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Field work and dispersion relations of excitations on fractals

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An approach to finding the dispersion law of excitations on fractals for an arbitrary interaction is developed. It is based on the scale invariance of the probe-field work and allows one to express exponents of the dispersion law in terms of the spectral and Hausdorff dimensions. The expressions obtained for the dispersion-law exponent are different for vibrational (Goldstone-type) and dipolar (non-Goldstone-type) excitations, in agreement with previous results of Alexander and Orbach and those of the authors of the present paper.

I. INTRODUCTION

It is now well established that excitations of fractals also possess fractal properties and that their density of states and dispersion law have scaling forms. Scaling of the eigenstate density ν for the vibrations on fractals has been introduced by Alexander and Orbach[1] in the form

$$\nu \propto \omega^{\bar{\bar{d}}-1} , \tag{1}$$

where ω is the excitation frequency and $\bar{\bar{d}}$ is an index called the fracton dimension[1] or vibrational spectral dimension.[2]

The dispersion relation for the vibrations of fractals ("fractons") is also governed by $\bar{\bar{d}}$ and has the form[1]

$$\omega \propto L^{-D/\bar{\bar{d}}} , \tag{2}$$

where L is the coherence length of the excitation and D is the fractal (Hausdorff) dimension. In the trivial limit ($D=\bar{\bar{d}}=d$, where d is the dimension of the embedding space), Eq. (2) reproduces the dispersion law of a wave with constant speed, $\omega \propto L^{-1}$, with L as the wavelength. In fractals, L plays the role simultaneously of the wavelength and localization radius (the strong localization hypothesis[1,3]).

The dispersion relation (2) has been proven in Ref. 4 from very general mode-counting arguments, independently of any model-based considerations. However, it is clear *a priori* that Eq. (2) can only be valid for excitations of the Goldstone type, which are characterized by the absence of a gap in the spectrum and become running waves in the trivial limit ($D \to d$). For non-Goldstone excitations on fractals (e.g., of the plasmon type), one can expect a dispersion law different from Eq. (2), in particular, possessing a spectral gap for $D \to d$. Such a law for dipolar excitations on fractals has been derived in Ref. 5, and in fact, it is different from Eq. (2).

The problem of finding the dispersion relations for fractal excitations has also been highlighted by recent experimental developments. A detailed experimental investigation of light[6-8] and neutron scattering[9,10] from aerogels has given rich information about vibrational fractal excitations and crossovers between fracton and phonons, which are governed by the coherence length. On the other hand, an experimental study of the polar fractal excitations, which determine the fractal optical absorption and nonlinear responses in the visible and near-uv regions, has given results[11] confirming the theoretical prediction of enhanced optical nonlinearities.

It is clear from comparison of vibrational and dipolar excitations (see above) that the dispersion relation (2) is not universal, though it can describe a class of excitations. Also, either the vibrational or dipolar interactions are only approximations to describing the effect of microscopic interactions (including the Coulomb and exchange potentials). In particular, there always is the dipolar interaction between the oscillating groups in the vibrational problem. Therefore, there is clearly a need for an approach allowing one to find the dispersion relations of different branches of fractal excitations for interactions of a general type. This paper suggests such an approach and tests it for two models of vibrational and polar (plasmon) excitations, correctly reproducing the dispersion laws of Refs. 4 and 5.

Our approach is based on the self-similarity of a fractal and the idea that collective excitations of large coherence length L are insensitive to the details of the fractal structure at small scales. Thus one can change the spatial resolution R_0 of the fractal structure without changing the observable quantities. We choose the work of the probe field as the quantity whose invariance with respect to the change of R_0 is the condition to find the required dispersion laws.

In Sec. II the basic equations for the vibrational and polar excitations are considered and the corresponding expressions for the field work are obtained. Section III is devoted to examination of the predictions of scale invariance, where the scaling forms of the field work and the required dispersion relations are derived for the two models under consideration. The results are discussed in Sec. IV.

II. BASIC EQUATIONS

A. Vibrational excitations

We consider a fractal cluster consisting of N material particles positioned at points $\mathbf{r}_i$. For the sake of brevity, we designate the cluster simply as a fractal and the constituent particles as monomers. The small displacements $\mathbf{d}_i$ of the monomers induced by an external driving force $\mathbf{F}_i$, where $i=1,2,\ldots,N$ is the ordinary number of a monomer, obey the system of equations

$$\frac{\partial^2 d_{i\alpha}}{\partial t^2} = -u^2 \sum_j T_{i\alpha j\beta}(d_{i\beta}-d_{j\beta}) - \omega_0^2 d_{i\alpha} - \gamma \frac{\partial d_{i\alpha}}{\partial t} + F_{i\alpha} \ . \tag{3}$$

Here the Greek letters denote vector indices, with implied summation over repeated ones; u is a characteristic frequency (of the order of the Debye frequency); T is the interaction matrix, whose elements do not depend on the interaction strength (in the simplest case $T_{i\alpha j\beta} \propto \delta_{\alpha\beta}$ is unity for i,j indexing the nearest neighbors and zero otherwise); ω_0 is the restoring frequency, which may originate from the interaction of the monomers with the embedding host medium, and for a self-supporting fractal $\omega_0=0$; γ is a relaxation constant; and $F_{i\alpha}$ is the component of the weak (probe) force acting upon the ith monomer. Assuming the probe force to have the temporal dependence $\mathbf{F} \propto e^{-i\omega t}$ with frequency ω, one can, in a routine manner, introduce for each quantity the corresponding amplitude. Such amplitudes will be used below denoted by the same letters.

In treating the system (3), we shall employ the formalism of Ref. 5. Let us define the vector $|d)$ in a $3N$-dimensional linear space with components

$$(i\alpha|d) = d_{i\alpha} \ ,$$

and similarly for other quantities. Then the system of equations (3) can be rewritten as

$$(Z+W)\,|d) = |E) \ , \tag{4}$$

where the following definitions are made:

$$Z \equiv -X - i\delta, \quad X = \frac{\omega^2-\omega_0^2}{u^2}, \quad \delta = \frac{\omega\gamma}{u^2}; \qquad |E) = \frac{1}{u^2}|F) \ , \tag{5}$$

with X and δ being real parameters, and the operator W is determined by its matrix elements

$$(i\alpha|\,W|j\beta) = T_{i\alpha j\beta} - \delta_{ij} \sum_k T_{i\alpha k\beta} \ . \tag{6}$$

This operator is symmetrical and obeys the condition

$$\sum_j (i\alpha|\,W|j\beta) = 0 \ . \tag{7}$$

It follows from Eq. (7) that the homogeneous vectors $|0\alpha)$ with components

$$(i\alpha|0\beta) = \delta_{\alpha\beta} N^{-1/2} \tag{8}$$

are eigenvectors of W with zero eigenvalues. This is an exact condition for the excitations to be of the Goldstone type. It means that the homogeneous excitation is simply a shift of the system as a whole and brings about a gapless excitation spectrum with the excitation frequency tending to zero for $L \to \infty$ (the Goldstone theorem).

Let us introduce the eigenvectors $|n)$ of the operator W corresponding to the eigenvalues w_n. Since W is a symmetric operator, all the eigenvalues and the components of the eigenvectors[12] $(i\alpha|n)$ are real. The solution of Eq. (4) is expressed in terms of the linear response function (polarizability) χ,

$$d_{i\alpha} = \sum_j \chi_{i\alpha j\beta} E_{j\beta} \ , \tag{9}$$

where

$$\chi_{i\alpha j\beta} = \sum_n (i\alpha|n)(j\beta|n)(Z+w_n)^{-1} \ . \tag{10}$$

From this expression and the completeness of the set $|n)$, an exact sum rule[5] follows:

$$\int \chi_{i\alpha j\beta}\, dX = \pi \delta_{ij} \delta_{\alpha\beta} \ . \tag{11}$$

The contributions of the uniform eigenvectors (8) should be excluded from (10), since these contributions correspond to the movement of the system as a whole, and not to the internal excitations. Thus, in Eq. (10), we shall put $n \neq 0$. The condition of orthogonality to the eigenvectors (8) has the form[4]

$$\sum_i (i\alpha|n) = 0 \quad (n \neq 0) \ . \tag{12}$$

As a consequence of Eq. (12), the polarizability (10) obeys the condition

$$\sum_j \chi_{i\alpha j\beta} = 0 \ . \tag{13}$$

This condition shows that the constant component of the probe field $\mathbf{E}$ (or $\mathbf{F}$) vanishes from the solution (9) [or (10)]. Therefore, we can exclude this constant component, i.e., impose the condition

$$\sum_i E_{i\alpha} = 0 \quad \left[\sum_i F_{i\alpha} = 0 \right] \ . \tag{14}$$

Excluding from (10) the term with $n=0$ and taking (8) into account, one can obtain the sum rule for the polarizability of a Goldstonian system as

$$\int \chi_{i\alpha j\beta}\, dX = \pi \delta_{\alpha\beta} \left[\delta_{ij} - \frac{1}{N} \right] \ . \tag{15}$$

The work of the probe field A (more exactly, the power of the external field dissipated by the system) is given by

$$A = \overline{\left\langle \sum_i \frac{\partial \mathbf{d}_i(t)}{\partial t} \mathbf{F}_i(t) \right\rangle} \ , \tag{16}$$

where $\langle \cdots \rangle$ denotes averaging over the randomness of the fractal and $\overline{(\cdots)}$ over the randomness of the driving field; the observable quantity $\mathbf{d}_i(t)=\mathbf{d}_i e^{i\omega t}+\text{c.c.}$ and similarly for $\mathbf{F}_i(t)$. Using the definitions (5), Eq. (9), and the rotational symmetry of the cluster as a whole, we obtain, from Eq. (16),

$$A=\frac{2\omega}{\omega^2-\omega_0^2}X\left\langle \sum_{i,j} \operatorname{Im}\chi_{ij}\overline{\mathbf{F}(\mathbf{r}_i)\mathbf{F}^*(\mathbf{r}_j)}\right\rangle , \tag{17}$$

where $\chi_{ij}=\frac{1}{3}\chi_{i\alpha j\alpha}$, and we assumed the driving field to be a potential one, which means that the force $\mathbf{F}$ actually depends upon the coordinate of the corresponding particle, i.e., $\mathbf{F}_i=\mathbf{F}(\mathbf{r}_i)$.

B. Polar (plasmonlike) excitations

Let us consider the model of Ref. 5, which describes polar excitations, governing, in particular, optical absorption. In this model the fractal consists of polarizable monomers with the dipole interaction between them at the driving frequency ω. The fractal is subjected to an external electric field, whose amplitude at the site of the ith monomer is equal to $\mathbf{E}_i$. The induced (transition) dipole moments of the monomers $\mathbf{d}_i$ $(i=1,2,\ldots,N)$ obey the system of equations

$$Zd_{i\alpha}=E_{i\alpha}-\sum_j (i\alpha|W|j\beta)d_{j\beta} , \tag{18}$$

where $Z=\chi_0^{-1}$ and χ_0 is the polarizability of an isolated monomer,

$$(i\alpha|W|j\beta)=\left[\delta_{\alpha\beta}-\frac{(r_{ij})_\alpha(r_{ij})_\beta}{r_{ij}^2}\right]\frac{1}{r_{ij}^3} , \tag{19}$$

with $\mathbf{r}_{ij}\equiv\mathbf{r}_i-\mathbf{r}_j$.

The system (18) in the $3N$-dimensional vector space acquires exactly the form (4), though the meaning of the quantities is different: The operator W is determined by (19), $Z\equiv -X-i\delta=\chi_0^{-1}$, and $\mathbf{E}_i$ is the external field, which is not renormalized as distinct from (5). Correspondingly, the linear response expressions (9) and (10) are valid. However, the polar excitations are not Goldstonian, there exist no conditions such as (7), (12), and (13), and in Eq. (10) the sum is extended over all the eigenstates $|n\rangle$. As a consequence, a constant component (with zero wave vector) of the field $\mathbf{E}$ excites internal modes and in general should be kept in Eq. (9).

The field work has a form similar to Eq. (16), with the difference that the field performing the work is the original field $\mathbf{E}_i$,

$$A=\overline{\left\langle \sum_i \frac{\partial \mathbf{d}_i(t)}{\partial t}\mathbf{E}_i(t)\right\rangle} . \tag{20}$$

Using Eqs. (9) and (10), we obtain from (20)

$$A=2\omega\left\langle \sum_{i,j} \operatorname{Im}\chi_{ij}\overline{\mathbf{E}(\mathbf{r}_i)\mathbf{E}^*(\mathbf{r}_j)}\right\rangle . \tag{21}$$

This expression is different from its counterpart (17) for the Goldstonian (vibration) excitations.

III. SCALING AND DISPERSION LAWS

A. General relations

By the dispersion relation we understand the dependence of the eigenvalue of the interaction operator W on the excitation coherence radius L. The dependence upon the minimum scale R_0 of the fractal is also important for this paper. Thus the form of the dispersion relation is $w=w(L,R_0)$.

Scale invariance holds for the fractal excitation if its coherence radius L is well between the maximum and the minimum scales of the fractal,

$$R_0 \ll L \ll R_c , \tag{22}$$

where R_c is the total radius of the cluster and R_0 is the characteristic spacing between the closest monomers. Note that these two parameters and the Hausdorff dimension determine the number of monomers in the fractal,

$$N\sim(R_c/R_0)^D . \tag{23}$$

If the condition (22) is met, the excitation extends over many monomers and is not sensitive to details of the fractal at small scales. Therefore, the change of the minimum scale should not modify the functional form of the dispersion relation, while there should be no dependence whatsoever on the maximum scale R_c. In the usual way, to meet these requirements, w should be a power (scaling) function of L and R_0.

Let us introduce the metrical dimensionality a of W: When all the linear dimensions (including R_0, R_c, and L) are changed by the same factor, then w is changed proportionally to, say, R_0^a. For the vibrational model (6) considered above, the dimension a can be arbitrary, with the usual choice as $a=0$; for the dipolar model (19), $a=-3$. The change of w under changing only the coherence length L (without affecting R_0 or any other length) is given by a nontrivial index, which we denote by $-\kappa$: $w\propto L^{-\kappa}$. Thus, in the scaling region (22), the dispersion relation has the form

$$w\sim(L/R_0)^{-\kappa}R_0^a . \tag{24}$$

It is physically understandable that when L becomes large (i.e., for considerably delocalized excitations), then w becomes small. This means that $\kappa>0$. In some simple cases, the index κ can be determined analytically from the well-known decimation procedure.[2,13] For example, for vibrational excitations with the interaction between the nearest neighbors on the Sierpinski gasket in d dimensions,

$$\kappa=\ln(d+3)/\ln 2,\quad D=\ln(d+1)/\ln 2 . \tag{25}$$

The main idea of employing scale invariance to determine the dispersion law is in using the invariance of the field work A with respect to the change of the minimum scale R_0 (without changing any other linear dimensions). Such a transformation is equivalent to unifying a group of monomers entering a fractal "blob" to form a renormalized monomer and, therefore, has been called the renormalization transformation.[5] It is evident that the renormalized monomers constitute a fractal which is characterized by the same critical indices as the original fractal, though the strength of the interactions may change.

As one can see from the expressions (10), (17), and (21), the necessary condition for the field work to have a scaling form is the smallness of the dissipation,

$$\delta \ll X \ . \tag{26}$$

In what follows we shall consider Eq. (26) satisfied. Then the dissipation parameter δ cancels from Eq. (10), and the imaginary part of the polarizability acquires the form

$$\mathrm{Im}\,\chi_{i\alpha j\beta}=\pi \sum_n (i\alpha|n)(j\beta|n)\delta(X-w_n) \ . \tag{27}$$

Note that in this expression the term $n=0$, which should be excluded for the case of the Goldstonian excitations, along with (8) acquires the form

$$\frac{1}{N}\delta_{\alpha\beta}\delta(X) \ . \tag{28}$$

This term describes the movement of the system as a whole, and it vanishes if $X \neq 0$.

B. Dispersion relation for vibrational excitations

The fractal under the applied probe field is characterized by an additional large-scale distance, the field correlation length L_f. To bring about scaling of the field work, one should assume the excitation coherence radius to be much less than not only R_c, but also L_f; i.e., in addition to (22), we require

$$L \ll L_f \ . \tag{29}$$

We shall assume the probe field $\mathbf{F}(\mathbf{r})$ to be homogeneously random, which means that the field correlator $C \equiv \mathbf{F}(\mathbf{r}_i)\mathbf{F}^*(\mathbf{r}_j)$ depends only on the coordinate difference $\mathbf{r}_{ij}$.

As a consequence of Eq. (29), the field is weakly changing at relevant distances characteristic of the fractal excitations. Therefore, it is possible to expand the field correlation function in Eq. (17) into a series:

$$C(\mathbf{r}_{ij}) \approx C(0)+\mathbf{r}_{ij}\frac{\partial C}{\partial \mathbf{r}_{ij}}+\frac{1}{2}r_{ij\alpha}r_{ij\beta}\frac{\partial^2 C}{\partial r_{ij\alpha}\partial r_{ij\beta}} \ . \tag{30}$$

When substituted into Eq. (17), the first term of (30) cancels because of (13), and the second one vanishes by averaging over the orientation of the fractal as a whole. The third term can be estimated as $-(r_{ij}/L_f)^2|\mathbf{F}|^2$ [the minus sign follows from the fact that the correlator $C(\mathbf{r})$ has a maximum at $r=0$, and therefore, its second derivative should be negative]. Thus the field work (17) can be estimated as

$$A \sim -\frac{\omega}{\omega^2-\omega_0^2}X\overline{|\mathbf{F}(\mathbf{r})|^2}\frac{1}{L_f^2}\left\langle \sum_{i,j}\mathrm{Im}\,\chi_{ij}r_{ij}^2\right\rangle \ . \tag{31}$$

Let us identically transform the mean in Eq. (31):

$$\left\langle \sum_{i,j}\mathrm{Im}\,\chi_{ij}r_{ij}^2\right\rangle=\left\langle \sum_{i,j}\mathrm{Im}\,\chi_{ij}(r_i^2+r_j^2-2\mathbf{r}_i\mathbf{r}_j)\right\rangle \ . \tag{32}$$

The first two terms in parentheses vanish because of the condition (13), and Eq. (32) is reduced to the form

$$\left\langle \sum_{i,j}\mathrm{Im}\chi_{ij}r_{ij}^2\right\rangle=-2\left\langle \sum_i \mathrm{Im}\,\chi_{ii}r_i^2+\sum_{\substack{i,j\\(i\neq j)}}\chi_{ij}\mathbf{r}_i\mathbf{r}_j\right\rangle \ . \tag{33}$$

To estimate the relative magnitude of the two terms in Eq. (33), we shall employ an approach of Ref. 5 and consider the integrals over X of these terms, which can be calculated from the sum rule (15):

$$\begin{gathered}\int\left\langle \sum_i \mathrm{Im}\,\chi_{ii}r_i^2\right\rangle dX=\pi N\left\langle \frac{1}{N}\sum_i r_i^2\right\rangle \ , \\ \int\left\langle \sum_{\substack{i,j\\(i\neq j)}}\mathrm{Im}\chi_{ij}\mathbf{r}_i\mathbf{r}_j\right\rangle dX=-\pi N\left\langle \frac{1}{N}\sum_i \mathbf{r}_i\right\rangle^2 \ .\end{gathered} \tag{34}$$

One can easily see that the second integral in Eq. (34) is always smaller in absolute value than the first one, and it vanishes when the center of mass of the fractal is chosen as the origin of the coordinate system. Assuming this and taking into account a power-law dependence on X, we conclude that in the scaling region the second term in Eq. (33) is negligibly small.

To proceed further, we introduce formal definitions of two important quantities. The first is the density of eigenmodes $\nu(X)$, i.e., the number of eigenmodes per unit interval of X and per monomer:

$$\nu(X)=N^{-1}\left\langle \sum_n \delta(X-w_n)\right\rangle \ . \tag{35}$$

Second, the coherence radius $L=L(X)$ of the eigenmodes with a given eigenvalue $w_n=X$ is defined by

$$L^2=\frac{\langle \sum_i \chi_{ii}r_i^2\rangle}{\langle \sum_i \chi_{ii}\rangle}\equiv\frac{\langle \sum_{i,n}r_i^2(i\alpha|n)^2\delta(X-w_n)\rangle}{\langle \sum_{i,n}(i\alpha|n)^2\delta(X-w_n)\rangle} \ . \tag{36}$$

This definition has a clear quantum-mechanical analogy (cf. Ref. 14), with $(i\alpha|n)$ as the wave function.

Using the completeness of the vector set $|i\alpha)$ and Eq. (35), we obtain from Eq. (27) the equality

$$\left\langle \sum_i \chi_{ii}\right\rangle=\frac{\pi}{3}N\nu(X) \ , \tag{37}$$

which is exact for $X \neq 0$ [see (28)]. Taking into account Eqs. (36) and (37) and neglecting the second term in Eq. (33) on the ground of the above arguments, we obtain

$$\left\langle \sum_{i,j}\mathrm{Im}\,\chi_{ij}r_{ij}^2\right\rangle=-2NL^2\nu(X) \ . \tag{38}$$

Finally, substituting (38) into Eq. (31), the required estimate of the field work is

$$A \sim \frac{\omega}{\omega^2-\omega_0^2}\overline{|\mathbf{F}|^2}\left[\frac{L}{L_f}\right]^2 NX\nu(X) \ . \tag{39}$$

This expression possesses a feature generally characteristic of Goldstone excitations: The field work tends to zero when the field correlation length becomes large. This can be understood since the field in the limit $L_f \rightarrow \infty$ becomes homogeneous, and such a field induces a shift of the system as a whole and cannot excite internal Goldstone modes.

If X is in the scaling region, i.e., the length $L(X)$ obeys (22), then the probe field interacts with fractal excitations delocalized over many monomers, and changes of the minimum scale R_0 of the fractal should not affect the field work. Hence A [Eq. (39)] should not depend on R_0. The external (driving) frequency ω, obviously, does not depend on R_0. The restoring frequency ω_0 also does not depend on R_0: When some monomers are unified to a new (renormalized) monomer, their total mass and restoring force are increased in proportion to their number. The excitation coherence length L itself should not depend on R_0. Thus, taking (23) into account, we obtain, from Eq. (39),

$$R_0^{-D} X \nu(X) \propto R_0^0 \; . \tag{40}$$

As suggested in a number of papers (e.g., Refs. 1–5) and confirmed by model calculations,[5,13,14] in the intermediate region (22) the density of states obeys scaling, which, accounting for the metrical dimensionality of W and X, has the form

$$\nu(X) \sim R_0^{-a} |R_0^{-a} X|^{d_X - 1} \; , \tag{41}$$

where the index d_X is called the spectral dimensionality. Substituting (41) into (40), we obtain the transformation law

$$|X| \propto R_0^{(D + a d_X)/d_X} \; . \tag{42}$$

Accordingly to Eqs. (21) and (27), the probe field excites the fractal modes with $w = X$. Taking this into account and comparing the power of R_0 in Eqs. (24) and (42), one obtains $\kappa = D/d_X$ or

$$|X| \sim (L/R_0)^{-D/d_X} R_0^a \; . \tag{43}$$

This relation, with the choice of X as independent variable instead of ω and the notation of the spectral dimensionality d_X instead of $\bar{\bar{d}}$, is the same as Alexander and Orbach's law (2). However, let us point out that the value of the critical index d_X depends on the definition of the critical variable X (see also below). Note that κ does not depend on the metrical dimension a.

When expressed in terms of frequency with the use of (5), Eq. (43) becomes

$$|\omega^2 - \omega_0^2| \sim (L/R_0)^{-D/d_X} u^2 R_0^a \; . \tag{44}$$

In the long wavelength limit ($L \rightarrow \infty$), $\omega \rightarrow \omega_0$, and for $\omega_0 \neq 0$, from (44) one obtains the scaling law

$$|\omega - \omega_0| \propto L^{-D/d_X} \; . \tag{45}$$

Let us consider the case $\omega_0 = 0$, which holds, in particular, for vibrations in a self-supporting fractal. Then $X \propto \omega^2$, and it is natural to use ω as the independent variable. Transforming the density of states using the equality $\nu(\omega) d\omega = \nu(X) dX$, from (41) one obtains Eq. (1), where

$$\bar{\bar{d}} = 2 d_X \; . \tag{46}$$

With this value and $\omega_0 = 0$, Eq. (44) is exactly equivalent to Alexander and Orbach's relation (2). We note that in the trivial limit ($\bar{\bar{d}} = D = d$), the dispersion law (44) along with (46) acquires the familiar form

$$\omega = (\omega_0^2 + c k^2)^{1/2} \; , \tag{47}$$

where the wave vector $k \sim L^{-1}$ and $|c| \sim u^2 R_0^{2+a}$, which describes different branches of elementary excitations in condensed media (see also Sec. IV).

C. Dispersion relation for polar excitations

Unlike Goldstone excitations, in the case of polar excitations the polarizability does not obey the condition (13). Thus the relevant expression (21) for the field work does not vanish for the uniform field $\mathbf{E}$. Physically, such a field corresponds to the excitation of a fractal by electromagnetic radiation with wavelength much longer than the cluster size R_c. In this case the first term $C(0) = \overline{|\mathbf{E}|^2}$ in the expansion (30) gives the main contribution, and the field work (21) becomes

$$A = 2\omega N \chi \overline{|\mathbf{E}|^2}, \quad \chi = \left\langle \frac{1}{N} \sum_{i,j} \chi_{ij} \right\rangle \; , \tag{48}$$

where χ is the polarizability of the cluster per monomer. To express this in terms of the density of states, we again follow Ref. 5 in using the sum-rule approach.

We decompose χ into two terms,

$$\chi = \frac{1}{N} \left\langle \sum_i \chi_{ii} + \sum_{\substack{i,j \\ (i \neq j)}} \chi_{ij} \right\rangle \; . \tag{49}$$

From the sum rule (11), one sees that the second term in Eq. (49) vanishes when integrated over X. Taking into account the power dependence of χ in the scaling region, we conclude that this term should be small in this region, and using Eq. (37), we obtain from (49) the relation[5] $\chi(X) = (\pi/3)\nu(X)$. Taking this into account, we get from (48) the required expression for the field work:

$$A = \frac{2\pi}{3} \omega \overline{|\mathbf{E}|^2} N \nu(X). \tag{50}$$

Using Eqs. (23) and (41), we arrive at the conclusion that the A-invariance requirement applied to Eq. (50) leads to the transformation law

$$|X| \propto R_0^{(D + a d_X)/(d_X - 1)} \; . \tag{51}$$

Comparing (51) to (24), we find $\kappa = -(D+a)/(d_X - 1)$ and the dispersion relation is

$$|X| \sim \left[\frac{L}{R_0} \right]^{-(D+a)/(d_X - 1)} R_0^a \; , \tag{52}$$

which clearly differs from Eq. (43) for vibrational excitations. Note that now κ depends on a.

Equation (52) gives X as a function of L and, through the dependence $X = -\mathrm{Re}\chi_0^{-1}(\omega)$, the relation between the eigenfrequency ω and L. The last relation, distinct from $X(L)$, does not necessarily scale. Note that at the resonance of a monomer, the polarizability χ_0 has a pole in frequency, and in its vicinity $X \approx K(\omega-\omega_0)$, where ω_0 is the resonance frequency and $K=\text{const}$.

For the fractal response to the optical (electromagnetic) field, the interaction is given by Eq. (19), so that its metrical dimension is $a=-3$. In this case the dispersion law (52) becomes

$$|X| \sim \left[\frac{L}{R_0}\right]^{-(3-D)/(1-d_o)} R_0^{-3} , \tag{53}$$

where d_o is the notation of Ref. 5 for d_X in the dipole case.[15] The dispersion relation (53) is in full agreement with the corresponding one of Ref. 5. For the interaction (19), the constant $K=\hbar/|\mathbf{d}_{12}|^2$, where $\mathbf{d}_{12}$ is the dipole matrix element of the resonant transition in the monomer, and in the vicinity of the resonance, Eq. (53) expressed in terms of frequency becomes [cf. Eq. (44)]

$$|\omega-\omega_0| \sim \left[\frac{L}{R_0}\right]^{-(3-D)/(1-d_o)} R_0^{-3}|\mathbf{d}|_{12}^2/\hbar . \tag{54}$$

For the trivial limit ($D=3$), we obtain from (53) the dispersion relation

$$|X| \sim R_0^{-3} , \tag{55}$$

or, in terms of frequency,

$$|\omega-\omega_0| \sim R_0^{-3}|\mathbf{d}|_{12}^2/\hbar . \tag{56}$$

In this case the excitations are dispersionless (no dependence on L), and their eigenfrequency ω is shifted from that ω_0 of the isolated monomer by an amount which depends on the interaction [unlike Eq. (47)].

The relation (55) receives a simple electrodynamic interpretation. For $D=3$ the cluster is characterized by its mean density $\rho=R_0^{-3}$ and the dielectric permittivity $\epsilon=1+4\pi\rho\chi_0$. Assuming for simplicity the cluster to have a spherical form, its polarizability (per monomer) χ is found from the usual electrodynamic expression $\chi=R_0^3(\epsilon-1)(\epsilon+2)^{-1}$. The dispersion relation of eigenmodes follows from $\mathrm{Re}\chi=0$, which yields the well-known relation for surface plasmons, $\mathrm{Re}\epsilon=-2$ or $\mathrm{Re}\chi_0=-3R_0^3/4\pi$. This expression, considering the definition of $X=-\mathrm{Re}\chi_0^{-1}$ and the implied smallness of $\mathrm{Im}\chi_0$, agrees with Eq. (55). In consequence, Eq. (56) correctly reproduces the spectral gap of surface plasmons, toward which the polar excitations of a fractal tend in the trivial, $D\rightarrow 3$, limit. Hence the present theory agrees with the corresponding electrodynamics results for continuous media in the case under consideration, as expected.

IV. DISCUSSION

The dispersion relation (2) has been obtained in Ref. 4 for vibrations of fractals from very general mode-counting arguments. Initially, one could think that it is universal and that its form would not depend on the interaction between monomers. However, a different form of the dispersion relation [see Eq. (53)] has been derived in Ref. 5 using another approach. The resolution of this apparent controversy is that the expression for the exponent of the dispersion law (denoted above as κ) in terms of the spectral and Hausdorff dimensions is not universal. It depends on the interaction, though it may be the same for a class of interactions. One of the motivations for this work was to obtain the above two dispersion relations from a single approach.

In this paper we have suggested a general method for finding the dispersion relation of fractal excitations, which is a generalization of the approach of Ref. 5. This method is based on the idea that the work of a probe field cannot depend on the choice of a minimum scale in the fractal: This choice is arbitrary and should not affect the long-correlation-length excitations, which determine the field work.

Since always $D>0$ and $\bar{\bar{d}}>0$ is finite, Alexander and Orbach's dispersion relation (2) predicts that $\omega\rightarrow 0$ in the long-wavelength limit $L\rightarrow\infty$. This feature for the trivial limit ($D,\bar{\bar{d}}\rightarrow d$) means that the excitations considered always possess a gapless spectrum and, consequently, belong to the Goldstone type. Therefore, it was natural to suggest that for any Goldstone-type excitation, which tends to a uniform shift of the system in the infinite-wavelength limit, the dispersion law should be given by Eq. (2) or (43), while for non-Goldstone-type excitations different dispersion relations should be valid.

To examine this conjecture, it seemed natural to add to the "normal" vibration force a restoring component [the term proportional to ω_0^2 in Eq. (3)], which brings about the spectral gap [cf. (47)] and therefore makes the excitation to be non-Goldstonian. However, by an obvious change of the spectral variable from ω to X [Eq. (5)], the problem has been again reduced to a Goldstone one, for which the present approach correctly reproduces the Alexander and Orbach's relation in a generalized form [see Eqs. (43)–(45)]. Note that the spectral dimension d_X used above is one-half of the vibrational ("fracton") spectral dimension $\bar{\bar{d}}$ from Ref. 1 [see Eq. (46)]. For the dipole problem, the present approach yields the relation (52), which is different from that (43) for vibrations and is in accord with the dispersion law (53) obtained in Ref. 5.

Let us discuss the formal and physical origins of the difference of the dispersion relations for the two problems considered. By the choice of the spectral variable $Z=-X-i\delta$ [see Eqs. (5) and (18)], the basic equations for the two problems are reduced to the common equation (4) in $3N$-dimensional vector space. However, there are two differences between the two problems. First, for vibrations the field $\mathbf{E}$, which appears in the basic equation (4), is not the physical field ($\mathbf{F}$) [see Eq. (5)], while for polar excitations it is. Second, for vibrations the interaction obeys the Goldstone requirement (7), which brings about the conditions (12)–(14). As a result of these differences, the expressions for the field work for the two problems under consideration are different [cf. Eq. (17) with (21) and also Eq. (39) with (50)]. In particular, for vibrations, as for any Goldstone excitation, the field work (39) tends to zero in the limit of constant exciting field ($L_f\rightarrow\infty$), which is not the case for non-Goldstone excitations.

Therefore, it is not surprising that the dispersion laws (43) and (52), obtained from the scale-invariance requirements for the corresponding field-work expressions, are different. What is surprising is that this difference stems only from the difference in fields **E** and **F** and has explicitly nothing to do with the Goldstone requirements mentioned above. Technically, the difference in the fields brings about an extra power of X in Eq. (39) as compared to Eq. (50), which accounts for all the difference between the dispersion laws.

However, there does exist an intimate connection of the form of the dispersion relation with the Goldstone requirements. The simplest way to demonstrate this is to compare the results (47) and (56) for long-wavelength ($L \rightarrow \infty$) excitations in the trivial limit. For the vibration problem, the eigenfrequency ω in this limit coincides with the frequency $\omega_0=\text{const}$ independently from the interaction, whose effect cancels because of the requirement (7). This feature is inherent to a Goldstone mode. On the contrary, for the dipole problem [see Eq. (53)] the eigenfrequency ω is shifted from the resonant frequency ω_0 of the individual monomer by an amount determined by the interaction and proportional to the density of monomers $\rho=R_0^{-3}$ and to the coupling constant $|\mathbf{d}_{12}|^2$. Such a property, namely the noncancellation of the effect of interaction in the limit $L \rightarrow \infty$, is characteristic of non-Goldstone modes. Thus at least a relaxed form of the above conjecture is valid: In the case of general position, non-Goldstone modes [in the sense of the absence of the requirement (7)] cannot be described by the dispersion law (2).

To summarize briefly, the approach suggested is capable of reproducing in a unified manner the dispersion laws for fractal vibrations and polar excitations, which are different. Further applications of this approach to other fractal excitations seem promising. The conjecture that a connection exists between the Goldstone (or non-Goldstone) type of the excitation and the dispersion law remains to be examined.

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