref. 7). In order to find the saddle-

point solution of the above integral after projection to the LLL, Affleck proposed the ansatz $s_{\rm cl}(\mathbf{r})=A\varphi^{(0)}(\mathbf{r})$. In 2D this is a real multiple of the $m=0$ eigenfunction of the unperturbed Hamiltonian H_0 in symmetrical gauge [see Eq. (3.2)]. In 3D, it has to be multiplied by

$$h^{(0)}(z)=(m|\varepsilon|/2)^{1/4}\cosh^{-1}(\sqrt{2m|\varepsilon|}z)$$

(cf. Ref. 13, m is the mass). The variation has to be performed with respect to the coefficient A. Inserting this ansatz into Eq. (3.4) and after integrating out the fermionic variables, the DOS becomes proportional to

$$b^2\exp\left[\varepsilon A^2+\tfrac{1}{2}\lambda\iota_0 A^4+\sum_m \ln(\varepsilon+\lambda\iota_m A^2)\right], \tag{3.5}$$

where we have used the notation $b=|\varphi^{(0)}(0)|$ and $\iota_0=I_{0000}$, $\iota_m=I_{00mm}$. The corresponding saddle-point condition now reads

$$2A\varepsilon+2\lambda\iota_0 A^3+\sum_m \frac{2\lambda\iota_m A}{\varepsilon+\lambda\iota_m A^2}=0 . \tag{3.6}$$

There is, of course, one trivial saddle point at the origin, i.e., $A=0$. However, this saddle point does not contribute to the quantities under consideration. The other saddle-point manifold is characterized by $A^2=|\varepsilon|/\lambda\iota_0$, provided that $|\varepsilon|^2\gg\lambda\iota_0$. This condition means that the energy has to be situated in the deep tail of the DOS. After explicit calculation of the integrals ι_0,ι_m, we obtain

$$A^2=\begin{cases}\dfrac{4\pi l^2}{\lambda}|\varepsilon| & \text{for 2D},\\ \dfrac{16\pi l^2}{\lambda}\sqrt{|\varepsilon|/2m} & \text{for 3D}.\end{cases} \tag{3.7}$$

Calculating the one-instanton contribution to the DOS now means to approximate the action S in Eq. (3.1) by the quadratic expression

$$S\to S_0+\int\bar\chi(\mathbf{r})\frac{\partial^2 S}{\partial\bar\chi\,\partial\chi}\chi(\mathbf{r})d\mathbf{r}+\int s_i(\mathbf{r})\frac{1}{2}\frac{\partial^2 S}{\partial s_i\,\partial s_j}s_j(\mathbf{r})d\mathbf{r} \tag{3.8}$$

with

$$S_0=\begin{cases}\left(\dfrac{|\varepsilon|}{\Gamma_2}\right)^2, & \Gamma_2=\left(\dfrac{2\pi l^2}{\lambda}\right)^{-1/2} \quad \text{for 2D},\\ \left(\dfrac{|\varepsilon|}{\Gamma_3}\right)^{3/2}, & \Gamma_3=\left(\dfrac{32\pi l^2}{3\lambda\sqrt{2m}}\right)^{-2/3} \quad \text{for 3D},\end{cases} \tag{3.9}$$

and

$$\frac{\partial^2 S}{\partial\bar\chi\,\partial\chi}=-\varepsilon-\lambda s_{\rm cl}^2(\mathbf{r}),\qquad \frac{1}{2}\frac{\partial^2 S}{\partial s_i\,\partial s_j}=-\varepsilon-\lambda s_{\rm cl}^2(\mathbf{r})\begin{pmatrix}3&0\\0&1\end{pmatrix}. \tag{3.10}$$

Consequently, in the one-instanton approximation, we are left with two Gaussian integrations, one with respect to the fermionic modes $\bar\chi,\chi$ and the other over the real bosonic modes s_1,s_2. Explicit calculations can be carried out, provided that the eigenfunctions and the corresponding eigenvalues of the above operators are known. In the following we will refer to the eigenfunctions of the operators of (3.10) as fermionic (bosonic) modes and to the corresponding eigenvalues as $\lambda_F^{(m)}(\lambda_B^{(m)})$.

There is one bosonic mode with negative eigenvalue and we have to rotate back the contour of functional integration with respect to this mode so that the integral becomes imaginary, as required. Furthermore, there are bosonic modes with vanishing eigenvalue (henceforth called *zero modes*) corresponding to invariance of the action S with respect to translations of the spacial coordinates $\Phi(\mathbf{r})\to\Phi(\mathbf{r}-\mathbf{r}_0)$ and to rotation in functional space, i.e., $\Phi\to e^{i\varphi}\Phi$. Instead of integrating with respect to these modes, we transform to the corresponding continuous parameters $(\mathbf{r}_0,\varphi)$ of the invariance group and integrate with respect to the latter.

Finally, we have to take into account that there is also one fermionic zero mode with the eigenvalue $\lambda_F^{(0)}=0$ and the coefficients $g_0,\bar g_0$. Due to the Grassmann integration rules

$$\sum_m\int\prod_{l=0}d\bar g_l\,dg_l\,g_m\bar g_m\exp\left[-\sum_n\lambda_F^{(n)}\bar g_n g_n\right]=\int\prod_{l=1}d\bar g_l\,dg_l\exp\left[-\sum_n\lambda_F^{(n)}\bar g_n g_n\right], \tag{3.11}$$

we get a nonvanishing contribution in Eq. (3.4) only if there is a fermionic zero mode in the preexponential. The density of states is thus determined by the saddle-point value of the action S_0 by the determinants arising from the Gaussian integrations and the Jacobians Δ_i from the transformation of the bosonic zero modes to the invariance parameters:

$$\rho(E)=e^{-S_0}\left|\det{}'\frac{1}{2}\frac{\partial^2 S}{\partial s_i\,\partial s_j}\right|^{-1/2}\det{}'\frac{\partial^2 S}{\partial\bar\chi\,\partial\chi}\left[\prod_i\frac{\Delta_i}{\sqrt{\pi}}\right]\times\int|\varphi^{(0)}(\mathbf{r}-\mathbf{r}_0)|^2\,d\mathbf{r}_0 , \tag{3.12}$$

where the primes at the determinants denote the omission of zero modes. Due to normalization, the last integral equals unity. Inserting the numerical values of the nonzero eigenvalues $\lambda_F^{(m)},\lambda_B^{(m)}$ of the operators of Eq. (3.10) (cf. Ref. 7 for 2D) into Eq. (3.12) yields

$$\rho(E)=\begin{cases}\dfrac{2}{\sqrt{\pi}}\dfrac{1}{2\pi l^2}\dfrac{1}{\Gamma_2}\left(\dfrac{|\varepsilon|}{\Gamma_2}\right)^2\exp\left[-\left(\dfrac{|\varepsilon|}{\Gamma_2}\right)^2\right] & \text{for 2D},\\ \dfrac{9}{\pi}\dfrac{1}{2\pi l^2}\sqrt{2m/\Gamma_3}\left(\dfrac{|\varepsilon|}{\Gamma_3}\right)^{5/2}\exp\left[-\left(\dfrac{|\varepsilon|}{\Gamma_3}\right)^{3/2}\right] & \text{for 3D}.\end{cases} \tag{3.13}$$

Of course, for 2D this agrees with Wegner's exact result[8,9] in the limit of large $|\varepsilon|/\Gamma_2$.

Both results, although with different numerical prefactors, have been obtained by Ioffe and Larkin[13] using the optimal fluctuation method and the 3D case has recently been reconsidered by Hayn *et al.*[14] within the framework of supersymmetry.

Although the method presented above does not allow us to obtain the exact solution for the DOS in the one-band model, it is more useful for our purpose than the method of Refs. 8 and 9 because it can be generalized to kinetic quantities to be calculated in the one-instanton approximation. For different averages of Green's functions, only minor modifications are necessary which concern mainly the different preexponential and thus the spatial integration. In the following section we will extensively use the above-described method with special emphasis on the simplifications due to fermionic zero modes in the preexponential.

IV. THE CONDUCTIVITIES

It has been shown in Sec. II that the conductivities can be decomposed into two contributions $\sigma_{\mu\nu}^{(+)}$ and $\sigma_{\mu\nu}^{(-)}$, the first of which contains an energy integral whereas the second is determined by states at the Fermi energy only. As already mentioned, these two contributions are equal to each other in the one-instanton approximation, i.e.,

$$\sigma_{\mu\nu}^{(-)}(\omega)=\sigma_{\mu\nu}^{(+)}(\omega) , \tag{4.1}$$

to leading order in ω. The proof of this equality will be given in Sec. IV B. To begin with, we want to determine the low-frequency asymptotics of the conductivities $\sigma_{\mu\nu}^{(+)}$ in analogy to the previous calculation of the density of states.

A. The energy integrals $\sigma^{(+)}$

Let us first consider $\sigma_{xx}^{(+)}$ given in Eqs. (2.16). All states below the Fermi energy contribute to this quantity. We obtain the leading-order contribution of the averaged product of two retarded Green's contribution if we choose the bosonic fields in the preexponential [see Eq. (2.23)] to be the saddle-point solution, i.e.,

$$\overline{G^+(\mathbf{r},\mathbf{r}')G^+(\mathbf{r}',\mathbf{r})} = -\int [d\bar{\Phi}][d\Phi] s_{\rm cl}(\mathbf{r}) s_{\rm cl}^*(\mathbf{r}') \chi(\mathbf{r}') \bar{\chi}(\mathbf{r}) \exp(-S) . \tag{4.2}$$

In the one-instanton approximation, the determinants and Jacobians are exactly the same as those in Eq. (3.12). Consequently,

$$I_2(\mathbf{r}_1,\mathbf{r}_2;E)=-\pi\rho(E)\cdot A^2 \int |\varphi^{(0)}(\mathbf{r}_1-\mathbf{r}_0)|^2 |\varphi^{(0)}(\mathbf{r}_2-\mathbf{r}_0)|^2 dr_0 . \tag{4.3}$$

With A and ρ from Eqs. (3.7) and (3.13), one can show that, to leading order,

$$\int_{-\infty}^{E_F} A^2(E)\rho(E)dE=\rho(E_F) \tag{4.4}$$

and

$$\frac{1}{V}\int (x_1-x_2)^2 |\varphi^{(0)}(\mathbf{r}_1-\mathbf{r}_0)|^2 |\varphi^{(0)}(\mathbf{r}_2-\mathbf{r}_0)|^2 dr_0 dr_1 dr_2 = 2l^2 , \tag{4.5}$$

so that the energy integral contribution to the longitudinal conductivity reads

$$\sigma_{xx}^{(+)}(\omega)=-i\omega e^2 l^2 \rho(E_F) . \tag{4.6}$$

This result confirms the previous statement that the range of validity of the one-instanton approximation is the localization regime which is characterized by a vanishing longitudinal dc conductivity and a finite dc polarizability $\chi=i\lim_{\omega\to 0}\sigma_{xx}(\omega)/\omega$. We would like to stress that this statement is equivalent to the criterion of the finite return probability of localized states usually formulated in terms of dc quantities (cf. Refs. 15 and 16), i.e. there are only localized states at a given energy E, provided that

$$\lim_{\eta\to 0^+} (\eta|\overline{G^+(\mathbf{r},\mathbf{r}';E)|^2})>0 , \tag{4.7}$$

with uniform convergence in $\mathbf{r},\mathbf{r}'$. In Sec. IV B we will indeed confirm that G^+G^- has a $(i\omega)^{-1}$ singularity and it is this fact which leads to $\sigma_{xx}^{(-)}=\sigma_{xx}^{(+)}$.

We have seen that the dominant low-frequency behavior of the longitudinal conductivity can be expressed in terms of the averaged product of two retarded Green's functions at the same energy. As has been shown above, $\sigma_{yx}^{(+)}$ cannot contain a term linear in ω because, due to translational and rotational invariance of S, the integral I_2 depends on $(x_1-x_2)^2+(y_1-y_2)^2$ and z_1-z_2 only and thus after spatial integration the corresponding contribution to $\sigma_{yx}^{(+)}$ vanishes. We want to illustrate this result for the Hall conductivity as follows:

$$\prod_{i=1}^{N} G^{-1}(\mathbf{r}_{i-1},\mathbf{r}_i;E)$$

with $\mathbf{r}_0=\mathbf{r}_N=\mathbf{r}$ describing the transition from a given point $\mathbf{r}$ via given intermediate positions $\mathbf{r}_i$ back to the initial place $\mathbf{r}$. The way in which the magnetic fields enters into the calculation is that it distinguishes trajectories with identical traces by their orientation. However, for $N=2$ this is not yet possible and only for $N\geq 3$ will the

orientation of the magnetic field manifest itself in the relevant products of Green's functions. This is the necessary condition for having a nonvanishing Hall conductivity and the leading contribution is indeed given by Eq. (2.19).

The arguments which allow us to determine J from Eq. (2.24) are only slightly more complicated than those yielding I_2. The first step of the calculation remains the same as previously, i.e., the bosonic fields in the preexponential are chosen to be the instanton solution

$$J(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3;E)=i\int[d\bar{\Phi}][d\Phi]\chi(\mathbf{r}_1)\bar{\chi}(\mathbf{r}_2)s_{\rm cl}(\mathbf{r}_2)s^*_{\rm cl}(\mathbf{r}_3)\chi(\mathbf{r}_3)\bar{\chi}(\mathbf{r}_1)\exp(-S)\ . \tag{4.8}$$

One pair of fermionic fields in the above preexponential has to come from zero modes, whereas, for the remaining pair, only the choice of modes with $m\geq 1$ yields a nonvanishing contribution to the functional integral. In analogy to Eq. (3.12), the result for I_3 reads

$$\begin{aligned}I_3(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3;E)=e^{-S_0}&\left|\det{}'\frac{1}{2}\frac{\partial^2 S}{\partial S_i\,\partial s_j}\right|^{-1/2}\det{}'\frac{\partial^2 S}{\partial\bar{\chi}\,\partial\chi}\left[\prod_i\frac{\Delta_i}{\sqrt{\pi}}\right]A^2\\&\times 2\pi\,\mathrm{Im}\left[\int dr_0\sum_{m=1}^{\infty}\frac{1}{\lambda_F^{(m)}}[\varphi^{(m)}(\mathbf{r}_1-\mathbf{r}_0)\varphi^{(m)*}(\mathbf{r}_2-\mathbf{r}_0)\varphi^{(0)}(\mathbf{r}_2-\mathbf{r}_0)\right.\\&\left.\times\varphi^{(0)*}(\mathbf{r}_3-\mathbf{r}_0)\varphi^{(0)}(\mathbf{r}_3-\mathbf{r}_0)\varphi^{(0)*}(\mathbf{r}_1-\mathbf{r}_0)-(\mathbf{r}_2\leftrightarrow\mathbf{r}_3)-(\mathbf{r}_1\leftrightarrow\mathbf{r}_3)]\right]\ .\end{aligned} \tag{4.9}$$

Again, the prefactors may be compared to the density of states. From the fermionic integration we obtain the product of all fermionic eigenvalues which do not have their corresponding eigenmodes in the preexponential. Therefore, $\lambda_F^{(m)}$ occurs as a denominator in Eq. (4.9) because, in the determinant $\det'\partial^2 S/\partial\bar{\chi}\,\partial\chi$, all nonzero eigenvalues are taken into account, although there is a pair of fermionic $m\neq 0$ modes in the preexponential.

Only a few terms of the above sum contribute to the Hall conductivity. This is due to the fact that the x,y dependence of I_3 is determined by contributions of the type

$$\mathrm{Im}[u^{(m)}(\mathbf{r}_i)u^{(m)*}(\mathbf{r}_2)u^{(0)}(\mathbf{r}_2)u^{(0)*}(\mathbf{r}_3)u^{(0)}(\mathbf{r}_3)u^{(0)*}(\mathbf{r}_1)]=\frac{r_1^m r_2^m}{(2l^2)^m m!}\sin[m(\alpha_1-\alpha_2)]|u^{(0)}(\mathbf{r}_1)|^2|u^{(0)}(\mathbf{r}_2)|^2|u^{(0)}(\mathbf{r}_3)|^2\ , \tag{4.10}$$

where α_i denotes the polar angle of $\mathbf{r}_i$. Since the sine functions $\sin(m\phi)$ are orthogonal for different positive integers m, on the one hand, and the orientated area $\Delta(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)$ contains only terms proportional to $\sin(\alpha_i-\alpha_j)$ on the other hand, only the $m=1$ modes contribute to $\sigma_{yx}^{(+)}$ in Eq. (2.19), i.e.,

$$\begin{aligned}\sigma_{yx}^{(+)}(\omega)&=\frac{e^2\omega^2}{6\pi V}\int\int\int\int f(E)\Delta(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)I_3^{(1)}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3;E)dr_1\,dr_2\,dr_3\,dE\ ,\\ I_3^{(1)}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3;E)&=\frac{2\pi A^2}{l^2\lambda_F^{(1)}}\rho(E)\Delta(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)\int\prod_{i=1}^{3}|\varphi^{(0)}(\mathbf{r}_i-\mathbf{r}_0)|^2\,dr_0\ .\end{aligned} \tag{4.11}$$

One can check by direct calculation that

$$\frac{1}{V}\int\int\int\int\Delta^2(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)\prod_{i=1}^{3}|\varphi^{(0)}(\mathbf{r}_i-\mathbf{r}_0)|^2 dr_0\,dr_1\,dr_2\,dr_3=\tfrac{3}{2}l^4\ . \tag{4.12}$$

In 2D, the fermionic $m=1$ eigenvalue is $\lambda_F^{(1)}=|\varepsilon|/2$ and, after inserting Eq. (4.12) into Eq. (4.11), the Hall-conductivity part $\sigma_{yx}^{(+)}$ finally reads

$$\sigma_{yx}^{(+)}(\omega)=\frac{e^2\omega^2}{\Gamma_2^2}\times 2l^2 n\ , \tag{4.13}$$

where n is the number of particles, i.e., $n=\int_{-\infty}^{E_F}\rho(E)dE$. In 3D, we obtain, correspondingly,

$$\sigma_{yx}^{(+)}(\omega)=\mathrm{const}\times\frac{e^2\omega^2}{\Gamma_3^2}l^2\sqrt{\Gamma_3/|\varepsilon|}n\ . \tag{4.14}$$

We already stated in Eqs. (2.11) and (2.20) that the conductivities have two contributions of equal magnitude, one of which we calculated above. We present the explicit calculation of $\sigma_{\mu\nu}^{(-)}$ in Sec. IV B.

Finally, we have to comment on the corrections to the leading ω dependence. Whereas above we have calculated the imaginary part of the longitudinal conductivity, the real part is logarithmic and originates from tunneling between two instantons

$$\sigma_{xx}(\omega)=c_1\times i\omega\rho(E_F)+c_2\omega^2\ln^{\nu}\left[\frac{1}{\omega^2}\right]\ . \tag{4.15}$$

For $B=0$, Houghton *et al.*[17] derived $\nu=d+1$ in the hy-

drodynamic limit. For the lower tail of the LLL, Apel[18] obtained $\nu=1$ in 2D ($d=2$). Both approaches were based on the optimal fluctuation method. Recently the result of Houghton *et al.* were generalized for $d=1$ to the limiting case of weak disorder using the method of supersymmetric functional integrals (see Ref. 19).

The leading-order corrections to the real part of the Hall conductivity also come from $\sigma_{yx}^{(+)}$. The relative correction is proportional to ω^2 and to the cyclic combination of all products of five Green's functions. If, for convenience, we express this product in terms of two pairs of superfields, we see that the instanton solutions of both fields are involved in the preexponential which gives rise to an additional factor A^2 as compared to the leading order. Another factor A^2 comes from the additional Jacobian due to the rotation between Φ_1 and Φ_2. We thus obtain a relative correction proportional to $\omega^2 A^4$ which, in two dimensions, reads

$$\Delta \mathrm{Re}\sigma_{yx}(\omega)/\mathrm{Re}\sigma_{yx}(\omega) \propto \Omega^2, \quad \Omega=\omega\varepsilon_F/\Gamma_2^2 . \tag{4.16}$$

We again see that the characteristic energy scale is determined by the disorder-induced broadening.

B. The Fermi-surface contributions $\sigma^{(-)}$

In this section we want to prove explicitly that, in the lower tail of the LLL Eq. (4.1) is valid. For this purpose we explicitly calculate $\sigma_{\mu\nu}^{(-)}$. The $G^{\pm}G^{\pm}$ terms in $K^{(-)}$ are analytical in ω and thus can only contribute to $\sigma_{xx}^{(-)}$ in the order ω^n, $n\geq 2$ [see Eq. (2.12)]. The leading term (i.e., for $\omega=0$) is rotational invariant and does not contribute to the Hall conductivity. Only higher-order terms contribute and give rise to terms proportional to ω^n, $n\geq 3$.

Consequently, we are just left with $\overline{G^+G^-}$. In the functional-integral representation of the mixed two-particle Green's function, two pairs of supervectors are involved

$$\begin{aligned}\overline{G^+(\mathbf{r},\mathbf{r}')G^-(\mathbf{r}',\mathbf{r})}&=\int [d\bar{\Phi}_1][d\Phi_1][d\bar{\Phi}_2][d\Phi_2]\chi_1(\mathbf{r})\bar{\chi}_1(\mathbf{r}')\chi_2(\mathbf{r}')\bar{\chi}_2(\mathbf{r})\exp(-S_2) ,\\ S_2&=-i\int\bar{\Phi}[\sigma_3(E-H_0)+\tfrac{1}{2}\omega+i\eta]\Phi\, dr+\frac{\lambda}{2}\int(\bar{\Phi}\sigma_3\Phi)^2\, dr ,\end{aligned} \tag{4.17}$$

with

$$\Phi=(\Phi_1,\Phi_2),\sigma_3=\begin{bmatrix}1&0\\0&-1\end{bmatrix} .$$

After performing the same steps as those following Eq. (3.4) the one-instanton solution of the action S_2 can be parametrized as follows:

$$\begin{bmatrix}s_1\\s_2\end{bmatrix}_{\mathrm{cl}}(\mathbf{r})=\begin{bmatrix}e^{i\varphi_1}&0\\0&e^{-i\varphi_1}\end{bmatrix}\begin{bmatrix}\cosh(\vartheta/2)&\sinh(\vartheta/2)\\\sinh(\vartheta/2)&\cosh(\vartheta/2)\end{bmatrix}\begin{bmatrix}e^{i\varphi_2}&0\\0&e^{-i\varphi_2}\end{bmatrix}\begin{bmatrix}s_{\mathrm{cl}}(\mathbf{r})\\0\end{bmatrix} \tag{4.18}$$

with s_{cl} given in Sec. III. The corresponding saddle-point value of the action is

$$S_{20}=\frac{A^2|\varepsilon|}{2}+(\omega/2+i\eta)A^2\cosh\vartheta . \tag{4.19}$$

The angles $\varphi_1,\varphi_2,\vartheta$ parametrize SU(1,1) and the invariant measure of the group is

$$d\Omega=\sinh\vartheta\, d\vartheta\, d\varphi_1\, d\varphi_2 . \tag{4.20}$$

Integrating over the angles thus yields

$$\begin{aligned}&\int\exp(-S_{20})d\Omega\\&\quad=4\pi^2(\omega/2+i\eta)^{-1}A^{-2}\exp(-A^2|\varepsilon|/2) .\end{aligned} \tag{4.21}$$

The quadratic operators occurring in the saddle-point approximation are

$$\begin{aligned}\frac{1}{2}\frac{\partial^2 S_2}{\partial s_{1i}\,\partial s_{1j}}&=H_0-E-\omega/2-i\eta-\lambda s_{\mathrm{cl}}^2\begin{bmatrix}3&0\\0&1\end{bmatrix} ,\\ \frac{\partial^2 S_2}{\partial\bar{\chi}_1\,\partial\chi_1}&=H_0-E-\omega/2-i\eta-\lambda s_{\mathrm{cl}}^2 ,\\ \frac{1}{2}\frac{\partial^2 S_2}{\partial s_{2i}\,\partial s_{2j}}&=\frac{\partial^2 S_2}{\partial\bar{\chi}_2\partial\chi_2}=H_0-E+\omega/2+i\eta-\lambda s_{\mathrm{cl}}^2 .\end{aligned} \tag{4.22}$$

As in the previous calculations of the DOS and of $\sigma_{\mu\nu}^{(+)}$, the bosonic field s_1 has one negative mode. Besides the zero modes corresponding to the exact invariances of S_2, there are also almost zero modes. By the term *almost zero mode* we refer to the modes which become zero modes for a vanishing symmetry-breaking part and which have eigenvalues proportional to $\pm(\omega/2+i\eta)$. The additional almost zero modes compared to the calculation of the DOS are due to the invariance (at $\omega=\eta=0$) with respect to rotation in functional space [see Eq. (4.17)] parametrized by the three angles $\varphi_1,\varphi_2,\vartheta$ instead of just

one angle φ.

The functional integral in Eq. (4.16) is not more difficult to perform than the one for the density of states. First, note that the bosonic and fermionic modes corresponding to Φ_2 occur in pairs with the same eigenvalue so that the determinants cancel each other exactly. From the transformation of the bosonic zero modes to the angles of SU(1,1), an additional Jacobian factor proportional to A^2 arises as compared to the integration over one supervector only. Following this line of argument and noting that the leading contribution to Eq. (4.16) comes from the fermionic almost zero modes in the preexponential, we explicitly verify the validity of the Ward identity,

$$\rho(E)=\lim_{\eta\to 0^+}\left[\frac{\eta}{\pi V}\int\int |G^+(\mathbf{r},\mathbf{r}';E)|^2 dr\,dr'\right] . \tag{4.23}$$

Thus, we are able to deduce from Eqs. (2.11) and (2.12)

$$\sigma_{xx}^{(-)}(\omega)=\frac{e^2\omega}{2i}\rho(E_F)\frac{\int\int (x-x')^2\,|\varphi^{(0)}(\mathbf{r}-\mathbf{r}_0)|^2\,|\varphi^{(0)}(\mathbf{r}'-\mathbf{r}_0)|^2\,dr\,dr'}{\int\int |\varphi^{(0)}(\mathbf{r}-\mathbf{r}_0)|^2\,|\varphi^{(0)}(\mathbf{r}'-\mathbf{r}_0)|^2\,dr\,dr'} , \tag{4.24}$$

which completes the derivation of Eq. (4.1) for the longitudinal part.

Above, all Grassmann fields $\chi_1,\bar{\chi}_1,\chi_2,\bar{\chi}_2$ have been chosen as $m=0$ modes. However, in this approximation the two-particle Green's function from Eq. (4.16) is rotational invariant and cannot contribute to the Hall conductivity. With the same arguments as those given in Sec. IV A, we deduce that either $\chi_1,\bar{\chi}_1$ are $m=0$ modes and $\chi_2,\bar{\chi}_2$ are $m=1$ modes or vice versa. Since the almost zero modes corresponding to the two different pairs of fermionic fields have eigenvalues of equal modulus but opposite sign, the quantity G^+G^- contributes to $\sigma_{yx}^{(-)}$ in the form

$$\frac{\pi}{2l^2}\frac{1}{|\varepsilon|}\rho(E)(\mathbf{r}\times\mathbf{r}')_z\int |\varphi^{(0)}(\mathbf{r}-\mathbf{r}_0)|^2\,|\varphi^{(0)}(\mathbf{r}'-\mathbf{r}_0)|^2\,dr_0 . \tag{4.25}$$

As the last step of our derivation, we have to use the fact that, in the tails of the DOS $n(E)=\rho(E)\Gamma^2/2|\varepsilon|$. From this equality and a comparison of Eq. (4.24) with Eqs. (4.13) and (4.14), we finally get Eq. (4.1).

V. APPLICATIONS

As already mentioned in the Introduction, we want to apply our results to two sets of experiments, the first of which deals with magnetic-field-induced metal-insulator (MI) transitions in 3D uncompensatedly doped semiconductors. In these experiments the measured quantities are the dc resistivities ρ_{xx},ρ_{xy} which are defined by

$$\rho_{xx}=\lim_{\omega\to 0}\left[\frac{\sigma_{xx}(\omega)}{\sigma_{xx}^2(\omega)+\sigma_{yx}^2(\omega)}\right] ,$$
$$\rho_{xy}=\lim_{\omega\to 0}\left[\frac{\sigma_{yx}(\omega)}{\sigma_{xx}^2(\omega)+\sigma_{yx}^2(\omega)}\right] . \tag{5.1}$$

For the calculation of the dc resistivities it is necessary to introduce the frequency limit since, in the lowest localization region, both σ_{xx} and σ_{yx}, vanish and the conductivity tensor is not invertible. From Eqs. (4.1), (4.6), and (4.13), we deduce that, in the considered energy region, ρ_{xx} indeed diverges as ω^{-1} but the Hall resistivity remains finite and equals

$$\rho_{xy}=-\frac{B}{en}\times\begin{cases}\left(\dfrac{\Gamma_2}{2|\varepsilon|}\right)^2 & \text{for 2D ,}\\ \text{const}\times\left(\dfrac{\Gamma_3}{2|\varepsilon|}\right)^{3/2} & \text{for 3D .}\end{cases} \tag{5.2}$$

The finiteness of the Hall resistivity in localization regimes has already been observed in the quantum Hall regime of 2D systems. However, the present feature is essentially different because, for any plateau corresponding to $\sigma_{yx}=\nu e^2/h$, $\nu=1,2,3,\ldots$, the above result is trivial and we have $\rho_{xy}=\sigma_{yx}^{-1}$, whereas for the $\nu=0$ plateau in 2D and the lowest localization regime in 3D it depends essentially on the low-frequency asymptotics of both $\sigma_{xx}(\omega)$ and $\sigma_{yx}(\omega)$.

We conclude that, in degenerately doped semiconductors which exhibit metallical behavior at low magnetic fields and where freezing out occurs in the limit $B\to\infty$, there is, in general, an intermediate region of magnetic-field values where localization is the predominant effect; the localization and the freezing-out regime can be distinguished by the behavior of the Hall coefficient, which varies only a little in the first case but diverges in the latter.

The experiments of Hopkins *et al.*[2] agree with the above statements. In Ref. 1 we discussed the applicability of our model to a system characterized by the parameters given in Ref. 2,

$$\gamma=\omega_c/2E_B=a_B^2/l^2=0.2$$

and $\xi=\omega_c/E_F$, where E_B is the effective donor binding energy and a_B the effective Bohr radius, and found that the disorder-induced broadening obeys

$$\Gamma_3\gtrsim 10^3(na_B^3)^2\gamma^{-2}E_B .$$

After using the Mott criterion for the critical doping concentration $na_B^3\gtrsim 0.02$, we obtain $\Gamma_3\gg E_B$, so that freezing out can, at most, be a minor competitive effect in agreement with the conclusion of the authors of Ref. 2.

The second set of experiments for which our theory

turned out to be useful are transmission measurements of GaAs-$Al_xGa_{1-x}As$ heterostructures in microwave waveguides (see Refs. 4 and 5). The measured quantity was $\sigma_{yx}^2(\omega)$ as a function of the magnetic field up to magnetic-field values corresponding to a filling vector $\nu=2$ of non-spin-resolved LL's. Obviously, we cannot directly apply the results of Sec. IV to this case because they have been obtained for the lower tail of the LLL only. For higher localization regions, the situation is more difficult due to the presence of extended states below the Fermi energy. A direct application of the instanton approximation is impossible as discussed in Ref. 11. However, assuming that, for the completely filled LLL, the conductivities are the same as in the disorder-free case—which is reasonable, provided that the overlap to adjacent LL is negligible, i.e., $\hbar\omega_c/\Gamma \gg 1$—we can formulate the following symmetry relation:

$$\sigma_{yx}(\omega;\hbar\omega_c/2+\varepsilon_F)=\frac{e^2}{h}\frac{1}{1-\omega^2/\omega_c^2} -\sigma_{yx}(\omega;\hbar\omega_c/2-\varepsilon_F) \tag{5.3}$$

between the Hall conductivity in the lower tail of the LLL and the upper localization region of the LLL, i.e., for $\Gamma \ll \varepsilon_F < \hbar\omega_c/2$, $\varepsilon_F = E_F - \hbar\omega_c/2$. In the Appendix, a proof will be given within the one-band model. Inserting the Hall conductivity from Eqs. (2.4), (2.13), and (4.11) yields

$$\sigma_{yx}(\omega)=\frac{e^2}{h}\left[\frac{2}{1-\omega^2/\omega_c^2}-\left[\frac{2\hbar\omega}{\Gamma}\right]^2(2-\nu)\right], \tag{5.4}$$

where $\nu=2\pi l^2 n$ is the filling factor of the LL and $1-\nu/2$ gives the fraction of unoccupied states in the upper localization regime. Let b denote the magnetic field measured from the center of the Hall plateau at zero frequency in T and $\omega=\tilde{\omega}\times(10^3$ GHz). Furthermore, we use that the bandwidth Γ is related to the lifetime τ and the corresponding mobility at zero magnetic field $\mu=e\tau/m$, $m=0.067m_e$ for GaAs

$$\mu=\tilde{\mu}\times(10^5\ \mathrm{cm^2/V\,s}),$$

by $\Gamma^2=2\omega_c/\pi\tau$. Inserting into Eq. (5.4) the density $n=2\times10^{11}\ \mathrm{cm^{-2}}$ as a typical experimental parameter (cf. Ref. 5) we obtain, in the vicinity of the $\nu=2$ plateau,

$$\sigma_{yx}^2(b;\tilde{\omega})\simeq 4\frac{e^4}{h^2}(1-\tilde{\mu}\tilde{\omega}^2 b). \tag{5.5}$$

It is the square of the Hall conductivity which has to be compared to the experimental results since the bolometer signal in microwave transmission measurements is proportional to σ_{yx}^2. Instead of a plateau at $\omega=0$, the Hall conductivity exhibits a finite slope, which increases with the frequency and the mobility.

VI. CONCLUSIONS

We have demonstrated that, in the lowest localized region, the Hall conductivity is proportional to ω^2 and to the number of localized states per unit area. The knowledge of the low-frequency asymptotics of the Kubo conductivities in the lowest localization region enabled us to invert the conductivity tensor, which is singular at zero frequency. We thus obtained that in spite of a diverging longitudinal resistivity the dc Hall resistivity remained finite. This feature differs from the one that is familiar from the quantum Hall regime of a 2D EG because we consider the lower tail of the LLL where all states are localized and edge states do not exist.

For 2D systems there are many localization regions where the instanton approximation is also valid. We were able to derive a relation between the Hall conductivity below and above the LLL and thereby to study the deformation of the Hall plateaus at low frequencies.

Both results could be related to recent experiment and we found good qualitative agreement of the main features. Nevertheless, the range of validity of our predictions is still rather restricted since we had to assume that the Fermi energy is situated in a region where the density of states is exponentially small. A considerable effort and probably some additional ideas will be necessary to obtain similar results for regions of larger density of states.

APPENDIX

In this appendix we will show that the force-force correlation

$$\gamma_F(\omega)=\beta\int_0^\infty e^{i\omega t-\eta t}\langle F;F(t)\rangle dt\ , \tag{A1}$$

$$F=\frac{1}{\sqrt{2}}\frac{e}{m}\left[\frac{\partial V}{\partial x}+i\frac{\partial V}{\partial y}\right],$$

which is related to the conductivity

$$\sigma(\omega)=\beta\int_0^\infty e^{i\omega t-\eta t}\langle j;j(t)\rangle dt\ , \tag{A2}$$

$$j=\frac{1}{\sqrt{2}}(j_x+ij_y)\ ,$$

by the identity[20]

$$\sigma(\omega)=\frac{e^2 n}{m}\frac{i}{\omega-\omega_c}+\frac{\gamma_F(\omega)}{(\omega-\omega_c)^2}\ , \tag{A3}$$

vanishes in the one-band model for complete filling of the LL. Throughout the following proof, the potential is assumed to have finite range so that the averages of expressions containing the special derivatives of V exist. Since the result is independent of the potential range, it is valid also in the limit of a white-noise potential and we can deduce from Eq. (A3) that

$$\sigma_{yx}(\omega)=\frac{i}{2}[\sigma(\omega)-\sigma^*(-\omega)]=\frac{e^2}{h}\frac{1}{1-\omega^2/\omega_c^2} \tag{A4}$$

for the completely filled LLL and thus Eq. (5.3) is valid in the upper localization regime. In analogy to Eq. (2.7), we may express the force-force correlation in the form

$$\gamma_F(\omega)=\frac{1}{2\pi V}\int dE\, f(E)\mathrm{Tr}[\overline{F(G^- - G^+)F^+G^-_{-\omega}G_-} - \overline{FG^+_{\omega}G^+F^+(G^- - G^+)}] , \tag{A5}$$

where we have used the notation $G^{\pm}_{\omega}=G^{\pm}(E+\omega)$. Shifting, as before, the energies by $\pm\omega/2$, the correlation can again be decomposed into two contributions, the first of which depends on states near the Fermi energy only and thus vanishes with the DOS as $E_F\to\infty$. The second contribution is the energy integral

$$\gamma_F^{(+)}(\omega)=\frac{1}{4\pi}\int f(E)\mathrm{Tr}(\overline{FG^+_{\omega/2}F^+G^+_{-\omega/2}G^+} - \overline{FG^+_{\omega/2}G^+F^+G^+} + \overline{FG^-_{\omega/2}G^-F^+G^-_{-\omega/2}} - \overline{FG^-F^+G^-_{-\omega/2}G^-})dE . \tag{A6}$$

Note that, in the LLL, the involved Green's functions read

$$G^{\pm}(\varepsilon;\mathbf{r},\mathbf{r}')=\mp i\int[d\bar{\Phi}][d\Phi]\chi(\mathbf{r})\bar{\chi}(\mathbf{r}')\exp(-S) , \qquad S=\mp i\int\bar{\Phi}(\varepsilon-V\pm i\eta)\Phi\, d^2r . \tag{A7}$$

The transformation $V\to\kappa V$ and $\Phi\to\kappa^{-1/2}\Phi$ maps $G^{\pm}(\varepsilon)$ onto $\kappa G^{\pm}(\kappa\varepsilon)$. Consequently, the energy integral (A6) satisfies the scaling relation

$$\gamma_F^{(+)}(\omega)=\gamma_{\kappa F}(\kappa\omega) \quad \text{as } E_F\to\infty \tag{A8}$$

already, before averaging. Consequently, for the completely filled LLL, γ_F is independent of the disorder strength and thus vanishes.

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