Spin glass with two replicas on a Bethe lattice

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We consider a mechanism in a spin-glass system with two interacting replicas in which the two replicas interact via a hypothetical replica coupling field that is set to be zero after thermodynamic quantities are evaluated. A symmetry-breaking role of the replica coupling field is manifested by appearance of a negative overlap. Calculation on a Bethe lattice shows that a negative overlap is possible even for a nonzero magnetic field. The result, however, leads to negative entropy at zero temperature on the Bethe lattice with infinite coordination number, presumably due to the assumption of uncorrelated boundary condition. However, the study in this paper opens an interesting question about existence of a negative overlap for a nonzero magnetic field.

I. INTRODUCTION

The infinite-range spin-glass model, known as the Sherrington-Kirkpatrick (SK) model,[1,2] is well understood. In studies using the replica trick the replica-symmetry breaking was recognized as the signal of the spin-glass phase transition at the de Almeida–Thouless (AT) line.[3] The Parisi solution $q(x)$ was obtained by a replica-symmetry-breaking scheme[4] and has passed all stability tests.[5] But the physical meaning of the replica-symmetry breaking and the abstract Parisi solution was not understood at first. An alternative mean-field theory was developed by Thouless, Anderson, and Palmer[6] (TAP) without using the replica trick. Later it was found that the TAP equation has infinitely many solutions.[7] It is believed that the spin-glass phase has infinitely many pure states, identified as the TAP solutions with the lowest free energy. With this astonishing discovery the Parisi solution gained a proper physical interpretation,[8] $q(x)$ is the order parameter function for an overlap between two pure states and $P(q)=dx/dq$ is the probability distribution function for overlap q. It was found that the Parisi solution leads to a hierarchical structure of pure states, ultrametricity.[9] A relaxation dynamics of the SK model was developed by Sompolinsky and Zippelius.[10] Sompolinsky's assumption of hierarchy of many long-time scales were crucial to find the solution with long-time behavior due to relaxation through many pure states[11] and the result was found to be equivalent to Parisi's[12] except for a discrepancy in interpreting the mathematically identical $q(x)$ between the two theories.[13]

Let us consider a system with two real replicas, say α and β, with an interaction energy, $-u\sum_i S_i^\alpha S_i^\beta$, added to the Hamiltonian for two independent replicas. The limit of the hypothetical replica coupling field u is eventually taken to be zero. Then a spin-glass order parameter is given as $-\overline{\partial F/\partial u}|_{u\to 0}$ for the free energy F, where the overbar denotes a random bond average. This system was studied by Blandin and his collaborators, and also by Toulouse.[14] Earlier Blandin *et al.* recognized that u plays the role of symmetry-breaking field in that the solution in the limit of $u\to 0$ is different from that for zero u, the SK solution. As the Parisi solution had been well known, Toulouse considered this two-replica system to project the moments of the Parisi order parameter, $\int dx[q(x)]^k$, in the limit of $u\to 0$, where the above spin-glass order parameter becomes $\int dx\, q(x)$.

In this paper we discuss that the introduction of replica coupling might give rise to a different projection; the order parameter obtained from the first derivative of the free energy with respect to u is not $\int q(x)dx$ but a maximum overlap $q(1)$. This does not mean that the process in $u\to 0$ distorts the problem, but that it forces the two replicas into a single pure state. A more interesting consequence of the replica coupling is that the spin-glass order parameter can have a negative value. This is not a new fact for zero magnetic field where the system possesses the spin reversal symmetry. A similar symmetry breaking can be seen in a ferromagnet where magnetization is either positive or negative for zero magnetic field and its sign is determined according to how a magnetic field approaches zero. In the two-replica system $u\to 0^+$ leads to a positive spin-glass order parameter and $u\to 0^-$ to a negative one. This was also observed by Blandin and his collaborators.[14] A new consequence found from the study on a Bethe lattice in this paper is that a negative overlap is seen even for a nonzero magnetic field. In previous studies a negative overlap for a nonzero field is implicitly neglected, but a thorough justification is not provided and there is no systematic scheme to observe a negative overlap and examine its stability. The study of two interacting replicas might be a proper approach to reexamine this problem.

The spin glass on a Bethe lattice has been extensively studied.[15–18] It may serve as a more realistic mean-field theory than the SK model since a Bethe lattice has the same local lattice structure as the real lattice with the same coordination number has and the effect of boundary conditions can be examined.[17,18] Also the replica trick can be avoided because thermodynamic quantities can be expressed for an arbitrary distribution of exchange coupling J_{ij}. Equivalence to the Bethe approximation,[19] to the TAP approach,[15,19] and to the dynamic theory in infinite dimension[20] has been found. The replica-symmetry-breaking with the identical AT line was satisfactorily found in the study of two noninteracting replicas by Thouless.[16] But the order parameter of a single system, $\langle S_i^\alpha \rangle^2$, was found to be identical to the SK replica-symmetric solution. One of the motivations of this paper is to find a new solution from the interacting two-replica system in the limit of zero replica coupling field and, in fact, there are new solutions found in this paper. One of them is in good agreement with our expectation of a physical solution based on the symmetry-breaking mechanism in the system of interacting two replicas. Unfortunately the property at low temperature presents an unphysical result, negative entropy which is equal to that obtained by Blandin and his collaborators.[14] It is noted that they assumed the replica symmetry in obtaining a solution using the replica trick. Throughout this paper we use the assumption that thermodynamic quantities between different subtrees of a Bethe lattice are not correlated which is presumably equivalent to the assumption of replica symmetry.[21,22] However, we expect that the result in this paper may bring us a qualitative understanding of the interacting two-replica problem.

In Sec. II effective fields for the two replicas on a Bethe lattice are defined and the recursive relation for the effective fields at a site and its outer sites is derived. A solution is considered to be a fixed point in the iterative process given by the recursive relation. There are various solutions found below the critical temperature including those for noninteracting two replicas. In Sec. III the stability for the fixed points in the iterative process is examined. Below the AT line every solution except the replica symmetric solution is found to be a stable fixed point. In Sec. IV a new mechanism of the interacting two replicas is discussed. Especially one of various solutions found in Sec. III is found to support our argument. In Sec. V the free energy density in a translationally invariant deep region of a Bethe lattice is derived. In Sec. VI the exact probability distribution for effective fields in the limit of infinite coordination number is derived. Using this result the free-energy densities for various solutions are compared. The solution in in agreement with our argument in Sec. IV is found to have the lowest free-energy density, but also to have negative entropy at zero temperature. Section VII is for summary of our work and suggestion of future study of the interacting two replicas.

II. SOLUTIONS FOR RECURSIVE EQUATIONS

Let us consider two replicas on a Bethe lattice with coordination number $K+1$. The Hamiltonian is written as

$$H=-\sum_{\langle i,j\rangle} J_{ij}(S_i^\alpha S_j^\alpha+S_i^\beta S_j^\beta)-h\sum_i (S_i^\alpha+S_i^\beta)$$
$$-u\sum_i S_i^\alpha S_i^\beta\ , \tag{2.1}$$

where α and β label two replicas. For a spin glass the J_{ij} are randomly distributed with probability $P(J_{ij})$ given as

$$P(J_{ij})=\left[\frac{K+1}{2\pi\tilde{J}^2}\right]^{1/2}\exp\left[-\frac{(K+1)J_{ij}^2}{2\tilde{J}^2}\right]\ . \tag{2.2}$$

A different type of $P(J_{ij})$, for example, $\pm J$ distribution, can be used and the formalism does not depend on any specific $P(J_{ij})$. In this paper Ising spins are considered. The angular brackets $\langle\ \rangle$ appearing on spin variables denote the ensemble average due to the given Hamiltonian and the overbar on the ensemble averaged quantities denotes the average over random bond J_{ij}. u is a replica coupling field, generates an order parameter $q_{\alpha\beta}=N^{-1}\sum_i \overline{\langle S_i^\alpha S_i^\beta\rangle}$, and will be set to be zero at the final step of calculation.

The partition function can be written in terms of conditional partition functions which satisfy the recursive equations

$$\begin{aligned} Z_{i\pm\pm}&=\prod_j (e^{u+2h\pm 2J_{ki}}Z_{j++}+e^{u-2h\mp 2J_{ki}}Z_{j--}\\ &\qquad +e^{-u}Z_{j+-}+e^{-u}Z_{j-+})\ ,\\ Z_{i\pm\mp}&=\prod_j (e^{u+2h}Z_{j++}+e^{u-2h}Z_{j--}\\ &\qquad +e^{-u\pm 2J_{ki}}Z_{j+-}+e^{-u\mp 2J_{ki}}Z_{j-+})\ , \end{aligned} \tag{2.3}$$

where k denotes the inner neighboring site of a site i, the j denotes the K outer neighboring sites, and Z_{i++} is the conditional partition function for $S_k^\alpha=+1$ and $S_k^\beta=+1$, Z_{i--} for $S_k^\alpha=-1$ and $S_k^\beta=-1$, and so forth. Effective fields are defined from the ratios of conditional partition functions. For a single system there is one effective field due to Z_{i+}/Z_{i-}. For two replicas there are three independent ratios of conditional partition functions. Then three effective fields ξ_i, η_i, and ζ_i are defined such that

$$\begin{aligned} e^{2(\xi_i+\eta_i)}&=\frac{Z_{i++}}{Z_{i--}}\ ,\\ e^{2(\xi_i+\zeta_i)}&=\frac{Z_{i++}}{Z_{i-+}}\ ,\\ e^{2(\eta_i+\zeta_i)}&=\frac{Z_{i++}}{Z_{i+-}}\ . \end{aligned} \tag{2.4}$$

ξ_i and η_i are effective magnetic fields acting from a site i to its inner site k for the two replicas α and β. ζ_i is the effective replica coupling field acting from a site i to its inner site k. ζ_i is equal to zero for noninteracting two replicas. If $x_i=\xi_i+\eta_i$ and $y_i=\xi_i-\eta_i$ are used, the recursive equations for x_i, y_i, and ζ_i can be written as

$$\tanh x_i = \frac{e^{U_i}\sinh(2J_{ki})\sinh(X_i)}{e^{U_i}\cosh(2J_{ki})\cosh(X_i)+e^{-U_i}\cosh(Y_i)} , \tag{2.5a}$$

$$\tanh y_i = \frac{e^{-U_i}\sinh 2J_{ki}\sinh Y_i}{e^{U_i}\cosh(X_i)+e^{-U_i}\cosh(2J_{ki})\cosh(Y_i)} , \tag{2.5b}$$

$$e^{4\zeta_i} = \frac{e^{2U_i}[\cosh^2(2J_{ki})+\sinh^2(X_i)]+e^{-2U_i}\cosh^2(Y_i)+2\cosh(2J_{ki})\cosh(X_i)\cosh(Y_i)}{e^{2U_i}\cosh^2(X_i)+e^{-2U_i}[\cosh^2(2J_{ki})+\sinh^2(Y_i)]+2\cosh(2J_{ki})\cosh(X_i)\cosh(Y_i)} , \tag{2.5c}$$

where $U_i = u + \sum_j \zeta_j$, $X_i = 2h + \sum_j x_j$, and $Y_i = \sum_j y_j$.

In order to have thermodynamic quantities which are translationally invariant, we are only interested in a deep region of a Bethe lattice far from the boundary. In this region the iterative process given by the set of recursive equations for effective fields in Eqs. (2.5a)–(2.5c) will yield a fixed point probability distribution for the effective fields. As Bowman and Levin noticed an effective field cannot have a fixed point value but a fixed point probability distribution due to a given probability distribution for J_{ij}.[15] In Sec. VI the probability distribution for effective fields will be exactly obtained in the limit of infinite K. For a general K where the probability distribution for effective fields cannot be calculated exactly, the moment expansion can be used near the critical temperature. Then a fixed point can be represented by a set of the moments of effective fields instead of the probability distribution for them.

Throughout this paper we assume that the effective fields at different subtrees, which are not located on the same path from the central site to a site at boundary, are not correlated. As seen in Eqs. (2.5a)–(2.5c) the effective fields at site i are determined by the effective fields at its outer neighboring sites j. Two different subtrees have no outer neighboring sites in common. So this assumption seems to be reasonable and most of the studies on the spin glass on a Bethe lattice have used this assumption. When an iterative process has multiple fixed points different fixed points are mapped into by different classes of initial conditions. On a Bethe lattice boundary conditions play the role of initial conditions and are given by distributions of unrecognizable random external fields on boundary sites.[15,17] An uncorrelated boundary condition in which a random external field at each site of the boundary is given independently of other sites leads to uncorrelated effective fields at different subtrees inside.

Mottishaw pointed out that this assumption might be equivalent to the assumption of the replica symmetry.[21] Recently Lai and Goldschmidt found in a numerical study that a correlated boundary condition leads to a solution which is in qualitative agreement with the Parisi solution.[22] But as far as we know no analytic method has been developed to find a fixed point solution with correlation among different subtrees. In Sec. VI it will be found that the result with this assumption of uncorrelated boundary condition leads to an unphysical result. An important purpose of this paper is to present a possible new mechanism of the interacting two replicas which is absent in the previous theories for a single system. We expect that the solution obtained with this assumption might bring us an interesting result in the qualitative level.

Near critical temperature Eqs. (2.5a)–(2.5c) can be expanded in terms of various powers of effective fields. Then powers of effective fields at site i can also be written recursively in terms of those at its outer neighboring sites j. When the random bond average is executed on the powers of effective fields the odd powers of x_i, y_i vanish due to the symmetric bond distribution. The moments of effective fields are the bond-averaged powers of effective fields such as $\overline{x_i^2}$, $\overline{y_i^2}$, $\overline{\zeta_i}$, $\overline{x_i^2 y_i^2}$, $\overline{\zeta_i x_i^2}$, and so forth where the overbar denotes the random bond average. As a result the infinite set of recursive equations for the moments of effective fields can in principle be obtained. A fixed point is then a solution for those recursive equations where the moments at site i and sites j are set to be equal, i.e., $\overline{x^2}=\overline{x_i^2}=\overline{x_j^2}$, $\overline{y^2}=\overline{y_i^2}=\overline{y_j^2}$, $\overline{\zeta}=\overline{\zeta_i}=\overline{\zeta_j}$, and so forth. The assumption of uncorrelated moments at different subtrees can break coupled moments between different subtrees into moments at the same subtrees, for example, $\overline{x_j^2 x_{j'}^2}=\overline{x_j^2}\,\overline{x_{j'}^2}$ for sites j,j' at different subtrees. Then for a fixed point the moments of higher orders can be written in terms of powers of the moments of the lower order such as $\overline{x^2}$, $\overline{y^2}$, and $\overline{\zeta}$.

Resultant equations for a fixed point are three equations for $\overline{x^2}$, $\overline{y^2}$, and $\overline{\zeta}$, and can be expressed in terms of $\overline{\xi^2}$, $\overline{\xi\eta}$, and $\overline{\zeta}$ by use of $\overline{\xi^2}=(\overline{x^2}+\overline{y^2})/4$ and $\overline{\xi\eta}=(\overline{x^2}-\overline{y^2})/4$. When the three equations for $\overline{\xi^2}$, $\overline{\xi\eta}$, and $\overline{\zeta}$ for $h=u=0$ are expanded up to second order in themselves, using $t_{2n}=K\tanh^{2n}2J_{ij}$ as parameters, it can be shown that for $t_2<1$ there exists a unique paramagnetic solution where $\overline{\xi^2}=\overline{\xi\eta}=\overline{\zeta}=0$, and that for $t_2>1$ there are various solutions with nonzero $\overline{\xi^2}$ where $\overline{\xi^2}$ is the same to the first order in t_2-1, given as $(t_2-1)/2(K-1)$, and $\overline{\zeta}$ is exactly equal to zero or vanishes to first order in t_2-1. The critical temperature T_c is determined from the condition $t_2=1$. In the limit of infinite K, $T_c=\tilde{J}$ and $K\overline{\xi^2}=1-T/T_c$ to the lowest order.

Treating $\overline{\zeta}$ to be of order of $\overline{\xi^2}^2$, the three equations for $\overline{\zeta}$, $\overline{\xi^2}$, and $\overline{\xi\eta}$ for nonzero h and u are written to next order as

$$0=(t_2-1)\overline{\zeta}-\frac{2(t_2-t_4)(K-1)}{(1-t_4)}\overline{\xi^2}\,\overline{\zeta}+4\kappa(\tfrac{5}{4}\overline{\xi^2}^2+\tfrac{1}{2}\overline{\xi\eta}^2)\overline{\zeta}+\frac{u}{K}\ , \tag{2.6a}$$

$$0=(t_2-1)\overline{\xi^2}-\frac{2(t_2-t_4)(K-1)}{(1-t_4)}\overline{\xi^2}^2+\frac{2(t_2-t_4)(K-1)}{(1-t_4)}\overline{\xi\eta}\,\overline{\zeta}+\frac{17}{3}\kappa\overline{\xi^2}^3+\frac{h^2}{K}\ , \tag{2.6b}$$

$$0=(t_2-1)\overline{\xi\eta}-\frac{2(t_2-t_4)(K-1)}{(1-t_4)}\overline{\xi^2}\,\overline{\xi\eta}+\frac{2(t_2-t_4)(K-1)}{(1-t_4)}\overline{\xi^2}\,\overline{\zeta}+5\kappa\overline{\xi^2}^2\,\overline{\xi\eta}+\tfrac{2}{3}\kappa\overline{\xi\eta}^3+\frac{h^2}{K}\ , \tag{2.6c}$$

where $\kappa=(K-1)(K-2+2Kt_4-t_4)/(1-t_4)$ is used. From these equations the three moments can be calculated up to the lowest orders in t_2-1, equivalently in $1-T/T_c$, from which each solution starts to differ. u is kept nonzero for the derivatives of the moments with respect to u required to evaluate a spin-glass susceptibility defined as $\partial q_{\alpha\beta}/\partial u$.

For $t_2>1$ and $u=0$ there are two solutions with $\overline{\zeta}=0$ which are nothing but the solutions for noninteracting two replicas.[16] One is the replica-symmetric solution with $\overline{\xi^2}=\overline{\xi\eta}$. It will be shown in Sec. VI that this solution becomes identical to the SK replica-symmetric solution in the limit of infinite K. The other is a replica-symmetry-breaking solution with $\overline{\xi^2}\neq\overline{\xi\eta}$ ($\overline{\xi\eta}=0$ for $h=0$). The two solutions become the same for a magnetic field given by $h_{\mathrm{AT}}^2=4\kappa K\overline{\xi^2}^3/3$ to the lowest order in $1-T/T_c$. In fact this line given by h_{AT} is identical to the AT line in the limit of infinite K. Hereafter we call this line the AT line even for a finite K.

For $t_2>1$ there are also two nontrivial solutions with $\overline{\zeta}\neq0$ even for $u=0$. For both solutions $\overline{\xi\eta}$ can be negative for $0\le h\le h'<h_{\mathrm{AT}}$ where $h'^2=(5/\sqrt{10}-14)h_{\mathrm{AT}}^2/54$. One gives rise to $\overline{\xi\eta}\simeq\pm\overline{\xi^2}/\sqrt{2}$ for $h=0$. The other solution looks most interesting. A positive $\overline{\xi\eta}$ is equal to $\overline{\xi^2}$ for every h below the AT line. A negative $\overline{\xi\eta}$, which is equal to $-\overline{\xi^2}$ for $h=0$, has less magnitude than $\overline{\xi^2}$ for a nonzero h, which shows an agreement with the argument about the mechanism of the interacting two replicas discussed in the next section. At the AT line both solutions with positive $\overline{\xi\eta}$ also coincides with the replica-symmetric solution.

The various solutions for $t_2>1$ can be grouped into two classes. One is replica symmetric in that $\overline{\xi^2}=\overline{\xi\eta}$ and $\overline{\zeta}=0$, where introduction of two replicas is meaningless and only the replica-symmetric solution belongs to this class. The other is replica symmetry breaking in that $\overline{\xi^2}\neq\overline{\xi\eta}$ or $\overline{\zeta}\neq0$. The rest of the solutions belong to this class. All the replica-symmetry-breaking solutions become equal to the replica-symmetric solution at the AT line. Above the AT line they give rise to a seemingly unphysical result that $\overline{\xi\eta}>\overline{\xi^2}$. So we can say, at least, that only the replica-symmetric solution is acceptable above the AT line. In the next section the stability analysis for these various fixed point solutions can show that the replica symmetric solution is unstable below the AT line.

III. STABILITY ANALYSIS FOR FIXED POINTS

The stability analysis for the fixed point solutions obtained in Sec. II can be examined by studying how a variation from its fixed point solution in a shell affects a variation in its inner shell.[16] Using a vector notation, the recursive equation for the moments of effective fields can be written as

$$\vec{\omega}_i=\mathcal{R}[\vec{\omega}_j]\ , \tag{3.1}$$

with

$$\vec{\omega}_i=(\overline{x_i^2},\overline{y_i^2},\overline{x_i^4},\overline{y_i^4},\overline{x_i^2y_i^2},\overline{\zeta_i},\overline{\zeta_ix_i^2},\overline{\zeta_iy_i^2},\ldots)\ ,$$

where i denotes the inner site of site j, $\vec{\omega}_i$ is the vector notation of moments of effective fields at site i, and $\mathcal{R}$ denotes the recursive relation provided by the infinite set of recursive equations for the moments of effective fields. The linear expansion of Eq. (3.1) due to small variations $\delta\vec{\omega}_i,\delta\vec{\omega}_j$ of $\vec{\omega}_i,\vec{\omega}_j$ from a fixed point $\vec{\omega}$ satisfying $\vec{\omega}=\mathcal{R}[\vec{\omega}]$, leads to

$$\delta\vec{\omega}_i=\mathbf{M}\cdot\delta\vec{\omega}_j \quad \text{with } \mathbf{M}=\left.\frac{\delta\mathcal{R}}{\delta\vec{\omega}_j}\right|_{\vec{\omega}_j=\vec{\omega}}\ , \tag{3.2}$$

where $\mathbf{M}$, an $\infty\times\infty$ matrix, is called the stability matrix. If all the eigenvalues of $\mathbf{M}$ are less than unity, then variation at the boundary causes an exponentially decaying response deep inside the tree; the fixed point is then a stable solution of the iterative process, otherwise the fixed point is not stable. The correlation length ξ_L associated with the eigenvalue λ can be defined[16,18] as

$$\lambda=\exp\left[-\frac{1}{\xi_L}\right] \quad \text{or } \xi_L=-\frac{1}{\ln\lambda}\ . \tag{3.3}$$

ξ_L can measure the characteristic steps of iteration required to reach a fixed point. At the critical line (AT line) ξ_L diverges, which resembles the critical slowing down in relaxation of a thermodynamic system toward an equilibrium state.

There are three relevant eigenvalues close to unity and others are order of $t_4,t_6,\ldots,$ which are order of $K^{-1},K^{-2},\ldots,$ in the limit of large K. $\overline{\xi^2}$ is a moment of the lowest order in $1-T/T_c$ and the relevant eigenvalues can be obtained to first order in $\overline{\xi^2}$ by solving an eigenvalue equation for an 8×8 matrix which is a small block matrix out of the original $\infty\times\infty$ stability matrix M. To this order the relevant eigenvalues depend only on $\overline{\xi^2}$. Above the critical temperature ($t_2<1$) the paramagnetic solution is unique and stable; for $u=h=0$, $\overline{\xi^2}=\overline{\xi\eta}=\overline{\zeta}=0$ and the three relevant eigenvalues are all equal to t_2 which is less than unity. This paramagnetic solution is unstable below the critical temperature where $t_2>1$. Below the critical temperature ($t_2>1$), as already noticed, every fixed point solution other than the paramagnetic solution has the same $\overline{\xi^2}$ to order of $1-T/T_c$. To this order every solution for $t_2>1$ has two marginal eigenvalues which are double rooted with the value of unity and the other is $1-2(K-1)\overline{\xi^2}\simeq 2-t_2$ which is less than unity. The two double-rooted eigenvalues differ in higher orders in $\overline{\xi^2}$. The stability of the various fixed point solutions for $t_2>1$ will be determined by correction to the two eigenvalues in higher orders.

Out of the huge stability matrix $\mathbf{M}$ only a small block matrix contributes to corrections to second order in $\overline{\xi^2}$ to the two eigenvalues. It can also be decomposed into two parts; $\mathbf{M}_0$ for which eigenvalues are known to first order in $\overline{\xi^2}$ and $\mathbf{M}'$ which gives second-order corrections to eigenvalues. $\mathbf{M}_0$ and $\mathbf{M}'$ are 15×15 matrices. Variations of moments involved are components of a vector given as

$$\delta\vec{\omega}_i=(\delta\overline{x_i^2},\delta\overline{y_i^2},\delta\overline{x_i^2y_i^2},\delta\overline{x_i^4},\delta\overline{y_i^4},\delta\overline{\zeta_i},\delta\overline{\zeta_i x_i^2},\delta\overline{\zeta_i y_i^2}, \delta\overline{x_i^6},\delta\overline{y_i^6},\delta\overline{x_i^2y_i^4},\delta\overline{x_i^4y_i^2},\delta\overline{\zeta_i x_i^4},\delta\overline{\zeta_i y_i^4},\delta\overline{\zeta_i x_i^2y_i^2})\ . \tag{3.4}$$

It is hardly appropriate to write down the matrix elements of the 15×15 matrices except noting that the stability matrix is not symmetric so that the perturbation theory used for a Hermitian matrix cannot be applied unless the completeness of eigenvectors is guaranteed.

For a nonsymmetric matrix $\mathbf{A}$, when eigenvalues are not distinct, as is the case of $\mathbf{M}_0$, eigenvectors may not be complete. In this case the incomplete eigenvector space can be extended to a complete vector space[23] which is the eigenvector space associated with $(\mathbf{A}-\lambda_m\mathbf{I})^n$ where λ_m is an eigenvalue of $\mathbf{A}$ with multiplicity m. In our problem $m=2$ and $\lambda_m=1$. It is found that the completeness of the eigenvectors of $\mathbf{M}_0$ depends on a parameter ϵ defined as $\epsilon=\overline{\xi\eta}/\overline{\xi^2}$. When $\epsilon^2=1$ there exist two different right (left) eigenvectors with the double-rooted eigenvalue so that eigenvectors of $\mathbf{M}_0$ are complete. When $\epsilon^2\neq 1$ there exists only one right (left) eigenvector, called $\vec{v}_1^R$ ($\vec{v}_1^L$), so that eigenvectors of $\mathbf{M}_0$ are not complete. However, there exists $\vec{v}_2^R$ satisfying $(\mathbf{M}_0-\mathbf{I})\cdot\vec{v}_2^R=c\vec{v}_1^R$ with $c=4(1-\epsilon^2)(K-1)\overline{\xi^2}$. Therefore the right eigenvectors of $(\mathbf{M}_0-\mathbf{I})^2$ are complete. This is also true for left eigenvectors. Let $\vec{v}_1^R$ and $\vec{v}_2^R$ ($\vec{v}_1^L$ and $\vec{v}_2^L$) be the two right (left) eigenvectors of either $\mathbf{M}_0$ for $\epsilon^2=1$ or $(\mathbf{M}_0-\mathbf{I})^2$ for $\epsilon^2\neq 1$, and $\vec{v}_3^R$ ($\vec{v}_3^L$) be the right (left) eigenvector of $\mathbf{M}_0$ whose eigenvalue is one of the three relevant eigenvalues equal to $1-2(K-1)\overline{\xi^2}$, called λ_3. As in a symmetric matrix the orthogonality of eigenvectors holds; $\vec{v}_\nu^L\cdot\vec{v}_\mu^R=\delta_{\nu\mu}$ where ν and μ are indices for different eigenvectors.

An eigenvector of $\mathbf{M}_0+\mathbf{M}'$ can now be expanded in terms of the complete set of eigenvectors of either $\mathbf{M}_0$ or $(\mathbf{M}_0-\mathbf{I})^2$. After a similar perturbation expansion used for a Hermitian matrix it can be shown that the second-order correction λ' to the two larger eigenvalues is given as follows:

for $\epsilon^2=1$

$$0=\det\begin{bmatrix} M'_{11}-\lambda' & M'_{12} \\ M'_{21} & M'_{22}-\lambda' \end{bmatrix}, \tag{3.5a}$$

for $\epsilon^2\neq 1$

$$\lambda'=\frac{1}{2}\left[M'_{11}+M'_{22}\pm\left[(M'_{11}-M'_{22})^2+4cM'_{21}+\frac{4cM'_{31}M'_{23}}{1-\lambda_3}\right]^{1/2}\right], \tag{3.5b}$$

where $M'_{\mu\nu}=\vec{v}_\mu^L\cdot\mathbf{M}'\cdot\vec{v}_\nu^R$. It is found that $M'_{11}=M'_{22}$, $M'_{21}=0$, and $M'_{23}=-M'_{31}/2$. λ' for both Eqs. (3.5a) and (3.5b) can be written in a common expression. So the eigenvalue $\lambda_{\max}$ closest to unity is given as

$$\lambda_{\max}=1-4(K-1)\epsilon\overline{\zeta}-\frac{h^2}{K\overline{\xi^2}}+\kappa\left[2\overline{\xi\eta}^2-\tfrac{2}{3}\overline{\xi^2}^2\right] \pm 2i(K-1)|\overline{\zeta}|\sqrt{1-\epsilon^2}\ . \tag{3.6}$$

Equation (3.6) reduces to the eigenvalue obtained in the noninteracting two-replica theory[16] when $\overline{\zeta}=0$, as expected.

Above the AT line only the replica-symmetric solution with $\overline{\xi^2}=\overline{\xi\eta}$ and $\overline{\zeta}=0$ has the eigenvalue $\lambda_{\max}$ less than unity. At the AT line this solution has $\lambda_{\max}$ equal to unity, so has the correlation length ξ_L divergent. However, $\lambda_{\max}$ is bigger than unity below the AT line. The replica symmetric solution is the unique stable solution above the AT line but unstable below it.

A replica-symmetry-breaking solution with $\overline{\zeta}=0$ is already known to be a stable fixed point below the AT line[16] in noninteracting two-replica theory. This solution is still stable even in the interacting two-replica theory because every term dependent on $\overline{\zeta}$ in Eq. (3.6), which is only different in $\lambda_{\max}$ between the two theories, vanishes.

Another replica-symmetry-breaking solution with $\overline{\xi\eta}\simeq\pm\overline{\xi^2}/\sqrt{2}$ for $h=0$ can be easily found in Eq. (3.6) to have complex eigenvalues below the AT line because $\epsilon^2<1$ and $\overline{\zeta}\neq 0$. The two complex eigenvalues are complex conjugate to each other and so are the eigenvectors since the stability matrix is real. Then a real vector representing a set of the moments of effective fields has complex conjugate coefficients for these complex eigenvectors. Each step of iteration simply gives complex conjugate multiplicative factors, which are the two complex eigenvalues, to each of the two coefficients. Therefore no complex moments of effective fields can be generated if the moments at the boundary are real. The magnitudes of the complex eigenvalues are less than unity below the AT line, so this solution is also a stable fixed point.

In the remaining replica-symmetry-breaking solution a positive $\overline{\xi\eta}$ leads to real $\lambda_{\max}$ since $\epsilon^2=1$,

$$\lambda_{\max}=1+\frac{h^2}{K\overline{\xi^2}}-\frac{4}{3}\kappa\overline{\xi^2}^2 \tag{3.7}$$

which is less than unity below the AT line. A negative $\overline{\xi\eta}$ yields complex $\lambda_{\max}$ the magnitude of which is less than unity, so is also stable.

IV. THE ROLE OF REPLICA COUPLING FIELD

The replica coupling field u plays the same role of generating field of the spin-glass order parameter $q_{\alpha\beta}$ as the magnetic field does in a ferromagnet; $q_{\alpha\beta}=N^{-1}\partial F/\partial u$. What $q_{\alpha\beta}$ really measures is, however, not so obvious. As Toulouse suggested this two-replica system, $q_{\alpha\beta}$ was considered to be the Parisi order parameter $\int dx\, q(x)$,[14] which is true for noninteracting two replicas if the Parisi solution and its interpretation[8] are accepted. But for the two interacting replicas even in the limit of $u \rightarrow 0 q_{\alpha\beta}$ might be different from its value for the two noninteracting replicas.

We conjecture that $q_{\alpha\beta}$ be a maximum overlap rather than the average of all possible overlaps. For a small u, expanding the free energy density for two replicas,

$$\overline{f(u)}=\overline{f(0)}-q_{\alpha\beta}(0)u + \cdots \ . \tag{4.1}$$

Since every pure state has the same free-energy density, $\overline{f(0)}$ is the same for any combination among pure states with an arbitrary overlap and is equal to two times the free-energy density of a single pure state. But for a small u a maximum overlap has the lowest free-energy density and the others become metastable. In fact in the absence of a magnetic field $q_{\alpha\beta}=q(1)$ for $u \rightarrow 0^+$ and $q_{\alpha\beta}=-q(1)$ for $u\rightarrow 0^-$ in terms of the Parisi solution. A similar argument holds in a ferromagnet where the magnetic field h replaces u so for $h \rightarrow 0^+$ the state with a positive magnetization is stable and vice versa. The sign of the order parameter depends on how u approaches zero, so u plays the role of symmetry-breaking field. Since u is not a realistic field, this mathematical resemblance to ferromagnetism alone might not confirm our conjecture, an explicit calculation must be provided. The result in the last section supports our argument. Let us return to this problem later in this section.

For zero magnetic field the system possesses spin reversal symmetry, so for a given pure state there always exists its mirror state obtained by reversing all spins and the magnetizations for these two mirror states have opposite signs to each other. Therefore there are the same number of negative overlaps as of positive overlaps. In the previous theories this is treated to be a matter of double counting so only positive overlaps are considered but there is a no systematic way to separate positive and negative overlaps. A reasonable explanation of this separation can be seen in the Monte Carlo study by Mackenzie and Young[24] where it was observed that the excitation from a pure state to its mirror state is thermodynamically most improbable, i.e., requires the longest relaxation time. The replica coupling between two real replicas, even if not realistic, provides a systematic way to separate the two distinct classes of overlaps.

This symmetry-breaking role of u might be thought of nothing but a way to reformulate a trivially understood property. However, if a negative overlap is found for a nonzero magnetic field, it would not be trivially understood. The following is a possible explanation of existence of a negative overlap for a nonzero magnetic field. In the presence of a sufficiently small magnetic field, spins start to align with the magnetic field. A pair of two mirror states built in the absence of the magnetic field will have small droplets with spins aligned with the magnetic field. If their typical spin configurations are not considerably changed the overlap between these two states is still negative, but there will be a small positive contribution due to such droplets. On the other hand the self-overlap of a single state has a positive value the magnitude of which is bigger than that of the negative overlap. The difference in magnitude between the positive and the negative overlap comes from small droplets excited in the two states.

There is a big qualitative difference in mirror states between the spin-glass and the ferromagnetic phase. In the presence of a magnetic field only the ferromagnetic state whose magnetization is aligned with the magnetic field is favorable; its mirror state is unfavorable. But no single state of two mirror states in the spin-glass phase is favorable, both are equally favorable or equally unfavorable. If they are equally unfavorable in the presence of the magnetic field they will be suddenly replaced by new pure states. However it is very hard to imagine this sudden appearance of new states. Two mirror states in the antiferromagnetic phase have rather similar properties, they coexist until a sufficient magnetic field makes a state with spins aligned with the magnetic field more stable. But the spin-glass phase is much more complicated. There are many pure states, some of them in the vicinity will fall into a new state, so the number of pure states will decrease as a magnetic field is applied. However, the dominant spin configuration and the number pure states will change continuously, not suddenly. So there might be a new pair of offsprings of two mirror states. A negative overlap for a nonzero magnetic field, presumably less than h_{AT}, cannot be satisfactorily explained by the argument we have given so far until an explicit calculation can be performed. However, we might say, at least, that the widely accepted argument that a negative overlap suddenly disappears or becomes metastable when a magnetic field is turned on is also questionable.

From solutions obtained in the last section new properties from the interacting two-replica theory can be seen. First it is noted that there are different expressions for spin-glass order parameters:

$$\tilde{q}_{\alpha\alpha}=\frac{1}{N}\sum_i \overline{\langle S_i^\alpha\rangle^2} \ \text{ and } \ \tilde{q}_{\alpha\beta}=\frac{1}{N}\sum_i \overline{\langle S_i^\alpha\rangle\langle S_i^\beta\rangle} \ . \tag{4.2}$$

Since we are interested in the thermodynamic property in a translationally invariant deep region of a Bethe lattice, the three spin-glass order parameters can be evaluated at the central site. It is convenient to have the following definition:

$$H_\alpha=h+\sum_j^{K+1}\xi_j, \quad H_\beta=h+\sum_j^{K+1}\eta_j, \quad U=u+\sum_j^{K+1}\zeta_j \ , \tag{4.3}$$

where the j are the $K+1$ neighboring sites of the central site. Then,

$$q_{\alpha\beta}=\frac{\overline{\tanh H_\alpha \tanh H_\beta+\tanh U}}{1+\tanh U \tanh H_\alpha \tanh H_\beta} ,$$
$$\tilde{q}_{\alpha\alpha}=\overline{m^2_{\alpha 0}} , \quad (4.4)$$
$$\tilde{q}_{\alpha\beta}=\overline{m_{\alpha 0}m_{\beta 0}} ,$$

where magnetizations at the central site $m_{\alpha,\beta 0}=\langle S_0^{\alpha,\beta}\rangle$ are given as

$$m_{\alpha,\beta 0}=\frac{\tanh H_{\alpha,\beta}+\tanh H_{\beta,\alpha}\tanh U}{1+\tanh U \tanh H_\alpha \tanh H_\beta} . \quad (4.5)$$

Near T_c expanding the order parameters in terms of effective fields leads to the following result, to the lowest orders in $1-T/T_c$:

$$\tilde{q}_{\alpha\alpha}=(K+1)\overline{\xi^2} ,$$
$$\tilde{q}_{\alpha\beta}=(K+1)\overline{\xi\eta} , \quad (4.6)$$
$$q_{\alpha\beta}-\tilde{q}_{\alpha\beta}=(K+1)\overline{\zeta} .$$

The solutions with nonzero $\overline{\zeta}$ show the symmetry breaking. $\overline{\xi\eta}$ and $\overline{\zeta}$ have the same sign, either positive or negative, so do $q_{\alpha\beta}$ and $\tilde{q}_{\alpha\beta}$. Also it can be found that $|q_{\alpha\beta}|>|\tilde{q}_{\alpha\beta}|$, which seems to support the assumption of the maximum overlap of $q_{\alpha\beta}$.

The replica coupling is assumed to allow only the combinations with maximum overlap among many pure states. Also in the presence of negative overlaps the whole phase space, the set of all possible spin configurations, for a single system can be divided into two distinct regions associated with positive and negative overlap, respectively. More precisely, the two distinct regions can be constructed for a given reference pure state in such a way that all the pure states which yield positive overlaps with the given state fall into one region and the rest into the other. The way of partitioning is not unique because of freedom in choosing reference states. However, if every reference pure state is to lead to the same physical result such partitioning can be well defined.

Let V and V' be such two distinct regions in phase space. For zero magnetic field if a pure state belongs to V, its mirror state belongs to V'. For a nonzero magnetic field the two states, if a negative overlap still exists, are no longer mirror states and the distinction between them in microscopic level is not clear. But for a given pure state in V, the other pure state which gives rise to a negative maximum overlap with the given state can, in principle, be found in V'.

Let a and $\tilde{a}$ be a pair of states with maximum overlap. For $u\to 0^+ a$ and $\tilde{a}$ are an identical state. For $u\to 0^-$, they should be found in different regions, V and V'. With the assumption of maximum overlap the ensemble average of a function $\mathcal{O}$ of spins of two replicas in the limit of $u\to 0$ can be given as

$$\langle\mathcal{O}\rangle=\frac{1}{Z}\sum_{a\in V}\sum_{\{S_i^\alpha\}\in a}\sum_{\{S_i^\beta\}\in\tilde{a}}\mathcal{O}\exp[-\mathcal{H}^\alpha-\mathcal{H}^\beta] , \quad (4.7)$$

where the previous two-replica Hamiltonian $\mathcal{H}$ is decoupled into two parts $\mathcal{H}^{\alpha,\beta}$ each of which is the Hamiltonian for a single system with name α or β. The memory of u is only contained in the summation over a restricted phase space. The partition function Z is given as

$$Z=\sum_{a\in V}\sum_{\{S_i^\alpha\}\in a}\sum_{\{S_i^\beta\}\in\tilde{a}}\exp[-\mathcal{H}^\alpha-\mathcal{H}^\beta]$$
$$=\sum_{a\in V}\exp[-2F_a] , \quad (4.8)$$

where F_a is the free energy of a pure state a and the factor 2 is due to the fact that $F_a=F_{\tilde{a}}$. F_a may be identified as the TAP free energy. Since the theory has no preferable pure state, the sum over a is performed. The a are restricted only in one of the two regions, say V, to have distinct combinations among pure states. More precisely, there must be a overall multiplicative factor 2, which will not change the result of ensemble average.

The order parameter $q_{\alpha\beta}$ is given in the limit of $u\to 0$ as

$$q_{\alpha\beta}=\frac{1}{N}\sum_i\overline{\frac{1}{Z}\sum_{a\in V}\sum_{\{S_i^\alpha\}\in a}S_i^\alpha\exp(-\mathcal{H}^\alpha)\sum_{\{S_i^\beta\}\in\tilde{a}}S_i^\beta\exp(-\mathcal{H}^\beta)}=\overline{\sum_{a\in V}P_a\frac{1}{N}\sum_i m_i^a m_i^{\tilde{a}}} . \quad (4.9)$$

m_i^a and $m_i^{\tilde{a}}$ are the magnetizations for states a and $\tilde{a}$, for example,

$$m_i^a=\frac{1}{Z_a}\sum_{\{S_i^\alpha\}\in a}S_i^\alpha\exp(-\mathcal{H}^\alpha) ,$$
$$Z_a=\sum_{\{S_i^\alpha\}\in a}\exp(-\mathcal{H}^\alpha)=e^{-F_a} . \quad (4.10)$$

So $P_a\propto\exp(-F_a)$. For $u\to 0^+ m_i^a=m_i^{\tilde{a}}$, so $q_{\alpha\beta}$ can be identified as $q(1)$ of the Parisi solution. For $u\to 0^- q_{\alpha\beta}$ is equal to $-q(1)$ for $h=0$. But for $h\neq 0$ it might have less magnitude than $q(1)$ due to a partial alignment of spins in the direction of the magnetic field as discussed before.

In a similar way the other two-order parameters $\tilde{q}_{\alpha\alpha}$, $\tilde{q}_{\alpha\beta}$ can be expressed as

$$\tilde{q}_{\alpha\alpha}=\overline{\sum_{a,b\in V}P_aP_b\frac{1}{N}\sum_i m_i^a m_i^b} , \quad (4.11a)$$

$$\tilde{q}_{\alpha\beta}=\overline{\sum_{a,b\in V}P_aP_b\frac{1}{N}\sum_i m_i^a m_i^{\tilde{b}}} . \quad (4.11b)$$

For $u\to 0^+\tilde{q}_{\alpha\alpha}=\tilde{q}_{\alpha\beta}=\int_0^1 dx\, q(x)$ for all h below the AT line. For $u\to 0^-\tilde{q}_{\alpha\beta}=-\tilde{q}_{\alpha\alpha}=-\int_0^1 dx\, q(x)$ only for $h=0$. But for $h\neq 0 |\tilde{q}_{\alpha\beta}|<\int_0^1 q(x)$. $\tilde{q}_{\alpha\beta}$ might also be written as $\int dx\, \hat{q}(x)$, but $\hat{q}(x)$ is unknown.

Of the various solutions below the AT line found in the last section only one solution is in agreement with the argument we have given, which can be seen in Eq. (4.6). In this solution there are two cases. For a positive $\overline{\xi}$, $\overline{\xi^2}=\overline{\xi\eta}$ for every h below the AT line. On the other hand, for a negative $\overline{\xi}$, $\overline{\xi\eta}$ is negative and $|\overline{\xi\eta}|<|\overline{\xi^2}|$. Also both cases leads to $|q_{\alpha\beta}|>|\tilde{q}_{\alpha\beta}|$, which supports the assumption of maximum overlap of $q_{\alpha\beta}$.

It is tempting to define a susceptibility χ_r of the spin-glass order parameter $q_{\alpha\beta}$ as

$$\chi_r=-\frac{1}{N}\frac{\partial^2\overline{F}}{\partial u^2}\bigg|_{u\to 0}=\frac{\partial q_{\alpha\beta}(u)}{\partial u}\bigg|_{u\to 0}=\frac{1}{N}\sum_{i,j}\overline{[\langle S_i^\alpha S_i^\beta S_j^\alpha S_j^\beta\rangle-\langle S_i^\alpha S_i^\beta\rangle\langle S_j^\alpha S_j^\beta\rangle]}\ . \tag{4.12}$$

Applying Eq. (4.7), for $u\to 0^+$,

$$\chi_r=\sum_{a\in V}P_a\frac{1}{N}\sum_{i,j}\overline{[(\langle S_iS_j\rangle_a)^2-(\langle S_i\rangle_a\langle S_j\rangle_a)^2]}\ , \tag{4.13}$$

where the angular brackets $\langle\ \rangle$ denote the ensemble average due to the Hamiltonian of a single system and the subscript a means that the average is performed in a single pure state a. Since the contributions from all a's are expected to be the same, the order parameter $q_{\alpha\beta}$ and its susceptibility χ_r are the quantities in a single pure state. A similar expression can be obtained for $u\to 0^-$.

This susceptibility can be evaluated by calculating the derivatives of the moments of effective fields with respect to u. An interesting result is that χ_r diverges at the AT line for every solution found in the last section. The spin-glass phase is signaled by the divergent susceptibility as in a ferromagnet.

When the most interesting solution with $\overline{\xi\eta}=\overline{\xi^2}$ and $\overline{\xi}\neq 0$ is chosen below the AT line and the replica symmetric solution above it, near the AT line, χ_r behaves symmetrically such that

$$\chi_r\simeq\left|\frac{4}{3}\kappa\overline{\xi^2}^2-\frac{h^2}{K\overline{\xi^2}}\right|^{-1}\ . \tag{4.14}$$

This seems to contradict the marginal stability below the AT line.

The usual spin-glass susceptibility $\chi_{\rm SG}$ is given as

$$\chi_{\rm SG}=\frac{1}{N}\sum_{i,j}\overline{(\langle S_iS_j\rangle-\langle S_i\rangle\langle S_j\rangle)^2}\ . \tag{4.15}$$

If it is evaluated in a single pure state, the difference between χ_r and $\chi_{\rm SG}$ is given as

$$\chi_r-\chi_{\rm SG}=\frac{2}{N}\sum_{i,j}\overline{\langle S_i\rangle\langle S_j\rangle(\langle S_iS_j\rangle-\langle S_i\rangle\langle S_j\rangle)}\ . \tag{4.16}$$

Since the susceptibility matrix χ_{ij} given as $\langle S_iS_j\rangle-\langle S_i\rangle\langle S_j\rangle$ is positive semidefinite,[25] the right-hand side in Eq. (4.16) becomes positive. So the positivity of the susceptibility χ_r can be used as a criterion in selecting a physical solution. In fact above the AT line the unique stable solution, the replica-symmetric solution, has a positive χ_r. Below the AT line, the solution with $\overline{\xi\eta}\simeq\pm\overline{\xi^2}/\sqrt{2}$ for $h=0$ as well as the replica symmetric solution has a negative χ_r. But the two remaining replica symmetry-breaking solutions below the AT line, including the solution obtainable from noninteracting two-replica system, have positive χ_r's.

The solution given by $\overline{\xi\eta}=\overline{\xi^2}$ everywhere below the AT line or $\overline{\xi\eta}<0$ up to a nonzero h below the AT line seems to be in most plausible agreement with our assumption of maximum overlap, a negative overlap for a nonzero h, positivity of χ_r. In the following sections it will be found that even this solution presents serious problems. However we hope that the above qualitative agreement is helpful to understand the possible new physics of the interacting two-replica theory.

V. FREE ENERGY DENSITY

We are interested only in a deep region of a Bethe lattice where the system is translationally invariant. The free energy in this region must be carefully defined. It cannot be obtained by directly evaluating the logarithm of the partition function in which the contribution from the outside of this region is not negligible. It is noted that the number of spins at the boundary is comparable to the number of interior spins.

Peruggi *et al.* found the expression for the free-energy density in a deep region of a Bethe lattice with a ferromagnetic or an antiferromagnetic interaction.[26] They expressed the internal energy density and the entropy density in terms of the probability for spin at a site and the joint probability for a pair of spins at neighboring sites. Generalization of their expression to the two-replica system is made in this paper. An equivalent expression may be obtained by writing the effective fields (ξ_i,η_i,ζ_i) in terms of the order parameters $(m_i^\alpha,m_i^\beta,q_i^{\alpha\beta})$ in the logarithm of the partition function and by extracting the contribution of a single site and half the contribution of a single bond without summing all terms. In this way Bowman and Levin derived the TAP free energy on the Bethe lattice with infinite coordination number.[15]

The free energy f_i at site i is expressed as $f_i=e_i-s_i$ where e_i is the internal energy assigned to site i and s_i the entropy. More explicitly,

$$\begin{aligned}e_i=&-2h(P_{i++}-P_{i--})-u(P_{i++}+P_{i--}-P_{i+-}-P_{i-+})\\&-2\sum_{j\,(\neq i)}J_{ij}(P_{i++,j++}+P_{i+-,j+-}+P_{i-+,j-+}+P_{i--,j--}-P_{i++,j--}-P_{i+-,j-+}-P_{i-+,j+-}\\&-P_{i--,j++})\ ,\end{aligned} \tag{5.1}$$

$$s_i = K \sum_{\sigma,\sigma'} P_{i\sigma\sigma'} \ln P_{i\sigma\sigma'} - \tfrac{1}{2} \sum_{j\,(\neq i)} \sum_{\sigma_1,\sigma_1'} \sum_{\sigma_2,\sigma_2'} P_{i\sigma_1\sigma_1',j\sigma_2\sigma_2'} \ln P_{i\sigma_1\sigma_1',j\sigma_2\sigma_2'} , \tag{5.2}$$

where $P_{i\sigma\sigma'}$ is the probability for two replica spins at site i given as $S_i^\alpha=\sigma$, $S_i^\beta=\sigma'$, and $P_{i\sigma_1\sigma_1',j\sigma_2\sigma_2'}$ the joint probability for two replica spins at neighboring sites i,j given as $S_i^\alpha=\sigma_1, S_i^\beta=\sigma_1'$, $S_j^\alpha=\sigma_2, S_j^\beta=\sigma_2'$. $\pm$ is a simple notation for $\sigma=\pm 1$. $\sum_j'$ denotes the sum over the neighboring sites j of site i.

The central site can be chosen for i most conveniently. Then the probabilities can be expressed in terms of the conditional partition functions defined in Eqs. (2.3) and (2.4), so in terms of the effective fields. For a simple example,

$$P_{0++} \propto e^{2h} \prod_{j=1}^{K+1} Z_{j++} \propto \exp\left[2h + \sum_{j=1}^{K+1} (\xi_j + \eta_j + \zeta_j)\right] \tag{5.3}$$

and the normalization factor is determined by the condition that $\sum_{\sigma,\sigma'} P_{0\sigma\sigma'} = 1$. The joint probabilities between neighboring spins are also given in a more complicated but a similar manner.

After some lengthy manipulation the complicated expression for the free-energy density can be written in a simplified form as

$$\begin{aligned} f_0 = & -h(m_{\alpha 0} + m_{\beta 0}) + \frac{K+1}{2} \ln(1-\tanh^2 J_{0j}) \\ & + K \ln[4\cosh U \cosh H_\alpha \cosh H_\beta (1+\tanh U \tanh H_\alpha \tanh H_\beta)] \\ & -(K+1)\ln[4\cosh U' \cosh H'_\alpha \cosh H'_\beta (1+\tanh U' \tanh H'_\alpha \tanh H'_\beta)] \\ & -\frac{K+1}{2} \ln[1+\tanh J_{0j}(m'_{\alpha 0} m'_{\alpha j} + m'_{\beta 0} m'_{\beta j}) + \tanh^2 J_{0j}\, q'_{\alpha\beta 0} q'_{\alpha\beta j}] - K[H_\alpha m_{\alpha 0} + H_\beta m_{\beta 0} + U q_{\alpha\beta 0}] \\ & +(K+1)[1+\tanh J_{0j}(m'_{\alpha 0} m'_{\alpha j} + m'_{\beta 0} m'_{\beta j}) + \tanh^2 J_{0j}\, q'_{\alpha\beta 0} q'_{\alpha\beta j}]^{-1} \\ & \times [H'_\alpha (m'_{\alpha 0} + \tanh J_{0j}\, m'_{\alpha j}) + H'_\beta (m'_{\beta 0} + \tanh J_{0j}\, m'_{\beta j}) \\ & \quad + U'(q'_{\alpha\beta 0} + \tanh^2 J_{0j}\, q'_{\alpha\beta j}) + \tanh J_{0j}\, U'(m'_{\alpha 0} m'_{\beta j} + m'_{\alpha j} m'_{\beta 0}) \\ & \quad + \tanh J_{0j} (H'_\alpha m'_{\beta j} + H'_\beta m'_{\alpha j}) q'_{\alpha\beta 0} + \tanh^2 J_{0j} (H'_\alpha m'_{\beta 0} + H'_\beta m'_{\alpha j}) q'_{\alpha\beta j}] , \end{aligned} \tag{5.4}$$

where j is one of the nearest-neighboring sites of the central site and the final result after the random bond average does not depend on a specific j. $H_{\alpha,\beta}$, U, $m_{\alpha 0}$, $m_{\beta 0}$, and $q_{\alpha\beta 0}$ are defined in the last section. The corresponding primed quantities have the same expressions as the unprimed except that the site j is not included in the summation over the neighboring sites of the central site 0. $m'_{\alpha j}$, $m'_{\beta j}$, and $q'_{\alpha\beta j}$ also have the same expressions as those at site 0 except that the summation over effective fields is done over the outer neighboring sites of the site j so the central site is not included. The difference between the primed and the unprimed quantity is small in the limit of large K, which will be used to calculate the free energy density in the limit of infinite K in the next section.

VI. IN THE LIMIT OF INFINITE COORDINATION NUMBER

In the limit of infinite coordination number the fixed point probability distribution for the three effective fields can be exactly found by using the method of Oliveira and Salinas.[27]

In this limit $K\to\infty$ we have

$$K\overline{J_{ki}^2} = \tilde{J}, \quad K\overline{J_{ki}^{2n}} = 0 \quad \text{for } n>2 , \tag{6.1}$$

where the overbar denotes the random bond average over J_{ki}. Then it is sufficient to expand ξ_i, η_i to first order in J_{ki} and ζ_i to second order such that

$$\xi_i = J_{ki} \frac{\tanh H'_{\alpha i} + \tanh H'_{\beta i} \tanh U'_i}{1+\tanh H'_{\alpha i} \tanh H'_{\beta i} \tanh U'_i} , \tag{6.2a}$$

$$\eta_i = J_{ki} \frac{\tanh H'_{\beta i} + \tanh H'_{\alpha i} \tanh U'_i}{1+\tanh H'_{\alpha i} \tanh H'_{\beta i} \tanh U'_i} , \tag{6.2b}$$

$$\zeta_i = J_{ki}^2 \frac{\tanh U'_i (1-\tanh^2 H'_{\alpha i})(1-\tanh^2 H'_{\beta i})}{(1+\tanh H'_{\alpha i} \tanh H'_{\beta i} \tanh U'_i)^2} , \tag{6.2c}$$

where $H'_{\alpha i} = h + \sum_j \xi_j$, $H'_{\beta i} = h + \sum_j \eta_j$, $U'_i = \sum_j \zeta_j$ are used. The j denote K outer neighboring sites of i and k the inner site. Then, in terms of the primed quantities in the last section,

$$\xi_i = J_{ki} m'_{\alpha i} , \tag{6.3a}$$

$$\eta_i = J_{ki} m'_{\beta i} , \tag{6.3b}$$

$$\zeta_i = J_{ki}^2 (q'_{\alpha\beta i} - m'_{\alpha i} m'_{\beta i}) . \tag{6.3c}$$

It is noted that in $m'_{\alpha i}$ or the other primed order parameters at site i the contribution from its inner site k is not included. Under the assumption that the effective fields at different subtrees are not correlated, J_{ki} is independent of $H'_{\alpha i}, H'_{\beta i}, U'_i$, so independent of $m'_{\alpha i}$ and the other two primed order parameters at site i. This property will yield the exact probability distribution for the effective fields in the following.

It is more useful to know the probability distribution for summed effective fields, for example, $\sum_i \xi_i$, than for a single effective field. The probability for x,y,z as general variables can be expressed in the Fourier representation as

$$P(x,y,z)=\frac{1}{(2\pi)^{3/2}}\int dk_1\int dk_2\int dk_3\, P(k_1,k_2,k_3)e^{(ik_1x+ik_2y+ik_3z)} . \tag{6.4}$$

If $x=\sum_i \xi_i$, $y=\sum_i \eta_i$, and $z=\sum_i \zeta_i$, then

$$P(k_1,k_2,k_3)=\frac{1}{(2\pi)^{3/2}}\overline{\prod_i^K \exp[-ik_1J_{ki}m'_{\alpha i}-ik_2J_{ki}m'_{\beta i}-ik_3J_{ki}^2(q'_{\alpha\beta i}-m'_{\alpha i}m'_{\beta i})]} . \tag{6.5}$$

Then the cumulant expansion gives

$$P(k_1,k_2,k_3)=\frac{1}{(2\pi)^{3/2}}\exp\left[-\frac{\tilde{J}^2}{2}[\tilde{q}_{\alpha\alpha}^2(k_1^2+k_2^2)+2\tilde{q}_{\alpha\beta}k_1k_2]-ik_3\tilde{J}^2(q_{\alpha\beta}-\tilde{q}_{\alpha\beta})\right] , \tag{6.6}$$

where it is used that in $K\to\infty$ limit $\tilde{q}_{\alpha\alpha}=\overline{(m'_{\alpha i})^2}=\overline{(m'_{\beta i})^2}$ and $\tilde{q}_{\alpha\beta}=\overline{m'_{\alpha i}m'_{\beta i}}$. By the Fourier transformation

$$P(x,y,z)=\frac{\delta(z-\tilde{J}^2(q_{\alpha\beta}-\tilde{q}_{\alpha\beta}))}{2\pi\tilde{J}^2\sqrt{\tilde{q}_{\alpha\alpha}^2-\tilde{q}_{\alpha\beta}^2}}\exp\left[-\frac{\tilde{q}_{\alpha\alpha}(x^2+y^2)-2\tilde{q}_{\alpha\beta}xy}{2\tilde{J}^2(\tilde{q}_{\alpha\alpha}^2-\tilde{q}_{\alpha\beta}^2)}\right] . \tag{6.7}$$

The distribution for $\sum_i \xi_i$ and $\sum_i \eta_i$ is the same as that for two noninteracting replicas.[27] The δ function distribution for $\sum_i \zeta_i$ is the difference between the interacting and the noninteracting two-replica system. For the noninteracting case $q_{\alpha\beta}-\tilde{q}_{\alpha\beta}$ is equal to zero, i.e., the ζ_i vanish in all equations.

In the limit of infinite K the free-energy density obtained in the last section can be further simplified with the known probability distribution for effective fields. We will not present the calculation in detail, except mentioning a basic idea. Since the probability distribution for the summed effective fields over K outer sites is known, for a quantity G at the central site, the following expansion can be used:

$$G(H_\alpha,H_\beta,U)=G(H'_\alpha,H'_\beta,U')+\frac{\partial G}{\partial H_\alpha}J_{0j}m'_{\alpha j}+\frac{\partial G}{\partial H_\beta}J_{0j}m'_{\beta j}+\frac{\partial G}{\partial U}J_{0j}^2(q'_{\alpha\beta j}-m'_{\alpha j}m'_{\beta j})+\cdots . \tag{6.8}$$

Then the bond averaged free-energy density $\bar{f}$, which is equal to $\overline{f_0}$, can be written as

$$\bar{f}=-\tfrac{1}{2}\tilde{J}^2(1-\tilde{q}_{\alpha\alpha})^2+\tfrac{1}{2}\tilde{J}^2(q_{\alpha\beta}-\tilde{q}_{\alpha\beta})(q_{\alpha\beta}+\tilde{q}_{\alpha\beta})-\overline{\ln[\cosh U^{\rm eff}\cosh H_\alpha^{\rm eff}\cosh H_\beta^{\rm eff}(1+\tanh U^{\rm eff}\tanh H_\alpha^{\rm eff}H_\beta^{\rm eff})]} , \tag{6.9}$$

where

$$U^{\rm eff}=\tilde{J}^2(q_{\alpha\beta}-\tilde{q}_{\alpha\beta}) ,$$
$$H_{\alpha,\beta}^{\rm eff}=h+\tilde{J}\left[\left[\frac{q_+}{2}\right]^{1/2}z_+\pm\left[\frac{q_-}{2}\right]^{1/2}z_-\right] , \tag{6.10}$$

and $q_\pm=\tilde{q}_{\alpha\alpha}\pm\tilde{q}_{\alpha\beta}$ is used. The overbar denotes the average over z_+,z_- with the Gaussian probability distribution given as

$$P(z_+,z_-)=\frac{1}{2\pi}\exp[-\tfrac{1}{2}(z_+^2+z_-^2)] . \tag{6.11}$$

For two noninteracting replicas $U^{\rm eff}=0$. Since the Gaussian average of a function of either $H_\alpha^{\rm eff}$ or $H_\beta^{\rm eff}$ alone over z_+,z_- becomes the Gaussian average of the function of $H=h+\tilde{J}\sqrt{\tilde{q}_{\alpha\alpha}}z$ over a single random variable z, $\bar{f}/2$ leads to the SK replica-symmetric free-energy density and $\tilde{q}_{\alpha\alpha}$ is equal to the SK replica-symmetric solution.[1,2] Also the replica-symmetry-breaking solution for noninteracting two replicas, where $q_{\alpha\beta}=0$ for $h=0$, shares the same unstable replica-symmetric free energy.

Below the AT line every solution except the replica-symmetric solution was found to be a stable fixed point. By comparing the free-energy densities for various stable fixed pints, we might determine the stability. Only the solution with the lowest free energy is acceptable. It is noted that the paramagnetic solution with $\overline{\xi^2}=\overline{\xi\eta}=\overline{\zeta}=0$ for $h=0$ has the lowest free energy, but it is an unstable fixed point.

It can be found that the free-energy densities for all stable fixed point solutions are the same up to fourth order in $1-T/T_c$ near T_c. The expansion of $\bar{f}$ to fifth order shows that the solution with $\tilde{q}_{\alpha\beta} \simeq +\tilde{q}_{\alpha\alpha}/\sqrt{2}$ for $h=0$ has a higher free energy than the other solutions; one is the replica-symmetry-breaking solution with $U^{\rm eff}=0$ and the other is the one with $U^{\rm eff}\neq 0$. It is from the sixth order that the two begin to differ. This is also the first order from which the Parisi solution and the replica-symmetric solution begin to differ.[28] The result is that the replica-symmetry-breaking solution newly obtained from the interacting two-replica theory has a lower free energy than the one from the noninteracting two-replica theory.

In the solution with the lowest free energy, it is interesting to compare the free-energy densities for a positive and a negative overlap for a nonzero magnetic field. They are the same for $h=0$ because of the spin reversal symmetry. The direct comparison is very involved. But for a small $h \ll h'$, where h' is the limiting field for negative overlap, we can easily compare them. Renaming the magnetic fields on the two replicas differently h_α and h_β, then the free-energy density for a small $h=h_\alpha=h_\beta$ can be expanded as

$$
\begin{aligned}
\bar{f}(h_\alpha=h,h_\beta=h)
&=\bar{f}(0,0)+\frac{1}{2}\left[\frac{\partial^2\bar{f}}{\partial h_\alpha^2}+\frac{\partial^2\bar{f}}{\partial h_\beta^2}+2\frac{\partial^2\bar{f}}{\partial h_\alpha \partial h_\beta}\right]\Bigg|_{h=0} h^2+\cdots \\
&=\bar{f}(0,0)-(1-\tilde{q}_{\alpha\alpha}+q_{\alpha\beta}-\tilde{q}_{\alpha\beta})|_{h=0}h^2+\cdots ,
\end{aligned}
\tag{6.12}
$$

where it is used that $\partial\overline{m_\alpha}/\partial h_\alpha=1-\tilde{q}_{\alpha\alpha}$ and $\partial\overline{m_\alpha}/\partial h_\beta=q_{\alpha\beta}-\tilde{q}_{\alpha\beta}$. Since $\tilde{q}_{\alpha\alpha}$ is the same for the two cases, the solution with a negative overlap has a higher free energy for a nonzero h, i.e., becomes metastable. A similar case can be seen in a ferromagnet where for a nonzero h the solution with magnetization in opposite direction to h becomes metastable. It might be said that the solution with a negative overlap is of no physical interest. However, we will see in the following that it is not conclusive yet.

The final task in this paper is the extension to zero temperature. The entropy density can be obtained by differentiating the free energy with respect to temperature T and is given as

$$
\begin{aligned}
s=&-\tfrac{1}{2}\tilde{J}^2(1-\tilde{q}_{\alpha\alpha})(1+3\tilde{q}_{\alpha\alpha}) \\
&-\tfrac{3}{2}\tilde{J}^2(q_{\alpha\beta}-\tilde{q}_{\alpha\beta})(q_{\alpha\beta}+\tilde{q}_{\alpha\beta})+\overline{\ln[4\cosh U^{\rm eff}\cosh H^{{\rm eff}\,\alpha}\cosh H^{{\rm eff}\,\beta}(1+\tanh U^{\rm eff}\tanh H^{{\rm eff}\,\alpha}\tanh H^{{\rm eff}\,\beta})]}\ .
\end{aligned}
\tag{6.13}
$$

A solution with $U^{\rm eff}=0$ ($q_{\alpha\beta}=\tilde{q}_{\alpha\beta}$) yields negative entropy at zero temperature no matter whether it is replica symmetric or not because the entropy is the same as that of the SK replica-symmetric solution.[2] It turns out that the replica-symmetry-breaking solution with $U^{\rm eff}\neq 0$ and $\tilde{q}_{\alpha\beta}=\pm\tilde{q}_{\alpha\alpha}$ for $h=0$ belongs the case for $U^{\rm eff}\gg 1$, i.e., $q_{\alpha\beta}-\tilde{q}_{\alpha\beta}$ is of order of T near zero temperature. As a result, for $h=0$ near $T=0$,

$$
\tilde{q}_{\alpha\alpha}=1-\frac{1}{\sqrt{2\pi}}\frac{kT}{\tilde{J}}-\frac{1}{4\pi}\left[\frac{kT}{\tilde{J}}\right]^2+\cdots , \tag{6.14a}
$$

$$
|q_{\alpha\beta}-\tilde{q}_{\alpha\beta}|=\frac{1}{\sqrt{2\pi}}\frac{kT}{\tilde{J}}+\frac{1}{4\pi}\left[\frac{kT}{\tilde{J}}\right]^2+\cdots , \tag{6.14b}
$$

where the conventional unit with $\beta=kT$ is recovered to see dependence on T. Unfortunately, even this replica-symmetry-breaking solution yields negative entropy at zero temperature. The entropy density per replica is equal to $-k/4\pi$, has less magnitude than $-k/2\pi$ for the SK replica-symmetric solution. The same negative entropy was obtained by Blandin *et al.*[14]

The problem seems to be in the assumption of uncorrelated effective fields at different subtrees. It is noted that Blandin *et al.* used the replica-symmetric assumption. The replica-symmetric assumption in the replica theory using the replica trick and the assumption of uncorrelated boundary condition on a Bethe lattice seem to be equivalent. At the present stage the stability of a negative overlap for nonzero h cannot be answered because even the solution with a positive overlap and a lower free energy presents unphysical property, negative entropy at zero temperature. It is interesting to see whether there exists a new solution with a negative overlap for a nonzero magnetic field, which may be accessible if the method to deal with correlated boundary condition or with replica-symmetry breaking is applied, and whether it is stable.

VII. SUMMARY AND FUTURE

The hypothetical replica coupling between two replicas is expected to project only maximum overlap. Since the replica-coupling field is set to be zero in the end, it might not distort the real physics. Instead it seems to bring us a mechanism for a possible new vision in the structure of many pure states in the spin-glass phase.

The system with two interacting replicas opens an interesting question in the spin-glass phase as to whether there is another party of equilibrium states for a nonzero magnetic field, i.e., whether a negative overlap is possible even for a nonzero magnetic field. Understanding within the half has been extended to a plausible level mainly by the Parisi static theory and the Sompolinsky dynamic theory. A negative overlap for zero magnetic field can be conceived from the spin reversal symmetry but things become not obvious for a nonzero magnetic field. A pair of mirror states for zero field, mapped to each other by overall spin reversal, are equally favorable or unfavorable in the presence of a small field. In this sense a simple thought that half the whole pure states suddenly become unstable for a small field turned on is not appealing. On the other hand the argument that the two parties of pure states coexist and yield negative overlaps with each other until a sufficient field is applied is worth being examined. Unfortunately, the assumption used in this paper that the effective fields at different subtrees are not correlated has given rise to an unphysical result, negative entropy at zero temperature. The question about negative overlap has not been answered yet.

A study on a Bethe lattice has a seeming advantage that the replica trick can be avoided and may serve as a more realistic mean-field theory than the SK model. But it turns out that there is the same level of problem encountered as in the SK model, negative entropy at zero temperature. If this problem is related to the assumption of the replica symmetry, it is likely that a theory should have a symmetry-breaking scheme, a scheme to deal with many pure states, such as the Parisi replica-symmetry-breaking scheme and the Sompolinsky assumption of many long-time scales. The two-replica coupling in this paper is the scheme to break the whole pure states into two parties possibly even for a nonzero magnetic field.

We expect that a previous symmetry-breaking scheme, Parisi's or Sompolinsky's, with the two-replica coupling scheme might lead to a true answer. The former deals with overlaps within each of two groups of pure states and the latter with overlaps between the two. Recently the dynamic spin glass with two interacting replicas has been studied.[20] Interestingly, the result without consideration of long-time behavior in relaxation through many pure states has been found to be exactly equivalent to the result on a Bethe lattice in this paper and to be dynamically unstable. The study of long-time behavior for two interacting replicas is now in progress.

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