Nonequilibrium statistical mechanics of the spin-$\frac{1}{2}$ van der Waals model. II. Autocorrelation function of a single spin and long-time tails

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The autocorrelation function of a single spin $\langle \mathbf{s}(t)\cdot\mathbf{s}(0)\rangle$ has been obtained from the time-evolution solutions given in the preceding paper by carrying out the ensemble averages explicitly. Slow decay is found in the transverse component and only at high temperatures ($T > T_c$). The exponent κ, where $\langle s^x(t)s^x(0)\rangle \sim t^{-\kappa}$, as $t\to\infty$, is found to depend discontinuously on the spin-spin interaction strength $R = J_z/J$: $\kappa=2$ if $R=0$, $\kappa=3$ if $0<R\le 2$ ($R\neq 1$), $\kappa=\infty$ if $R=1$, and $\kappa=\infty$ if $R>2$. Slow decay in this model is attributed to nondimensional effects, e.g., cooperativity. Physical and mathematical mechanisms of the slow decay are described.

I. INTRODUCTION

The existence of long-time tails in the velocity autocorrelation function (VAF) of a particle in a fluid ($\sim t^{-D/2}$, dimensionality D) appears to be strongly indicated by evidence, mainly from computer simulations and experimental measurements, obtained over the last two decades. What is much less well established are the origins and mechanisms of the slow decay behavior. (By slow decay, we shall mean a power-law form of the decay of the VAF.) Also, whether slow decay is a universal property of *Hermitian* many-body systems seems to be an open problem. Satisfactory understanding would probably ultimately require some detailed knowledge of the time evolution of a single particle in a given interacting environment. For realistic many-body models, obtaining time-evolution solutions from a canonical equation of motion represents a formidable task.

The possible existence of slow decay in fluids seems to have stimulated investigations of similar behavior in other systems. We may mention the Paley-Wiener criterion for relaxation functions,[1(a)] high-field electron transport in semiconductors,[2] anomalous spin diffusion in Heisenberg magnets,[3] heat conductivity in a one-component plasma,[4] and hydrogen-bond fluctuations in water dynamics.[5] Slow decay may be fundamentally linked to low-frequency phenomena in condensed matter.[1(b)]

Analytical studies show that slow decay in classical fluids originates from certain modes or excitations of a fluid or other basic properties.[6,7] Most commonly discussed are friction in Brownian-motion theory,[8] diffusive shear modes in hydrodynamic theory,[9] particle-number and momentum conservation in Fick's law,[10] etc. Although illuminating, they do not or cannot include collective effects at a microscopic level.[6] As a results, some have questioned the validity or relevance of these theories.[7,11]

The problem of the VAF of an electron in a system of fixed scatters (Lorentz gas) has drawn some attention recently.[10] Despite the progress made, there seems to be disagreement over the sources and mechanisms of long-time tails.[12]

In spin-diffusion theory, long-time tails are also understood only phenomenologically. It is based on an assumed pair correlation function, much as hydrodynamic theory is based on Fick's law.[13] This theory also yields an exponent $D/2$, but not surprisingly it has given no more understanding than hydrodynamic theory.

If the exponent for long-time tails has a seemingly universal value $D/2$, it would suggest that the observed slow decay is purely of geometric nature, a dimensional effect. Other physical effects (e.g., interaction or cooperative, quantum) evidently do not contribute at long times. That is, the VAF $v(t)$ behaves as follows:

$$v(t\to\infty) \sim At^{-a} + Bt^{-b} + \cdots ,$$

where A, B, ... are constants and $a \equiv D/2$, b, ... are exponents. The exponents satisfy an inequality: $0 < a < b < \cdots$.

There is a possibility that as the dimensions increase, the above inequality may not continue to hold. In particular, if $D\to\infty$ (hence, $a\to\infty$), but $b<\infty$, then slow decay in this high-dimension limit must arise from nondimensional effects. They are also likely to be present although not dominant when $D<\infty$.

The spin van der Waals model is known to be the $D=\infty$ limit of the NN anisotropic Heisenberg model.[14,15] Therefore, one cannot learn anything about purely dimensional effects from this model. But if there is a slow decay in this model, it must be of quantum cooperative nature of some limited generality. One may be able to identify the sources and mechanisms responsible for it.

In a realization of the canonical approach to dynamics, we shall obtain the spin autocorrelation function (SAF) by applying the time-evolution solutions given in the preceding article.[16] Our analysis is completely self-contained, i.e., no *a priori* assumptions or approximations invoked. It is based only on the requirement that $N \to \infty$, where N is the number of spins in our system. We find that slow decay exists in certain interaction regimes at high temperatures, but not at low temperatures. Slow decay in our model arises from a single spin coupling to some macroscopic modes of the system, which alone persists as $t \to \infty$.

Our mathematical results, first reported somewhat briefly several years ago,[17] are here given in full detail. In addition we describe the physical and mathematical mechanisms as well as identify the modes responsible for slow decay. Recently, Liu and Müller[18] have studied a classical version of the same model and obtained conclusions similar to ours for high temperatures.

II. SPIN van der WAALS MODEL AND TIME EVOLUTION OF A SINGLE SPIN: SUMMARY

In this section we shall present a brief summary of our solutions for the time evolution of a single spin in the spin van der Waals model obtained in the preceding article[16] (to be referred to as I). We have regarded a system of $N+1$ identical spins to be composed of two subsystems: One subsystem contains just one spin, which may be any one of the $N+1$ spins, and the other subsystem contains the remaining N spins. This division allows us to write the interaction energy of the system as a sum of the "self"-energy of the larger subsystem and the interaction energy of the two subsystems, i.e.,

$$H=H_0+V \; , \tag{1}$$

where

$$H_0=-(g\mathbf{S}^2-\omega S_z^2) \; , \tag{2}$$

$$V=-2(g\mathbf{s}\cdot\mathbf{S}-\omega s^z S_z) \; , \tag{3}$$

where $g=J/N$, $\omega=(J-J_z)/N=g-g_z$. Here $\mathbf{s}$ refers to the spin in the small subsystem and $\mathbf{S}$ to the spins of the large subsystem collectively. That is, the total spin $\mathbf{S}^{\rm tot}$ of our system of $N+1$ spins is

$$\mathbf{S}^{\rm tot}=\mathbf{s}+\mathbf{S} \; . \tag{4}$$

As in I, $\mathbf{s}$ and $\mathbf{S}$ will be referred to as a small spin and a large spin, respectively.

We have shown that the equation of motion for the small spin is expressible as a linear coupling of the small and large spins. Now, if $N \gg 1$, the large subsystem behaves classically and it appears somewhat like a "reservoir" to the small spin. In this limit the equation of motion for the small spin can be solved asymptotically. The asymptotic solutions are, of course, the only ones appropriate thermodynamically. They are given below:

$$s^x(t)=s^xL_{xx}(t)+s^yL_{xy}(t)+s^zL_{xz}(t) \; , \tag{5a}$$

$$s^y(t)=s^xL_{yx}(t)+s^yL_{yy}(t)+s^zL_{yz}(t) \; , \tag{5b}$$

$$s^z(t)=s^xL_{zx}(t)+s^yL_{zy}(t)+s^zL_{zz}(t) \; , \tag{5c}$$

where L's are quantities defined in terms of H_0 only. See Eqs. (35a)–(35c) in I. Our general solutions given in I may also be presented in the above form.

III. AUTOCORRELATION FUNCTION OF A SINGLE SPIN

We shall define the spin autocorrelation function SAF of a single spin as follows:

$$G_{ij}(t)=4\,{\rm Tr}[s^i(t)s^j(0)e^{-\beta H}]/{\rm Tr}e^{-\beta H} \equiv 4\langle s^i(t)s^j(0)\rangle, \quad i,j=x,y,z \; . \tag{6}$$

From our asymptotic solution, also from our general solution, we find that the diagonal components are

$$G_{xx}(t)=G_{yy}(t)=\langle L_{xx}(t)\rangle \; , \tag{7}$$

$$G_{zz}(t)=\langle L_{zz}(t)\rangle \; . \tag{8}$$

If $N \to \infty$, the ensemble averages may be carried out with respect to H_0 rather than H. Errors due to this replacement are of lower orders in N and become negligible in the large-N limit. It is consistent with our using the asymptotic solutions.

$L_{xx}(t)$ and $L_{zz}(t)$ may be obtained from I:

$$S^2L_{xx}(t)=[S_x^2+(S^2-S_x^2)\cos(\Omega t)]\cos(\Omega_z t) -[S_xS_y[1-\cos(\Omega t)]-SS_z\sin(\Omega t)]\sin(\Omega_z t) \; , \tag{9}$$

$$S^2L_{zz}(t)=S_z^2+(S^2-S_z^2)\cos(\Omega t) \; , \tag{10}$$

where $\Omega=2gS$, $\Omega_z=2\omega S_z$. For our asymptotic solutions we have replaced $\mathbf{S}^2$ by S^2, i.e., $\mathbf{S}^2=S(S+1)\approx S^2$.

Substituting (9) and (10) into (7) and (8), respectively, we obtain

$$G_{xx}(t)=\langle (S_x/S^2)\cos(\Omega_z t)\rangle +\langle [1-(S_x/S)^2]\cos(\Omega t)\cos(\Omega_z t)\rangle +\langle (S_z/S)\sin(\Omega t)\sin(\Omega_z t)\rangle \tag{11}$$

and

$$G_{zz}(t)=\langle (S_z/S)^2\rangle+\langle [1-(S_z/S)^2]\cos(\Omega t)\rangle \; . \tag{12}$$

In obtaining (11), we have dropped a term containing S_xS_y/S^2. It contributes to lower orders in N compared with those retained.[19(a),19(b)]

Taking advantage of the xy symmetry in H_0, we can write (11) as

$$G_{xx}(t)=\tfrac{1}{2}\langle [1-(S_z/S)^2]\cos(\Omega_z t)\rangle +\tfrac{1}{4}\langle (1+S_z/S)^2\cos(\phi_- t)\rangle +\tfrac{1}{4}\langle (1-S_z/S)^2\cos(\phi_+ t)\rangle \; , \tag{13}$$

where $\phi_\pm=\Omega\pm\Omega_z$. The above expression can be further simplified if $T>T_c$, where the sign of S_z is arbitrary. Hence, let $S_z\to -S_z$ in the third term on the right-hand side (RHS) of (13). Now from the definitions of Ω and Ω_z, $\phi_+(-S_z)=\phi_-(S_z)$. Thus,

$$G_{xx}(t)=\tfrac{1}{2}\langle[1-(S_z/S)^2]\cos(\Omega_z t)\rangle + \tfrac{1}{2}\langle(1+S_z/S)^2\cos(\phi t)\rangle \ , \tag{14}$$

where

$$\phi\equiv\phi_-(S_z)=\Omega-\Omega_z=2g[S-(1-R)S_z] \ , \tag{15}$$

$$R=g_z/g=J_z/J \ . \tag{15a}$$

The new parameter R that we have introduced here will be found to characterize the time-dependent behavior of the SAF. Hence, it is worth noting that $R=0$, 1, and ∞ denote, respectively, the pure XY, isotropic Heisenberg, and pure Ising interactions. The new frequency ϕ is neither even nor odd in S_z except when $R=\infty$, where it is odd in S_z. There are two frequencies, Ω_z and ϕ, in $G_{xx}(t)$, but only one frequency, Ω, in $G_{zz}(t)$. These frequencies play an essential role in determining the long time behavior of the SAF.

It turns out that (14) is also valid when $T<T_c$ if $R<1$, where $\langle S_z\rangle/N=0$. If $R>1$, where $\langle S_z\rangle/N\neq 0$, one must use (13). One, however, gets the same result from (14) if the linear term in the second term on the right-hand side of (14) is ignored, i.e.,

$$\langle(1+S_z/S)^2\cos(\phi t)\rangle\to\langle[1+(S_z/S)^2]\cos(\phi t)\rangle \ .$$

Observe that when $R=1$, where $\phi_+=\phi_-=\Omega$, the above-mentioned linear term does vanish. In any event, the ensemble averaging for $T<T_c$ and $R>1$ is straightforward and there is no difficulty here.

Now we shall turn to the remaining technical point—that of ensemble averaging in the asymptotic limit. We recall from our earlier work[19(a)] that if $F=F(S,S_z)$,

$$\langle F\rangle\simeq\langle F\rangle_{H_0}=Z^{-1}\sum_S\sum_{S_z}g(S)F(S,S_z)\exp(-\beta H_0) \to Z^{-1}\int_0^\infty dS\,g(s)\int_{-S}^{S}dS_z\,F(S,S_z)\exp(-\beta H_0) \ , \tag{16}$$

where Z is the partition function and $g(S)$ is the degeneracy factor. For N spin-$\frac{1}{2}$ particles,

$$g(S)=2^{N+1}s(1+s)^{-1}\exp[-NW(s)], \quad s=2S/N \ , \tag{17}$$

where

$$W(s)=\tfrac{1}{2}[(1-s)\ln(1-s)+(1+s)\ln(1+s)] \ . \tag{17a}$$

The partition function Z may be evaluated through the integral in (16) by setting $F=1$. If β is small, the phase factor in (16) is dominated by W, the entropy term at $S/N\approx 0$, which defines the high-temperature region. If β is large, the phase factor is at a maximum for $S/N\approx 1$. Hence, there exists an ordered state below a critical temperature T_c. This ordered state depends explicitly on whether $R<1$ or $R>1$. See Appendix A for additional detail.

IV. MATHEMATICAL MECHANISM OF LONG-TIME TAILS

Before carrying out a detailed analysis, it might be useful to have in mind a qualitative picture of the mathematical mechanism responsible for long-time tails in the SAF. The existence of long time tails implies a persistence of memory. That is, the initial condition of the small spin, say, must be lost only gradually. The memory of the small spin is lost through its coupling with modes of the "reservoir" or the large spin. Hence, the state of the reservoir plays a critical role in determining the nature of the decay of memory. Suppose that the reservoir is in an ordered state, i.e., there is one dominant macroscopic mode. The small spin coupled to it cannot long remain impervious to its influence. The small spin will likely lose its memory rapidly. Or suppose now that the reservoir is in a stationary state. The state of the small spin must be closely bound to it since the state of the reservoir is made up of individual states of spins like the small spin. The memory of the small spin will also likely be lost rapidly.

The most likely place to find a slow decay of memory is in the high-temperature phase of the reservoir. In this phase there are many fluctuating modes, none of which are, however, dominant as in the low-temperature phase. Over a long period of time, most of these modes will not lead to a slow decay since their fluctuations are large and random. Consider the x or y component of the small spin. If it is coupled coherently to some modes with small fluctuations, slow decay may emerge. Below we shall attempt to isolate these special modes.

We shall consider $G_{xx}(t)$ for $T>T_c$ when $R\neq 1$. Let us decompose $G_{xx}(t)$ given by (14) into two terms:

$$G_{xx}(t)=G_{xx}^{(1)}(t)+G_{xx}^{(2)}(t) \ , \tag{18}$$

where

$$G_{xx}^{(1)}(t)=\tfrac{1}{2}\langle[1-(S_z/S)^2]\cos(\Omega_z t)\rangle \tag{18a}$$

and

$$G_{xx}^{(2)}(t)=\tfrac{1}{2}\langle(1+S_z/S)^2\cos(\phi t)\rangle \ . \tag{18b}$$

The first term $G_{xx}^{(1)}$ has a classical structure [see the first term on the RHS of (11)]. It may be called a *direct* term. The second term $G_{xx}^{(2)}$ is a consequence of spin algebra, similar to the exchange effect in atomic physics. Hence, it may be called an *exchange* term. As in atomic physics, the two terms can "interfere."

To look for modes responsible for slow decay, we now separately examine (18a) and (18b) as $t\to\infty$.[20] If t grows, the phases on the RHS's of (18a) and (18b) become very large. They lead to rapid oscillations and cancellations upon ensemble averaging except if the frequencies vanish, i.e., $\Omega_z\to 0$ and $\phi\to 0$. Hence, the only modes important for slow decay are those that can satisfy the condition $\Omega_z=0$ or $\phi=0$ or both.

The first condition $\Omega_z=0$ (i.e., $S_z=0$) can always be satisfied independently of R since S_z is bounded: $-S\le S_z\le S$ and $0\le S<\infty$. The small spin coupled to the $S_z\approx 0$ modes (i.e., the large spin largely in planar configurations) will likely exhibit long-lived character. The second condition $\phi=0$ gives $S_z=S/(1-R)$. Since $|S_z|\le S$, the second condition can be satisfied if and only if $R\ge 2$ and $R=0$. $R=0$ and 2 are marginal or turning points. For $0<R<2$, the second condition cannot be met. Hence, there can be no slow decay in the exchange term. The direct term is the only possible source of slow decay for this range of R. It is more difficult to understand the physical meaning of the second condition (just as exchange phenomena in atomic physics). But modes satisfying the second condition have maximal S_z, i.e., $|S_z|\approx S$, in contrast to modes favored under the first condition, which have minimal S_z, i.e., $|S_z|\approx 0$. The small spin coupled to the large spin in largely axial configurations may also exhibit long-lived character.

An interference between the direct and exchange terms can be demonstrated as follows: Let $R\to\infty$, i.e., $\phi\to-\Omega_z$. Then,

$$G_{xx}^{(2)}(t)\to\tfrac{1}{2}\langle[1+(S_z/S)^2]\cos(\Omega_z t)\rangle\ . \tag{19}$$

Hence,

$$G_{xx}(t)\to\langle\cos(\Omega_z t)\rangle=e^{-ct^2},\quad c>0\ . \tag{20}$$

The $(S_z/S)^2$-mode terms cancel out exactly as if destructively interfered, removing any possibility of slow decay. See Appendix B for the proof of (20). If one lowers R from $R=\infty$, the same interference can take place just as completely as long as $|\Omega_z|>\Omega$. That is, ϕ, in effect, behaves discontinuously when $t\to\infty$:[21]

$$\phi=\Omega-\Omega_z\to-\Omega_z,\quad \text{if } |\Omega_z|>\Omega\ .$$

As far as the asymptotic behavior is concerned [i.e., $G_{xx}^{(2)}(t\to\infty)$], the time evolution of the small spin behaves as if $g=0$ (i.e., $R=\infty$) if $2<R\le\infty$. In this range of R, the exchange term is contributed by $\pm S_z$ equally for a given S (i.e., S_z symmetric).

The crossover occurs at $|\Omega_z|=\Omega$ or $R=2$. At this upper marginal or turning point, $\phi\to 0$ means $S_z\to -S$. Hence, only the negative components of S_z contribute (S_z symmetry broken).

Below this marginal point, ϕ again behaves discontinuously:

$$\phi=\Omega-\Omega_z\to\Omega,\quad \text{if } |\Omega_z|<\Omega \text{ or } 0<R<2\ .$$

Now $\phi\to 0$ if and only if $S\to 0$. Hence, the character of ϕ has changed completely. As a result, the exchange term can contribute only minimally now (i.e., no interference possible). Note that if $R=1$, $\phi=\Omega$ exactly since $\Omega_z=0$. Hence, the behavior of $G_{xx}(t)$ for $R=1$ should be quite similar to the asymptotic behavior of $G_{xx}^{(2)}(t)$ for $0<R<2$.

At the lower marginal point (also a turning point if negative R were to be included), $\phi\to 0$ means $S_z\to +S$. Now only the positive components of S_z can contribute. As in the upper marginal point, there is also a "symmetry break." However, the behavior of the time evolution of the small spin at the lower marginal point is *asymptotically* very different. For example, $\dot{s}^z(t)\neq 0$ if $R=0$ [see Eq. (10) of I] but one may regard $\dot{s}^z(t)=0$ if $R=2+\epsilon$, $\epsilon\to 0+$ since asymptotically the SAF behaves as when $R=\infty$ [compare (40a)–(40c) of I]. Hence, the time evolution of the small spin is three-dimensional if $R=0$, but is largely two-dimensional if $R=2$. Thus, the exchange term can have contributions from fluctuations in the z direction in spin space when $R=0$, but not when $R=2$.

The symmetry, or symmetry breaking, associated with R occurring in the asymptotic behavior of the exchange term, will be referred to later in Sec. V as *dynamic symmetry.*

V. $G_{xx}(t)$ FOR $T>T_c$: ANALYSIS

We shall now obtain the transverse component of the SAF of a single spin when $T>T_c$ by carrying out the ensemble averages explicitly. We look for an asymptotic solution in the form: $G_{xx}(t\to\infty)\sim t^{-\kappa}$, where $\kappa=\kappa(R)$.

A. Direct term

Writing out the ensemble average therein explicitly (see Appendix A), we can express (18a) as follows:

$$G_{xx}^{(1)}(t)=\frac{2}{\sqrt{\pi}}\int_0^1 du\left[1-\frac{b^2u^2}{1-\alpha u^2}\right]\int_0^\infty d\eta\, e^{-\eta^2}\eta^2\cos(2A^{1/2}u\eta)$$
$$=\tfrac{1}{2}\int_0^1 du\left[1-\frac{b^2u^2}{1-\alpha u^2}\right](1-2Au^2)e^{-Au^2}\ , \tag{21}$$

where $A=N(ab\omega t)^2=N(1-R)^2(abgt)^2$, $\alpha=1-b^2$. See Appendix A for the definitions of a and b. The time t is now contained entirely in our new parameter A. Dropping the part that vanishes rapidly as $t\to\infty$ (see Appendix B), we can obtain for $A\to\infty$ (i.e., $t\to\infty$ and $R\neq 1$)

$$G_{xx}^{(1)}(t\to\infty)=-(b^2/2)\int_0^{1\to\infty} du\frac{u^2}{1-\alpha u^2}(1-2Au^2)e^{-Au^2} \tag{22a}$$
$$=(\sqrt{\pi}b^2/4)A^{-3/2}+O(A^{-5/2})\sim t^{-3}\ . \tag{22b}$$

The upper limit may be increased as shown above, which introduces little error since most of the contribution to the integral comes from $u\approx 0$. As a result, the denominator term $(1-\alpha u^2)$ may also be replaced by unity for the leading contribution. Hence, the direct term yields slow decay with an exponent, say $\kappa_1=3$, independently of R except $R\neq 1$. To obtain the above asymptotic result, it would have been sufficient to take

$S_z = N^{1/2}abu\eta$ and $S = N^{1/2}a\eta$. (See Appendix A). Also, $u \approx 0$ implies that $S_z \approx 0$; i.e., the fluctuations of the large spin are small and largely of planar configurations. The small spin coupled to these small fluctuations is evidently long-lived.[22(a)]

If $R = 1$, then $A = 0$. Then, together with $b^2 = 1$ or $\alpha = 0$ (see Appendix A), we obtain from (21)

$$G_{xx}^{(1)}(t) = \tfrac{1}{3} . \tag{22c}$$

The same result is obtained just as readily from (18a). When $R = 1$, $\langle (S_Z/S)^2 \rangle = \frac{1}{3}$. At this isotropic interaction point, the direct term has no time dependence.

B. Exchange term

The exchange term (18b) may be expressed similarly. After carrying out the η integration, we obtain:

$$G_{xx}^{(2)}(t) = \tfrac{1}{4}\int_{-1}^{1} du \left[1 + \frac{bu}{\sqrt{1-\alpha u^2}}\right]^2 (1 - 2Az^2)e^{-Az^2} , \tag{23}$$

where

$$z = z(u) = u - \gamma(1-\alpha u^2)^{1/2}, \quad \gamma^{-1} = b\omega/g = b(1-R) . \tag{24}$$

Note that $z(u)$ is neither even nor odd in u. But if $\gamma \to 0$ (i.e., $R \to \infty$), $z \to u$. In this limit, (23) can be reduced to a form similar to (21).

For $t \to \infty$, (23) may be generally evaluated as follows: As $A \to \infty$, the above integral will be finite if and only if $z \to 0$ in the interval of $u = (-1, 1)$. Hence, we shall look for zeros of $z(u)$, defined as $z(u_0) = 0$. We find from (24) that the zeros are given by

$$u_0^2 = (\gamma^{-2} + \alpha)^{-1} .$$

Now the condition that these zeros lie in the allowed interval of u, i.e., $|u_0| \le 1$, gives the condition for the existence of slow decay in the exchange term:

$$(1-R)^2 \ge 1 . \tag{25}$$

That is, $R \ge 2$ and $R = 0$. $R = 0$ and 2 are the same marginal or turning points, earlier deduced from the frequency condition (see Sec. IV). The existence of slow decay in the exchange term is thus R dependent, not as in the direct term. We shall refer to the above condition (25) as *dynamic symmetry*.[23] In Fig. 1(a), the zeros of $z(u)$ are illustrated as a function of R. Observe that for $R = 0$ and $R \ge 2$, the zeros lie in the closed interval of u. But for $0 < R < 2$, the zeros lie outside, i.e, $z \not\to 0$. In this case, we shall see that the exchange term vanishes rapidly.

For R satisfying the dynamic symmetry (25), we can evaluate (23) by expanding the integrand about $u = u_0$. We can also proceed another way, perhaps more interesting. Let us change our variable from u to z, i.e., obtain $u = u(z)$ from (24). We can then transform (23) as shown below:

$$G_{xx}^{(2)}(t) = \tfrac{1}{4}\int_{z_1}^{z_2} dz\, f(z)(1 - 2Az^2)e^{-Az^2} , \tag{26}$$

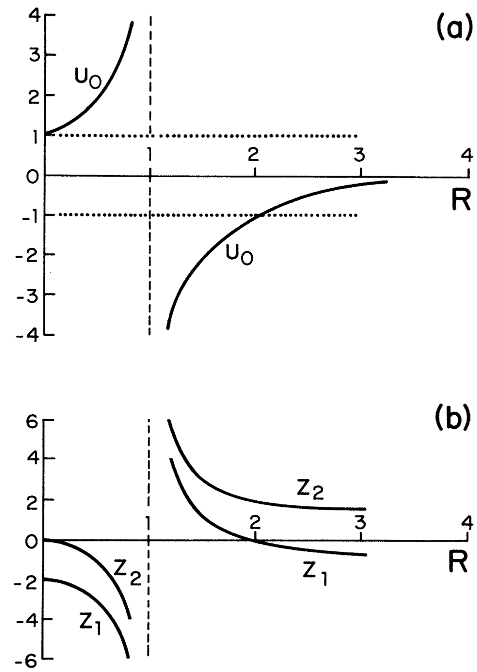

FIG. 1(a) u_0 vs R where $z(u_0) = 0$. See Eq. (24) for the definition of z. (b) z_1 and z_2 vs R, where $z_1 = -(R-2)/(R-1)$ and $z_2 = R/(R-1)$.

where

$$f(z) = \left[1 + \frac{bu}{\sqrt{1-\alpha u^2}}\right]^2 \frac{\partial u}{\partial z}\bigg|_{u=u(z)} , \tag{27}$$

$$z_1 = -(R-2)/(R-1) , \tag{28a}$$

$$z_2 = R/(R-1) . \tag{28b}$$

Now (26) is in the form of (21). Hence, the direct and exchange terms may be easily compared.

In Fig. 1(b), the new limits are illustrated as a function of R. For $0 < R < 2$, the limits z_1 and z_2 are both positive or both negative (i.e., $z = 0$ excluded). For $R > 2$, $z_1 < 0$ and $z_2 > 0$ (i.e., $z = 0$ included). For $R = 0$ and 2, one of them is zero. The new limits are no longer symmetric: $-1 < u < 1$ but $z_1 < z < z_2$. The length of the interval, however, remains the same: $z_2 - z_1 = 2$. The transformation $u = u(z)$ has put the dynamic symmetry into the limits of z. For asymptotic analysis, limits are often susceptible to manipulation.[22(b)]

After some algebraic manipulations (see Appendix C), we can express (27) in the following form:

$$f(z) = p^2 + \frac{pqz}{D(z)} + \frac{b^2z^2}{(1-\alpha z^2)} + \frac{\alpha b^2 u_0 z^3}{(1-\alpha z^2)D(z)} , \tag{29}$$

where

$$p = (2-R)bu_0 , \tag{30a}$$

$$q=2b-\alpha u_0 p \ , \tag{30b}$$

$$D(z)=(1-\alpha u_0^2\gamma^{-2}z^2)^{1/2} \ . \tag{30c}$$

Note that p and q are constant and $D(z)$ is even in z. The four terms on the RHS of (29) are alternately even and odd functions of z. At one marginal point ($R=2$), $p=0$. Hence, the first two terms do not contribute. But both are present at the other marginal point ($R=0$).

Using (29) in (26), we can now evaluate the exchange term in different regions of R indicated in Fig. 1(b).

1. $R>2$

If $A\to\infty$ (i.e., $t\to\infty$), the main contributions in (26) come from $z\approx 0$. Observe from Fig. 1(b) that $z=0$ lies within the interval (z_1,z_2). Hence, we can extend the upper and lower limits:

$$G_{xx}^{(2)}(t\to\infty)=\tfrac{1}{4}\int_{z_1\to-\infty}^{z_2\to\infty}dz\, f(z)(1-2Az^2)e^{-Az^2} \tag{31a}$$

$$=\frac{b^2}{2}\int_0^\infty dz\frac{z^2}{1-\alpha z^2}(1-2Az^2)e^{-Az^2} \ . \tag{31b}$$

To obtain (31b), we have noted that the constant term and the odd terms of $f(z)$ contribute nothing, leaving only the third term on the RHS of (29). Now from (22a), we see that

$$G_{xx}^{(2)}(t\to\infty)=-G_{xx}^{(1)}(t\to\infty) \ . \tag{32}$$

Hence, if $R>2$,

$$G_{xx}(t\to\infty)\sim e^{-ct^2},\ c>0 \ . \tag{33}$$

That is, $\kappa(R>2)=\infty$. In this region of R there is no slow decay at all. It is as if the direct and exchange terms have interfered completely destructively.

2. $R=2$

This is a marginal point of the dynamical symmetry. It is indicated by $z=0$ being coincident with one of the limits [see Fig. 1(b)]. One can thus write

$$G_{xx}^{(2)}(t\to\infty)=\tfrac{1}{4}\int_0^{z_2\to\infty}dz\, f(z)(1-2Az^2)e^{-Az^2}$$

$$=\frac{b^2}{4}\int_0^\infty dz\frac{z^2}{1-\alpha z^2}(1-2Az^2)e^{-Az^2} \ . \tag{34}$$

To obtain (34), we have noted that the first and second terms of $f(z)$ vanish identically since $p=0$ if $R=2$. The third term of $f(z)$ contributes $O(A^{-3/2})$ while the fourth term gives $O(A^{-2})$. Hence, we have retained only the third term on the RHS of (34). Now comparing (34) with (22a), we see that

$$G_{xx}^{(2)}(t\to\infty)=-\tfrac{1}{2}G_{xx}^{(1)}(t\to\infty) \ .$$

Hence,

$$G_{xx}(t\to\infty)=\tfrac{1}{2}G_{xx}^{(1)}(t\to\infty)\sim t^{-3} \ . \tag{35}$$

That is, $\kappa(R=2)=3$. At this marginal point the destructive interference is only partially complete, allowing the slow decay from the direct term to exist.

3. $R=0$

This is the other marginal point, where now $z_2=0$. Also, $p=2b$, $q=2b^3$, and $u_0=+1$. One can write

$$G_{xx}^{(2)}(t\to\infty)=\tfrac{1}{4}\int_{z_1\to-\infty}^{0}dz\, f(z)(1-2Az^2)e^{-Az^2}$$

$$=(b^4/2)A^{-1}+O(A^{-3/2}) \ . \tag{36}$$

In obtaining (36), we have noted that the first term of $f(z)$ does not contribute. But the second term does, which in fact gives the leading order. Thus, for $R=0$,

$$G_{xx}(t\to\infty)=G_{xx}^{(2)}(t\to\infty)\sim t^{-2} \ . \tag{37}$$

That is, $\kappa(R=0)=2$. For this pure XY interaction, the slowest of slow decay comes from the exchange term, not at all interfered.

4. $0<R<2$ ($R\neq 1$)

For this range of R the dynamic symmetry is not satisfied. Observe [see Fig. 1(b)] that $z=0$ is outside the interval (z_1,z_2). We can thus write

$$G_{xx}^{(2)}(t\to\infty)\le\begin{cases}U\exp(-Az_2^2), & 0<R<1 \ , \quad (38a)\\ U\exp(-Az_1^2), & 1<R<2 \ , \quad (38b)\end{cases}$$

where

$$U=\tfrac{1}{4}\left|\int_{z_1}^{z_2}dz\, f(z)(1-2Az^2)\right| \ . \tag{39}$$

Hence, regardless of the detail of $f(z)$, $G_{xx}^{(2)}(t\to\infty)$ vanishes rapidly. Thus,

$$G_{xx}(t\to\infty)=G_{xx}^{(1)}(t\to\infty)\sim t^{-3} \ . \tag{40}$$

That is, $\kappa(0<R<2)=3$. Slow decay arises entirely from the direct term.

5. $R=1$

When $R=1$, $b=1$ and $\alpha=0$. Also, $A=0$, but $Az^2=N(agt)^2\equiv B>0$. This follows from $z\to-\gamma$ and $\gamma=1/(1-R)$. See (24). Thus, from (23)

$$G_{xx}^{(2)}(t)=\tfrac{1}{4}\int_{-1}^{1}du(1+u)^2(1-2B)e^{-B}$$

$$=\tfrac{2}{3}(1-2B)e^{-B} \ . \tag{41}$$

The same result is also obtained from (18b) with $\phi=\Omega$, i.e., $\Omega_z=0$:

$$G_{xx}^{(2)}(t)=\tfrac{1}{2}\langle(1+S_z/S)^2\cos(\Omega t)\rangle$$

$$=\tfrac{1}{2}\langle(1+S_z/S)^2\rangle\langle\cos(\Omega t)\rangle \ . \tag{42}$$

Now it is shown in Appendix B that $\langle\cos(\Omega t)\rangle=(1-2B)\exp(-B)$. The above decoupling occurs if and only if $R=1$. See Appendix D.

Hence, together with the direct term (22c),

$$G_{xx}(t)=\tfrac{1}{3}+\tfrac{2}{3}(1-2B)e^{-B}\ . \tag{43}$$

The time-dependent part, i.e., $G_{xx}(t)-\frac{1}{3}$, decays very rapidly, which we shall thus denote by $\kappa(R=1)=\infty$. Excluding the constant term, there is no slow decay possible when the interaction is isotropic. We shall see in Sec. VII that the behavior of $G_{xx}(t;R=1)$ is essentially very similar to that of $G_{zz}(t;R\neq 1)$. By symmetry, $G_{xx}(t)=G_{zz}(t)$ if $R=1$.

Our results for the exponent κ may be summarized as follows: $\kappa(R=0)=2$, $\kappa(0<R<1)=3$, $\kappa(R=1)=\infty$, $\kappa(1<R\leq 2)=3$, $\kappa(R>2)=\infty$. The exponent behaves discontinuously. Slow decay does not exist if $R=1$ and $R>2$.

VI. NUMERICAL ANALYSIS OF $G_{xx}(t)$ AT $T>T_c$

It is of interest to know when the asymptotic behavior of $G_{xx}(t)$ begins to manifest. For this purpose, we have evaluated (21) and (23) numerically by applying the standard Simpson method to the u integration. The integration step was successively decreased until a reasonably good stabilization of the result was achieved. In Fig. 2, we give a double-logarithmic plot of $G_{xx}(t)$ versus t for $R=0.5$, 1.4, and 3 at various $T>T_c$. The time t is given in units of $\hbar/gN^{1/2}$ if $R<1$ and $\hbar/g_zN^{1/2}$ if $R>1$. See our earlier work[19(a)] for the origin of the time scales. In this plot the asymptotic slope of $G_{xx}(t)$ should give the values of the exponent κ, which may be compared with those obtained analytically in Sec. V.

First, observe that all the correlation curves show zero initial slope as is required of a Hermitian system.[24] For $t=1\sim 10$, there are structures in the correlation curves, evidently arising from various competing factors of quantum origin. As the time grows, these early patterns begin to vanish. They do not appear to have any influence on the asymptotic behavior.

Let us now examine the curves individually. For $R=3$ ($T=2T_c$), the downward curvature of $G_{xx}(t)$ steadily increases toward infinite slope, i.e., the slope never attains a constant value. This behavior is consistent with the Gaussian decay obtained for $R>2$ and $T>T_c$ (i.e., $\kappa=\infty$). At $R=1.4$ ($T=2T_c$), the character of the curve is decidedly different. It appears to reach a nonzero constant value of slope. Reducing R to $R=0.5$ and also T to $T=1.28T_c$ leaves the slope of the correlation curve virtually unchanged. The two lines appear to become essentially parallel, indicating the same exponent. This value is consistent with $\kappa=3$ obtained for $0<R\leq 2$ and $T>T_c$. Finally at $R=0$ ($T=4T_c/3$), the correlation curve also approaches a constant slope, but not in parallel with those of the previous. The slope is smaller, consistent with $\kappa=2$ for $R=0$ and $T>T_c$ given in Sec. V.[25]

Our numerical work clearly indicates that the correct power-law behavior of $G_{xx}(t)$ does not begin to emerge until $t\sim 10^2$. Any extrapolation taken earlier than this time could introduce no small error in the value of the exponent. For our model the asymptotic region may be said to lie in $t\gtrsim 10^2$.[26]

Our numerical results also indicate that the values of

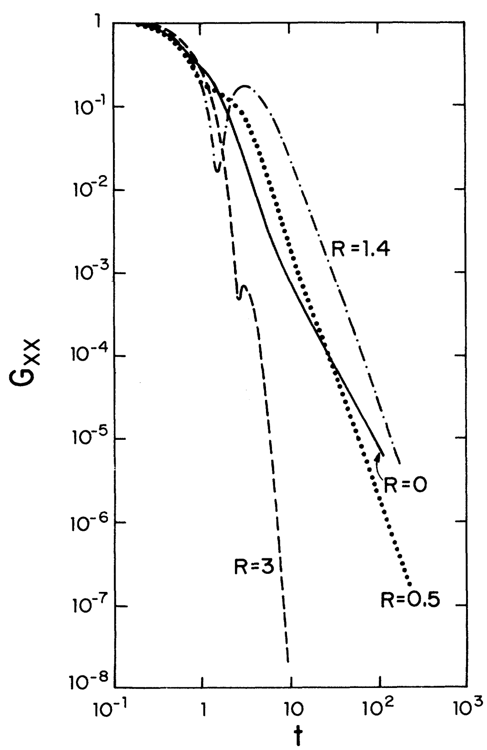

FIG. 2. Double-log plot of $G_{xx}(t)$ vs t for different values of R and $T>T_c$. The time t is given in units of $\hbar/gN^{1/2}$ if $R<1$ and $\hbar/g_zN^{1/2}$ if $R>1$. Value of R and T are $R=3$, $T=2$; $R=1.4$, $T=2$; $R=0.5$, $T=1.28$; $R=0$, $T=1.33$. T is given in units of T_c.

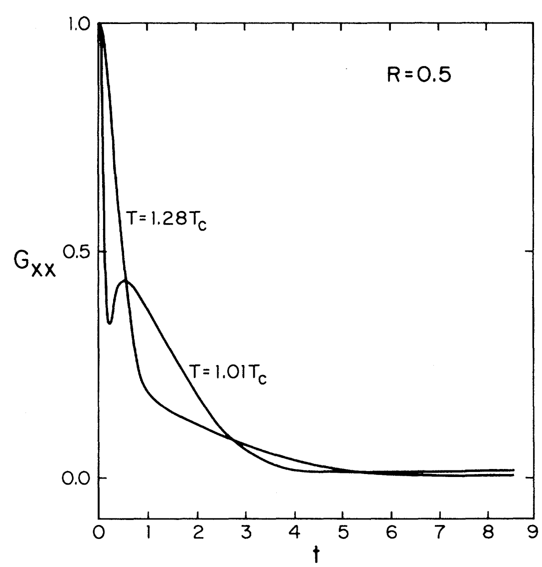

FIG. 3. $G_{xx}(t)$ vs t at $R=0.5$ and $T=1.01T_c$, $1.28T_c$.

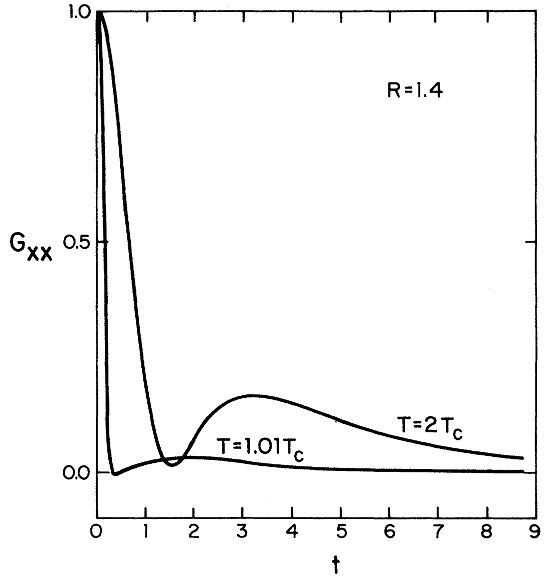

FIG. 4. $G_{xx}(t)$ vs t at $R=1.4$ and $T=1.01T_c$, $2T_c$.

the exponent are independent of T if $T>T_c$, just as shown by our asymptotic analysis. Long-time tails, if they exist, seem unaffected by the onset of critical fluctuations. That is, the exponent remains the same as $T\to T_c+$. The T dependence in $G_{xx}(t)$ is most noticeable in the short-time region. In Fig. 3, we have shown $G_{xx}(t)$ of $R=0.5$ at $T=1.01$ and 1.28 (T in units of T_c). In Fig. 4, we have shown $G_{xx}(t)$ of $R=1.4$ at $T=1.01$ and 2. The first temperature is almost within the neighborhood of the critical point; the second, well outside. For $t=0-1$, there are irregular structures, more pronounced if T is nearer T_c. But they disappear at $t=3-5$. What emerges thereafter at $t=4-6$ already begins to show asymptotic character, without much trace of the T dependence found in the earlier times.

VII. $G_{zz}(t)$ AT $T>T_c$

In this section we will evaluate the longitudinal component of the SAF of a single spin for $T>T_c$ given by (12). Let us write (12) as

$$G_{zz}(t)=G_{zz}^0+\Delta G_{zz}(t) . \tag{44}$$

The first term on the RHS of (44) is a constant, hence, not germane to long-time behavior. But it can be evaluated exactly. Converting the sums in the ensemble average therein into integrals, we obtain

$$G_{zz}^0\equiv\langle (S_z/S)^2\rangle=(b^2/\alpha)(\alpha^{-1/2}\tanh^{-1}\alpha^{1/2}-1) . \tag{45}$$

Similarly we obtain for the second term

$$\Delta G_{zz}(t)\equiv\langle[1-(S_z/S)^2]\cos(\Omega t)\rangle$$
$$=e^{-B}\int_0^1 du\frac{1-u^2}{1-\alpha u^2}[1-2B(1-\alpha u^2)]e^{\alpha B u^2} , \tag{46}$$

where $B=N(agt)^2$. If $t\to\infty$ (i.e., $B\to\infty$),

$$\Delta G_{zz}(t\to\infty)=-2Be^{-B}\int_0^1(1-u^2)e^{\alpha B u^2}$$
$$=-2Be^{-B}K . \tag{47}$$

The above integral K depends on the sign of α. Recall that $0<\alpha<1$ if $R<1$, but $\alpha<0$ if $R>1$. Hence, it must be handled separately.

If $0<\alpha<1$, the main contributions to K [see (47)] comes from $u\approx 1$. Changing the variable u to $x=1-u^2$, we can write

$$K=\tfrac{1}{2}e^{\alpha B}\int_0^{1\to\infty}dx\, xe^{-\alpha B x}+\cdots . \tag{48a}$$

If $\alpha<0$, the main contributions come from $u\approx 0$ in K. We can therefore write

$$K=\int_0^{1\to\infty}du\, e^{\alpha B u^2}+\cdots . \tag{48b}$$

Hence, together,

$$\Delta G_{zz}(t\to\infty)=\begin{cases}-(\alpha^2 B)^{-1}e^{-b^2 B}, & R<1 \quad (49a)\\ -(\pi B/|\alpha|)^{1/2}e^{-B}, & R>1 . \quad (49b)\end{cases}$$

If $\alpha=0$ (i.e., $R=1$), $\Delta G_{zz}(t)$ can be obtained exactly for any t. From (46) we have

$$\Delta G_{zz}(t)=\tfrac{2}{3}(1-2B)e^{-B},\quad R=1 . \tag{49c}$$

As noted earlier, indeed $G_{zz}(t)=G_{xx}(t)$ if $R=1$, as is required by symmetry. See (43).

There is thus no slow decay in $G_{zz}(t)$ when $T>T_c$. One can understand why long-time tails are absent here in $G_{zz}(t)$ if it is compared with, e.g., $G_{xx}^{(1)}(t)$. Equations (18a) and (46) show that both have the same structure differing only in the frequencies: Ω in $\Delta G_{zz}(t)$ and Ω_z in $G_{xx}^{(1)}(t)$. This difference makes their long-time behavior entirely different. Recall from our discussion in Sec. IV that as $t\to\infty$, what survives the ensemble averaging is that part which is coupled to smallest frequencies. If $\Omega\to 0$ (i.e., $S\to 0$) in (46), then $|S_z/S|\to 0$ or 1 since $|S_z|\le S$. (By this we mean that in the first case $S_z\to 0$ faster than $S\to 0$. In the second case, both go to zero at the same rate). The first possibility occurs when $R<1$ and the second when $R>1$. If $|S_z/S|\to 0$, then from (46) we see that $\Delta G_{zz}(t)$ is essentially determined by $\langle\cos(\Omega t)\rangle$. If $|S_z/S|\to 1$, then the leading term in $[1-(S_z/S)^2]$ vanishes. As a result, the next order term vanishes faster than $\langle\cos(\Omega t)\rangle$. In Appendix B we have shown that $\langle\cos(\Omega t)\rangle$ vanishes very rapidly. In contrast, in $G_{xx}^{(1)}(t)$, where $\Omega_z\to 0$, one picks up $S_z\approx 0$ for a *fixed* S.

Finally, comparison of Eq. (46) with Eq. (42) shows that the structure of $G_{zz}(t)$ is very similar to that of $G_{xx}(t;R=1)$. Thus, the time dependent behavior of $G_{xx}(t;R=1)$ can be explained essentially as described above for $G_{xx}(t;R\neq 1)$.

Illustrated in Figs. 5–8 is $G_{zz}(t)$, together with $G_{xx}(t)$. One can readily observe the different behavior between the two components. See especially Fig. 8. After the initial short time, $G_{zz}(t)$ is quickly dominated by G_{zz}^0. This

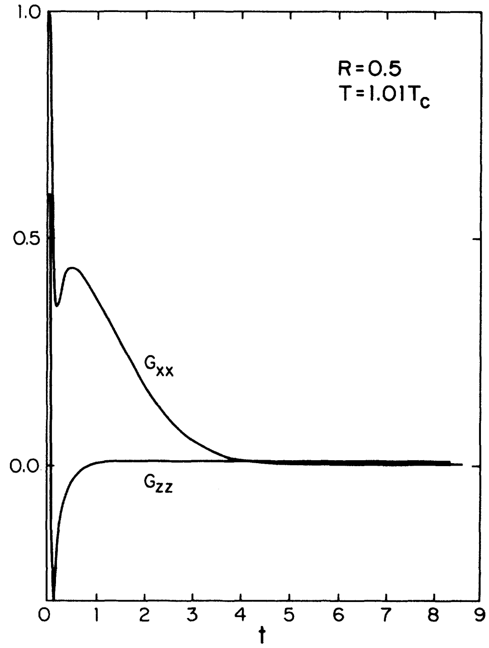

FIG. 5. $G_{xx}(t)$ and $G_{zz}(t)$ vs t at $R=0.5$, $T=1.01T_c$.

is no surprise. As noted earlier, the state of s^z is bound very close to a stationary state of the "reservoir" or the large spin. Recall that $\sum s^z(t)=S_z(t)=S_z(0)$. Hence, in the time-evolution process, the initial state of s^z never deviates "far" from one of the stationary states of H_0. The constant term may be regarded as representing that portion of s^z at t which still remains in the initial state. This portion depends on both R and T.

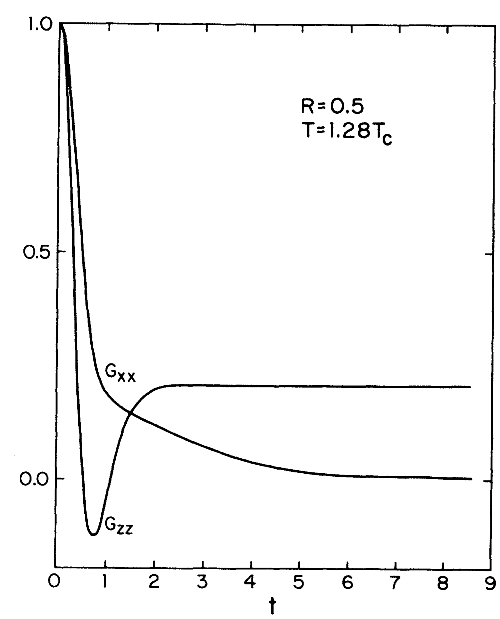

FIG. 6. $G_{xx}(t)$ and $G_{zz}(t)$ vs t at $R=0.5$, $T=1.28T_c$.

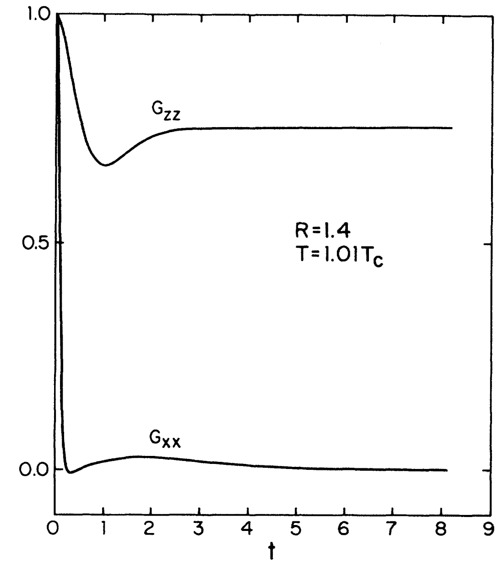

FIG. 7. $G_{xx}(t)$ and $G_{zz}(t)$ vs t at $R=1.4$, $T=1.01T_c$.

VIII. $G_{xx}(t)$ AND $G_{zz}(t)$ WHEN $T<T_c$

When T falls below T_c, the "reservoir" is dominated by a macroscopically ordered state. As a result, the analysis of the SAF of a single spin in the low-T region is much more straightforward than in the high-T region. In the low-temperature region we have shown[19(a)]

$$\langle S\rangle=0(N),\quad \langle S_z\rangle=O(N^x),\quad 0<x<1,\quad \text{if } R<1 \tag{50a}$$

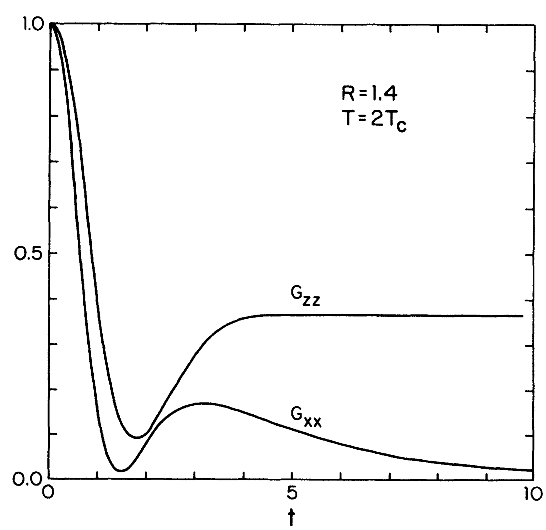

FIG. 8. $G_{xx}(t)$ and $G_{zz}(t)$ vs t at $R=1.4$, $T=2T_c$.

and

$$\langle S_z \rangle = \langle S \rangle = O(N), \quad \text{if } R > 1 \ . \tag{50b}$$

Hence, in the ensemble averaging, only those states very near an ordered state are significantly counted. As a result, the state of a single spin cannot evolve independently of such a dominant state of the "reservoir". It implies that slow decay is forbidden. We shall examine in this section both components of the SAF separately.

A. $G_{xx}(t)$

(i) R < 1. The SAF is given by [see (14)]

$$\begin{aligned} G_{xx}(t) &= \tfrac{1}{2}\langle [1-(S_z/S)^2]\cos(\Omega_z t)\rangle \\ &\quad + \tfrac{1}{2}\langle (1+S_z/S)^2 \cos(\phi t)\rangle \\ &\simeq \tfrac{1}{2}\langle \cos(\Omega_z t)\rangle + \tfrac{1}{2}\langle \cos[(\Omega-\Omega_z)t]\rangle \\ &= [\cos(\overline{\Omega}t/2)]^2 \langle \cos(\Omega_z t)\rangle \ , \end{aligned} \tag{51}$$

where $\overline{\Omega}=2g\langle S\rangle$. To obtain (51) we have used the low-temperature property (50a). In Appendix E, the remaining ensemble average is shown, i.e.,

$$\langle \cos(\Omega_z t)\rangle = \exp(-d^2t^2), \quad d^2 = 2\omega^2\langle S_z^2\rangle = \omega/\beta \ . \tag{52}$$

The above average also occurs in the SAF of the large spin:

$$\langle S_x(t)S_x\rangle = \langle S_x^2\rangle \langle \cos(\Omega_z t)\rangle \ .$$

See Appendix D. There are two time scales in (51): $1/\overline{\Omega}$ and $1/d$. But they are of different orders of magnitude, i.e., $1/\overline{\Omega}=0(1)$ and $1/d=O(\sqrt{N})$. Hence, for $t\sim 1/\overline{\Omega}$,

$$G_{xx}(t) \simeq [\cos(\overline{\Omega}/2)]^2 \ . \tag{53}$$

Thus, in time scales of $1/\overline{\Omega}$ the SAF is merely periodic. There is no slow decay as suggested earlier.

(ii) R > 1. Using the low-temperature property (50b), we find

$$\begin{aligned} G_{xx}(t) &= \tfrac{1}{2}\langle [1-(S_z/S)^2]\cos(\Omega_z t)\rangle \\ &\quad + \tfrac{1}{2}\langle [1+(S_z/S)^2]\cos(\phi t)\rangle \\ &\simeq \cos(\overline{\phi} t) \ , \end{aligned} \tag{54}$$

where $\overline{\phi}=2g_z\langle S\rangle$. The SAF is also oscillatory.

B. $G_{zz}(T)$

From (12) and using the low-temperature property (50a) and (50b) we obtain at once

$$G_{zz}(t) \simeq \begin{cases} \cos(\overline{\Omega}t), \quad \overline{\Omega}=2g\langle S\rangle, \quad R<1 \ , & (55a) \\ 1, \quad R>1 \ . & (55b) \end{cases}$$

C. $R=1$

When $R=1$, $G_{xx}(t)=G_{zz}(t)$. Hence, it follows directly that

$$G_{xx}(t)=G_{zz}(t)=\tfrac{1}{3}+\tfrac{2}{3}\cos(\overline{\Omega}t), \quad \overline{\Omega}=2g\langle S\rangle \ . \tag{56}$$

The oscillations found in the low-temperature regime refer to the oscillations of the small spin with respect to the direction of ordering of the large spin or the reservoir. The frequencies of oscillation are directly related to the magnitude of ordering. Also observe that these frequencies $\overline{\Omega}$ and $\overline{\phi}$ [see, e.g., (53) and (54)] are N independent, but d [see (52)] is N dependent.

IX. DISCUSSION

We have studied the long time behavior of the SAF of a single spin. We have demonstrated that in some physical regimes there can be slow decay, stemming from certain modes of the large spin to which the small spin is coupled. These modes have small fluctuations. Owing to the xy symmetry in our model, they assume xy planar or z axial configurations in spin space. In this section, we shall attempt to identify the modes responsible for slow decay by reexamining the time evolution solutions of the small spin obtained in I. The canonical approach, which we have realized for this problem, allows us to do so. We shall also reexamine the conditions that appear to be necessary for the existence of long time tails.

As we have seen, the SAF of a single spin is intimately tied to ensemble averaging. The long-time behavior is especially sensitive to the averaging processes used. We have shown that the ensemble average of, say, F may be written as

$$\langle F\rangle = \frac{\int du\, w(u)F(u)}{\int du\, w(u)} \tag{57}$$

where $w(u)$ is some weight function and u represents the order parameter of the system. For our model the weight function may be written as

$$w(u)=\exp[N\chi(u)] \ . \tag{58}$$

The phase factor $\chi(u)$ depends on the temperature of the system as follows:

$$\chi(u) = \begin{cases} -cu^2 - \cdots, \quad c>0, \text{ if } T>T_c & (59a) \\ \chi(\overline{u}) + \tfrac{1}{2}(u-\overline{u})^2\chi''(\overline{u}) + \cdots, \quad \text{if } T<T_c \ . & (59b) \end{cases}$$

where $\chi'(\overline{u})=0$ and $\chi''(\overline{u})<0$. Thus, the weight function is peaked at $u=0$ if $T>T_c$ and at $u=\overline{u}$ if $T<T_c$. Roughly speaking,

$$\langle F\rangle \approx \begin{cases} F(u\approx 0), \quad \text{if } T>T_c \ , & (60a) \\ F(u=\overline{u}), \quad \text{if } T<T_c \ . & (60b) \end{cases}$$

If $T>T_c$, the ensemble averaging (57) is reminiscent of a random Gaussian process.[27] It is stationary since our SAF satisfies

$$\langle \mathbf{s}(t+t')\cdot\mathbf{s}(t')\rangle = \langle \mathbf{s}(t)\cdot\mathbf{s}(0)\rangle \ .$$

It is, however, not Markovian since Doob's theorem is not satisfied,[27] i.e.,

$$\langle \mathbf{s}(t)\cdot\mathbf{s}(0)\rangle \neq \exp(-at), \quad a>0 .$$

Our Hamiltonian can never admit such a solution for the SAF.[24] If $T<T_c$, the weight function acts as a "filtering" function.[28] The ensemble averaging is essentially replaced by the value at the filter. In our problem, the processes of ensemble averaging above and below T_c are totally different. To obtain a slow decay, a Gaussian-like process is evidently necessary.

Now let us turn to the time evolution solutions of the small spin. As stated before, the time evolution of the small spin is brought about entirely by its coupling to the large spin, which itself evolves in time. The time evolution of the large spin **S** represents a simple plane rotation about the z direction in spin space, in which S_z is a constant of motion. The frequency of rotation is directly proportional to the magnitude of S_z. Illustrated in Fig. 9(a) and 9(b) are orientations of the large spin **S** for two possible values of S_z (e.g., $S_z\approx 0$ and $S_z \approx S$) at a given time. All possible different values of S_z $(-S\le S_z \le S)$ correspond to all possible allowed frequencies. An ensemble averaging of these different S_z values is a source of "fluctuations" in S_z.

At time t, let the small spin **s** be at an angle φ to the large spin **S**. As the large spin rotates about the z direction, the small spin follows it while at the same time rotating about the direction of the large spin. Since $\mathbf{s}\cdot\mathbf{S}$ is a constant of motion, one may regard the angle φ also to be a constant of motion. In this semiclassical context, the small spin precesses about a certain direction, identified as τ in I. Illustrated in Fig. 10(a) and Fig. 10(b) are relative directions of all three quantities **s**, **S**, and τ, projected in two dimensions.

The rotation of the small spin is thus a little more complex and more strongly R dependent than the rotation of the large spin. If $R\to\infty$, φ the angle between the two spins disappears. Also τ coincides with the z direction.

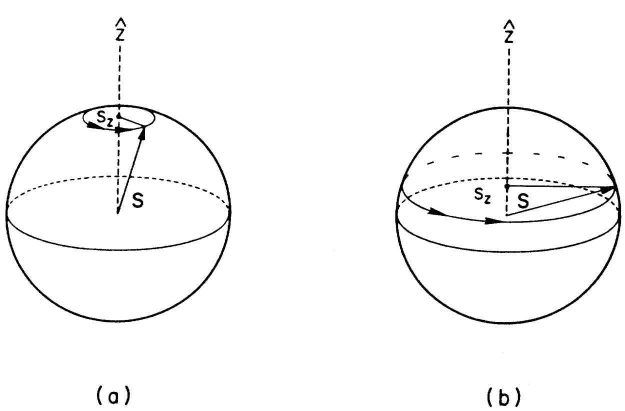

FIG. 9. Time evolution of the large spin **S** depicted in spin space at a given time and at two different values of S_z. The spin rotates about the z direction, as indicated by double arrows. Shown in (a) and (b) are, respectively, for $S_z\approx S$ and $S_z\approx 0$.

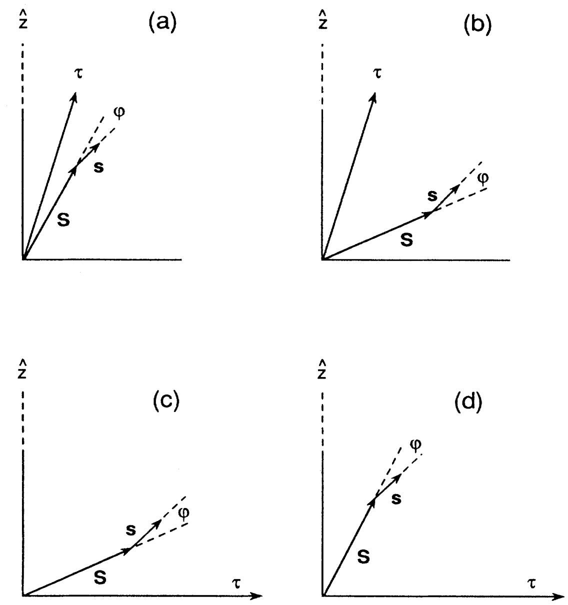

FIG. 10. Small spin **s**, large spin **S**, and precession axis τ, projected in two dimensions. The two spins make an angle φ, which is a constant of motion. Shown in (a) and (b) are for $S_z\approx S$ and $S_z\approx 0$, respectively, at arbitrary values of R. Shown in (c) and (d) are for $S_z\approx 0$ and $S_z \approx S$, respectively, at $R=0$. Observe that τ now lies in the xy spin plane.

In this limit the two spins have an identical rotation, both a pure rotation about the z direction. Consequently, they have an identical time evolution. This behavior may be regarded as a classical limit of the time evolution of the small spin since the large spin behaves like a classical vector.

If $R<\infty$, the rotation of the small spin is no longer a pure plane rotation about the z direction. It develops an extra rotation that amounts to oscillations of the xy plane about a direction normal to τ. If $S_z\approx 0$ [see Fig. 10(b)], the rotation causes the xy planar configuration to have small fluctuations.

If $R\to 0$, τ now lies in the xy plane [see Fig. 10(c) and 10(d)]. If $S_z\approx 0$ [see Fig. 10)(c)], the precession about τ causes fluctuations of the xy planar spin configuration, similar to that shown in Fig. 10(b). If $S_z\approx S$, the precession about τ causes fluctuations of the z axial spin configuration depicted in Fig. 10(d).

The slow decay that we have found in $G_{xx}(t)$ when $T>T_c$ is contained in the time evolution of the small spin described above. The slow decay emerges from certain modes, which we may term "slow decay modes", after a Gaussian-like process. There are two independent sources for these slow-decay modes: the direct term and exchange term. The direct term has the structure

$$\langle (S_x/S)^2\cos(\Omega_z t)\rangle \sim -\tfrac{1}{2}\langle (S_z/S)^2\cos(\Omega_z t)\rangle$$

if $t\to\infty$. The slow-decay modes in the direct term are quadratic, representing small fluctuations of the xy planar spin configuration, approximately depicted in Fig. 10(b). They are R independent. It is interesting to note that the quadratic modes in $\langle S_x^2\cos(\Omega_z t)\rangle$ are, however, not slow-decay modes.[29]

The exchange term has the structure $\frac{1}{2}\langle(1+S_z/S)^2\cos(\phi t)\rangle$, which contains both linear and quadratic modes. Dynamic symmetry governs the occurrence of slow-decay modes here. If $R>2$, only the quadratic modes can contribute in the form of $\frac{1}{2}\langle(S_s/S)^2\cos(\Omega_z t)\rangle$. There are thus slow-decay modes, which are of the xy planar spin configuration, identical to those found in the direct term except with an opposite phase. Hence, there can be no slow decay in this region of R. The origin of this remarkable result is found in the discontinuous behavior of the frequency of the exchange term when $t\rightarrow\infty$. The absence of slow decay in this regime $(2<R\leq\infty)$ invites the following interpretation: Asymptotically (i.e., $t\rightarrow\infty$) the time evolution of the small spin is of a pure plane rotation of $R=\infty$ [see Eqs. 40(a)–40(c) of I], or equivalently the small spin behaves as if the vector τ were fixed along the z axis, as long as $R>2$.

If $R<2$, the quadratic modes in the exchange term are no longer slow-decay modes. At the two marginal points ($R=0$ and 2), the quadratic modes are slow-decay modes, but they are not the dominant ones. At $R=0$ the dominant slow-decay modes are linear modes, allowed because of "broken symmetry," in the form of $\langle(S_z/S)\cos(\phi t)\rangle$. They represent small fluctuations of the z axial spin configuration, approximately depicted in Fig. 10(d). The same linear modes are allowed at the other marginal point $R=2$ since there is also the same "broken symmetry." But they do not contribute because of a vanishing amplitude. That is, $\dot{s}^z(t)\approx 0$ asymptotically, which is related to the asymptotic behavior of the time evolution when $R>2$.[30]

We have seen that when $R=\infty$, the small and large spins have the same time evolution, indicating that there can be no slow decay. In the SAF of a single spin, this particular result appears in the guise of a destructive interference between the direct and exchange terms. For $t\rightarrow\infty$, the interference persists until $R=2+\epsilon$, $\epsilon\rightarrow 0+$. We have described the absence of slow decay in this region of R as *classical* since the time evolution is that of a classical vector. The value of the exponent $\kappa(R>2)=\infty$ would appear to correspond to a finite-dimensional classical exponent $\kappa=D/2+\kappa'$, where κ' is some number, in which $D\rightarrow\infty$. The finite value of the exponent we have obtained for $R<2$, e.g., $\kappa(R=0)$-$=2$, thus must be of cooperative origin, unrelated to purely dimensional effects.

Finally, we shall turn to the behavior of $G_{xx}(t)$ when $R=1$. The time dependent part of the SAF shows a fast decay only, hence, ascribed $\kappa=\infty$ for its exponent, the same as for $R>2$. We shall see that there indeed is a similarity in the time evolution behavior when $R=1$ and $R>2$. First, the behavior of the SAF about $R=1$ has the appearance of a point "singularity" since $\kappa=3$ if $0<R\leq 2$ $(R\neq 1)$. In fact, we recall that the asymptotic region of the time is defined by $t_0=\sqrt{N}/(abJ|1-R|)$. See Ref. 22(a). As $R\rightarrow 1$, it takes longer and longer to enter the asymptotic domain. Now if $R\neq 1$, our system has a well-defined cylindrical spin symmetry, where there are two axes of rotation $\hat{z}$ and τ. (As a result, axial and planar fluctuations responsible for slow decay are possible.) The cylindrical symmetry persists as $R=1\pm\epsilon$, $\epsilon\rightarrow 0$. At $\epsilon=0$ or $R=1$, there is but one axis of rotation τ. This discontinuous behavior of spin symmetry in our system manifests itself as a point"singularity," also analogously as a strip "singularity" when $R>2$.

The absence of slow decay when $R=1$, and also when $R>2$, may be traced to the behavior of the vector $\tau(t)$. Recall that $\dot{\mathbf{s}}(t)=\mathbf{s}(t)\times\tau(t)$. See Eq. (44) of I. Now if $R=1$, $\tau(t)=2g\mathbf{S}(t)=2g\mathbf{S}$. If $R>2$, $\tau(t)\rightarrow 2g_zS_z(t)\hat{\mathbf{z}}=2g_zS_z\hat{\mathbf{z}}$, as discussed earlier. That is, in both cases $\tau(t)$ becomes independent of the time, unlike when $0<R\leq 2$ $(R\neq 1)$. Hence, the time evolution of the small spin is of pure rotation. If $R=1$, the small spin rotates about a fixed direction of the large spin. If $R>2$, the small spin rotates about the z axis just as the large spin. There are no precessions. The behavior of the SAF at $R=1$ as well as the behavior at $R>2$ stems from having one fixed axis of rotation.

If $0\leq R\leq 2$ $(R\neq 1)$, $\tau(t)$ is always time dependent. The motions of the small spin, as earlier noted, are thus precessional, as if brought about by some effective force applied on a simple rotation. We have found that lowest precession frequencies give rise to a slow decay of the SAF. In our model, slow decay cannot exist without a precession motion.

What will remain when our model is made finite-dimensional (e.g., NN anisotropic Heisenberg) cannot be answered. The values of the exponent are not expected to be the same. It would seem to us, however, that the basic mechanism for slow decay in our model may yet have some generality.

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APPENDIX A: PARTITION FUNCTION AND ENSEMBLE AVERAGING

We shall first obtain the partition function when $T>T_c$ in a slightly different way than previously given.[19(a)] The new way is found most suitable for the analysis of long-time tails. From (16) and (17), to leading order in N, we have

$$Z=\sum_S\sum_{S_z}g(s)e^{-\beta H}\rightarrow 2^{N+1}\int_0^\infty dS(2S/N)e^{-S^2/a^2N}\int_{-S}^{S}dS_z\,e^{-\alpha S_z^2/a^2b^2N} \quad \text{(A1)}$$

where $a^{-2}=(2-\beta J)$, $a^{-2}b^{-2}=(2-\beta J_z)$, $\alpha=1-b^2$. Note that $a^2,b^2>0$ when $T>T_c$. One can readily evaluate the above double integral. Instead, we shall now change the variables S,S_z to η,u, where

$$S=N^{1/2}a\eta(1-\alpha u^2)^{1/2} , \tag{A2}$$

$$S_z=N^{1/2}abu\eta . \tag{A3}$$

Then,

$$Z=2^{N+1}N^{1/2}2a^3b\int_{-1}^{1}du\int_0^{\infty}d\eta\,\eta^2e^{-\eta^2}$$
$$=2^{N+1}(N\pi)^{1/2}a^3b . \tag{A4}$$

Accordingly, the ensemble average of any "reservoir" quantity, say $F(S,S_z)$, may be given as follows:

$$\langle F\rangle=Z^{-1}\sum_S\sum_{S_z}g(S)F(S,S_z)e^{-\beta H}$$
$$=\frac{2}{\sqrt{\pi}}\int_{-1}^{1}du\int_0^{\infty}d\eta\,e^{-\eta^2}\eta^2\tilde{F}(u,\eta) , \tag{A5}$$

where

$$\tilde{F}=F(\sqrt{N}a\eta\sqrt{1-\alpha u^2},\sqrt{N}abu\eta) .$$

In Appendix B, several examples are given.

If $T<T_c$, the partition function is *not* dominated by $S\approx 0$. Instead one finds that the phase function is peaked at some macroscopic value of S, say S_0, which depends on whether $R<1$ or $R>1$. We shall briefly review our previous results.[19(a)]

(i) R < 1. In this regime, $c\equiv\beta\omega>0$. Hence,

$$\sum_{S_z}e^{-cS_z^2}\to 2\int_0^{S\to\infty}dS_z\,e^{-cS_z^2}=(\pi/c)^{1/2} . \tag{A6}$$

Therefore,

$$Z=2^{N+1}(\pi/c)^{1/2}\sum_S(2S/N)e^{N\mathcal{G}} \tag{A7}$$

where

$$\mathcal{G}=-W+\beta JS^2/N^2 . \tag{A8}$$

See (17a) for the definition of W. Now $\mathcal{G}(S)$ is sharply peaked at $S=S_0$, i.e., $\mathcal{G}'(S_0)=0$. Hence, one may expand the summand in (A7) about $S=S_0$.

(ii) R > 1. In this regime, $c<0$. Let us assume that the S_z symmetry is broken upward, i.e., $S_z/S=+1$ to order N. Then,

$$\sum_{S_z}e^{-cS_z^2}\to\int_0^{S}dS_z\,e^{-cS_z^2}\approx e^{-cS^2}/(-2cS) .$$

Hence,

$$Z=2^{N+1}\sum_S(2S/N)e^{N\mathcal{G}_z}/(-2cS) \tag{A9}$$

where

$$\mathcal{G}_z=-W+\beta J_zS^2/N^2 . \tag{A10}$$

Now $\mathcal{G}_z$ is also a sharply peaked function, i.e., $\mathcal{G}_z'(S_0)=0$ and one may expand the summand in (A10) about $S=S_0$ as in the previous case.

The ensemble averaging when $T<T_c$ is thus much more straightforward than when $T>T_c$. For example, $\langle F\rangle\approx F(S_0)$, where S_0 is determined by the extremal condition on the phase function: $\mathcal{G}'=0$ if $R<1$ and $\mathcal{G}_z'=0$ if $R>1$. Ensemble averaging appears as if processed by a filtering function.[28] A few examples are given in Appendix E.

APPENDIX B: $\langle\cos(\Omega_z t)\rangle$, $\langle\cos(\Omega t)\rangle$, AND $\langle\cos(\phi t)\rangle$ WHEN $T>T_c$.

Using this method outlined in Appendix A, we can evaluate the ensemble average of $\cos(\Omega_z t)$ as follows:

$$\langle\cos(\Omega_z t)\rangle=\frac{2}{\sqrt{\pi}}\int_{-1}^{1}du\int_0^{\infty}d\eta\,e^{-\eta^2}\eta^2\cos(2A^{1/2}u\eta) , \tag{B1}$$

where $A=N(ab\omega t)^2=N(1-R)^2(abgt)^2$. We now integrate over u first or integrate by parts. Then,

$$\langle\cos(\Omega_z t)\rangle=\frac{2}{\sqrt{\pi}}A^{-1/2}\int_0^{\infty}d\eta\,e^{-\eta^2}\eta\sin(2A^{1/2}\eta)$$
$$=e^{-A} , \tag{B2}$$

which is valid for any t, not just $t\to\infty$.

We next evaluate the ensemble average of $\cos(\Omega t)$. Let $C=B(1-\alpha u^2)$, $B=N(agt)^2$. Then

$$\langle\cos(\Omega t)\rangle=\frac{2}{\sqrt{\pi}}\int_{-1}^{1}du\int_0^{\infty}d\eta\,e^{-\eta^2}\eta^2\cos(2C^{1/2}\eta)$$
$$=\int_0^1du(1-2C)e^{-C}$$
$$=e^{-B}\left(e^{\alpha B}-2B\int_0^1du\,e^{\alpha Bu^2}\right) . \tag{B3}$$

where the last step was obtained on integration by parts. Denote the integral in (B3) by L. Now if $t\to\infty$ (i.e., $B\to\infty$), it may be evaluated asymptotically depending on the sign of α, where $\alpha=1-b^2$.

(i) $0<\alpha<1$ *(R < 1).* Let $x=1-u^2$. Then,

$$L=\tfrac{1}{2}e^{\alpha B}\int_0^1dx(1-x)^{-1/2}e^{-\alpha Bx}\simeq(\tfrac{1}{2}\alpha B)e^{\alpha B} . \tag{B4a}$$

(ii) $\alpha<0$ *(R > 1).*

$$L\simeq\int_0^{1\to\infty}du\,e^{\alpha Bu^2}=\tfrac{1}{2}(\pi/|\alpha|B)^{1/2} . \tag{B4b}$$

Hence,

$$\lim_{t\to\infty}\langle\cos(\Omega t)\rangle=\begin{cases}-(b^2/\alpha)e^{-b^2B}, & R<1 , \quad \text{(B5a)}\\ -(\pi B/|\alpha|)^{1/2}e^{-B}, & R>1 , \quad \text{(B5b)}\end{cases}$$

and

$$\langle\cos(\Omega t)\rangle=(1-2B)e^{-B},\quad R=1 . \tag{B5c}$$

In a similar way we can write down the ensemble average of $\cos(\phi t)$, where $\phi=\Omega-\Omega_z$:

$$\langle\cos(\phi t)\rangle=\frac{2}{\sqrt{\pi}}\int_{-1}^{1}du\int_{0}^{\infty}d\eta\, e^{-\eta^2}\eta^2\cos(2A^{1/2}z\eta)$$
$$=\tfrac{1}{2}\int_{-1}^{1}du(1-2Az^2)e^{-Az^2}\,, \quad \text{(B6)}$$

where $z=z(u)$ is defined by (24). Observe that if $R=1$, $\phi=\Omega$ and (B6) reduces to (B4). If $R=\infty$, $\phi=-\Omega_z$ and (B6) reduces to (B2). Changing the variable u to z in (B6) as in Eq. (26), we obtain

$$\langle\cos(\phi t)\rangle=\tfrac{1}{2}u_0^2\gamma^{-2}\int_{z_1}^{z_2}dz(1-\alpha u_0 zD^{-1})\times(1-2Az^2)e^{-Az^2}\,, \quad \text{(B7)}$$

where for the derivative of u we have used (C3) from Appendix C. See (24), (25), and (30c) for various quantities appearing here. We shall now evaluate the above integral in different regions of R following our ideas given in Sec. V, implicitly assuming that $A\to\infty$ ($t\to\infty$).

(1) $R>2$.

$$\langle\cos(\phi t)\rangle=u_0^2\gamma^{-2}e^{-A}\,. \quad \text{(B8)}$$

(2) $R=0,2$.

$$\langle\cos(\phi t)\rangle=\tfrac{1}{2}\alpha\gamma^{-2}\int_0^{\infty}dz\, z(1-2Az^2)e^{-Az^2}$$
$$=-\alpha b^2/4A\sim t^{-2}\,. \quad \text{(B9)}$$

One obtains the same result for both $R=0$ and 2. The only difference between the two appears in the sign of S_z, which is, however, immaterial since both signs occur in the ensemble averages.

(3) $0<R<2$ ($R\neq 1$).

$$\langle\cos(\phi t)\rangle\le\begin{cases}C_1e^{-Az_2^2}, & 0<R<1\,, \quad \text{(B10a)}\\ C_2e^{-Az_1^2}, & 1<R<2\,, \quad \text{(B10b)}\end{cases}$$

where C_1 and C_2 are constants. The results are similar to (36a) and (36b).

The ensemble average of $\cos(\Omega_z t)$, $\cos(\Omega t)$, and $\cos(\phi t)$ all lead to a rapidly vanishing result if $t\to\infty$, except for the third when $R=0$ and 2. The vanishing can be attributed to a cancellation arising from large phases, randomly occurring. When $R=0$ and 2, the frequency can be zero for an entire range of S whenever $|S_z|=S$. As a result, phases corresponding to very small frequencies can survive and contribute to slow decay.

APPENDIX C: TRANSFORMATION $u=u(z)$

Using (24) and (25), one can obtain the inverse transformation expressed in the following form:

$$u(z)=\gamma^{-2}u_0^2z+u_0D(z)\,, \quad \text{(C1)}$$

where

$$D(z)=(1-\alpha\gamma^{-2}u_0^2z^2)^{1/2}\,. \quad \text{(C2)}$$

In obtaining (C1) we have chosen the positive sign before $D(z)$ since $u=u_0$ if $z=0$.

We next turn to $f(z)$ given by (27). The algebra leading to (29) is somewhat involved but straightforward. Below we give an outline. The derivative of $u(z)$ may be expressed as

$$\frac{\partial u}{\partial z}=u_0^2\gamma^{-2}(D-\alpha u_0 z)D^{-1}\,. \quad \text{(C3)}$$

Also, using (24),

$$(1-\alpha u^2)^{1/2}=\gamma^{-1}(u-z)$$
$$=u_0\gamma^{-1}(D-\alpha u_0 z)\,, \quad \text{(C4)}$$

where we have used (C1) to eliminate u in the first expression. Hence,

$$1+\frac{bu}{\sqrt{1-\alpha u^2}}=B+\frac{bz}{u_0\gamma^{-1}(D-\alpha u_0 z)}\,, \quad \text{(C5)}$$

where $B=1+b\gamma$. Combining (C3) and (C5), we obtain

$$f(z)=B^2u_0^2\gamma^{-2}+Bu_0\gamma^{-1}(2b-\alpha u_0^2B\gamma^{-1})zD^{-1}+\frac{b^2z^2}{D(D-\alpha u_0 z)}\,. \quad \text{(C6)}$$

The last term of the RHS of (C6) can be split up as

$$\frac{b^2z^2}{D(D-\alpha u_0 z)}=\frac{b^2z^2}{1-\alpha z^2}+\frac{\alpha u_0 b^2z^3}{D(1-\alpha z^2)}\,, \quad \text{(C7)}$$

where we use the identity $D^2-(\alpha u_0 z)^2=1-\alpha z^2$. [See (C2)]. Substituting (C7) in (C6), we obtain

$$f(z)=p^2+pqzD^{-1}+\frac{b^2z^2}{1-\alpha z^2}+\frac{\alpha u_0 b^2z^3}{D(1-\alpha z^2)}\,, \quad \text{(C8)}$$

where

$$p=Bu_0\gamma^{-1}=(2-R)bu_0\,, \quad \text{(C9a)}$$
$$q=2b-\alpha u_0 p\,. \quad \text{(C9b)}$$

The constant p has the following special values: $p=2b$ if $R=0$; $p=0$ if $R=2$.

APPENDIX D: ABSENCE OF LONG-TIME TAILS IN THE LARGE SPIN

Let's define the SAF of the large spin as $\mathcal{S}_{\alpha\beta}(t)=\langle S_\alpha(t)S_\beta(0)\rangle$, $\alpha,\beta=x,y$, and z. Consider first $T>T_c$. We have shown previously[19(a)] that in the high-temperature region

$$\mathcal{S}_{xx}(t)=\mathcal{S}_{yy}(t)=\langle S_x(t)S_x\rangle=\langle S_x^2\cos(\Omega_z t)\rangle\,,$$
$$\Omega_z=2\omega S_z\,, \quad \text{(D1)}$$
$$\mathcal{S}_{zz}(t)=\langle S_z(t)S_z\rangle=\langle S_z^2\rangle\,. \quad \text{(D2)}$$

The z-component of the SAF is time independent since S_z is a constant of motion. That is, S_z denotes a stationary state of the "reservoir.' If $R\neq 1$,

$$\langle S_x^2\cos(\Omega_z t)=\langle S_x^2\rangle\langle\cos(\Omega_z t)\rangle \quad \text{(D3)}$$

and

$$\langle\cos(\Omega_z t)\rangle=e^{-A}\,, \quad \text{(D4)}$$

where $A=N(ab\omega t)^2=2\omega^2t^2\langle S_z^2\rangle$. See Appendix B. Because of decoupling, there is no long-time tail in the x and y components of the SAF for the large spin if $R\neq 1$. Now if $R=1$, $\mathcal{S}_{xx}(t)=\mathcal{S}_{zz}(t)$, hence, time independent.

Let us consider $T<T_c$. Here S_z is still a constant of motion. Now the x component is

$$\mathcal{S}_{xx}(t)=\langle S_x^2\cos(\Omega_z t)\rangle+(i/2)\langle S_z\sin(\Omega_z t)\rangle \ . \quad \text{(D5)}$$

The imaginary part would not be present if the SAF were symmetrized.[19(c)] If $R<1$, the first term on the RHS of (D5) also decouples. The second term may be neglected. That is,

$$\mathcal{S}_{xx}(t)=\langle S_x^2\rangle\langle\cos(\Omega_z t)\rangle$$
$$=\langle S_x^2\rangle\exp(-2\omega^2t^2\langle S_z^2\rangle) \ , \quad \text{(D6)}$$

where now $\langle S_z^2\rangle=1/(2\beta\omega)$. If $R>1$,

$$\mathcal{S}_{xx}(t)=\langle S_x^2\rangle\cos(\bar{\Omega}_z t)+(i/2)\bar{S}_z\sin(\bar{\Omega}_z t) \quad \text{(D7)}$$

where $\bar{\Omega}_z=2\omega\bar{S}_z$, $\bar{S}_z=\langle S_z\rangle$. Thus, the same decoupling may be said to occur.

The SAF of the large spin decays very rapidly if $T>T_c$. It behaves similarly if $T<T_c$ and $R<1$, but it is oscillatory if $R>1$. The frequency of oscillation is directly related to the long-range order. The absence of long-time tails in the SAF of the large spin may be attributed to decoupling of the modes of S_x^2 and S_z^{2n}, $n=1,2,\ldots$ [see (D3)].

When $R=1$ and $T>T_c$, there are other possibilities of decoupling. For example,

$$\langle S_z^2S^{2k}\rangle=\langle(S_z/S)^2\rangle\langle S^{2k+2}\rangle, \quad k=0,1,2,\ldots \ . \quad \text{(D8)}$$

Observe that we obtain $\langle(S_z/S)^2\rangle=\frac{1}{3}$ if we set $k=0$ in (D8). This identity (D8) was used in arriving at Eq. (42).

APPENDIX E: $\langle\cos(\Omega_z t)\rangle$, $\langle\cos(\Omega t)\rangle$, AND $\langle\cos(\phi t)\rangle$ WHEN $T<T_c$

Following the definition of the ensemble average, one may write

$$\langle\cos(\Omega_z t)\rangle=Z^{-1}\sum_S(2S/N)e^{N\mathcal{G}}\sum_{S_z}e^{-cS_z^2}\cos(p\,S_z) \ , \quad \text{(E1)}$$

where $c=\beta\omega$, $p=2\omega t$, $\mathcal{G}$ is the phase function (see Appendix A), and Z is the partition function.

(i) $R<1$. Here $c>0$. Hence,

$$\sum_{S_z}e^{-cS_z^2}\cos(p\,S_z)\to 2\int_0^{S\to\infty}dS_z\,e^{-cS_z^2}\cos(pS_z)$$
$$=(\pi/c)^{1/2}e^{-p^2/4c} \ .$$

Hence, substituting the above result in (E1), we obtain

$$\langle\cos(\Omega_z t)\rangle=e^{-p^2/4c} \ . \quad \text{(E2)}$$

(ii) $R>1$. Here $c<0$. Assume that the S_z symmetry is broken upward, i.e., $S_z/S=+1$ to order N. Then,

$$\sum_{S_z}e^{-cS_z^2}\cos(pS_z)\to\int_0^S dS_z\,e^{-cS_z^2}\cos(pS_z)$$
$$\simeq e^{-cS^2}\cos(pS)/(-2cS) \ .$$

Substituting the above result in (E1), we obtain

$$\langle\cos(\Omega_z t)\rangle=\cos(p\bar{S}) \ , \quad \text{(E3)}$$

where $\bar{S}=\langle S\rangle$. Together,

$$\langle\cos(\Omega_z t)\rangle=\begin{cases}e^{-2\omega^2t^2\langle S_z^2\rangle}, & R<1 \ , \quad \text{(E4a)}\\ \cos(2\omega t\bar{S}), & R>1 \ . \quad \text{(E4b)}\end{cases}$$

We shall next evaluate $\langle\cos(\Omega t)\rangle$. Using the same definition, we may write

$$\langle\cos(\Omega t)\rangle=Z^{-1}\sum_S(2S/N)e^{N\mathcal{G}}\cos qS\sum_{S_z}e^{-cS_z^2} \ , \quad \text{(E5)}$$

where $c=\beta\omega$ (same as before) and $q=2gt$. The evaluation is straightforward since the sign of c is now immaterial. We obtain directly,

$$\langle\cos(\Omega t)\rangle=\cos(\bar{\Omega}t), \quad \bar{\Omega}=2g\bar{S} \ , \quad \text{(E6)}$$

where $\bar{S}=\langle S\rangle$.

Finally, we shall calculate $\langle\cos(\Omega t)\rangle$. Using the definition,

$$\langle\cos(\phi t)\rangle=\langle\cos[(\Omega-\Omega_z)t]\rangle$$
$$=Z^{-1}\sum_S\frac{2S}{N}e^{N\mathcal{G}}\sum_{S_z}e^{-cS_z^2}\cos[(\Omega-\Omega_z)t] \ , \quad \text{(E7)}$$

where $\Omega=2gS$ and $\Omega_z=2\omega S_z$.

(i) $R<1$. Then, expanding the cos term, we obtain

$$\sum_{S_z}e^{-cS_z^2}\cos[(\Omega-\Omega_z)t]=\cos(\Omega t)\sum_{S_z}e^{-cS_z^2}\cos(\Omega_z t) \ .$$

Hence,

$$\langle\cos(\phi t)\rangle=\langle\cos(\Omega t)\rangle\langle\cos(\Omega_z t)\rangle$$
$$=\cos(\bar{\Omega}t)\langle\cos(\Omega_z t)\rangle \quad \text{(E8)}$$

where $\bar{\Omega}=2g\bar{S}$, $\bar{S}=\langle S\rangle$ and $\langle\cos(\Omega_z t)\rangle$ is given by (E4a).

(ii) $R>1$. Let us assume that the S_z symmetry is broken upward as before. Then,

$$\sum_{S_z}e^{-cS_z^2}\cos[(\Omega-\Omega_z)t]$$
$$\to\int_0^S dS_z\,e^{-cS_z^2}\cos[rS_z+q(S-S_z)]$$
$$=e^{-cS^2}\cos(rS)/(-2cS) \ ,$$

where $q=2gt$ and $r=2g_z t$.

Hence,

$$\langle\cos(\phi t)\rangle=\cos(\bar{\Omega}t) \ , \quad \text{(E9)}$$

where $\bar{\Omega}=2g_z\bar{S}$, $\bar{S}=\langle S\rangle=\langle S_z\rangle$.

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[19](a) R. Dekeyser and M. H. Lee, Phys. Rev. B **19**, 265 (1979). (b) In Ref. 19(a) we have shown that such terms are of lower order in N when their ensemble averages are evaluated. Similarly, if one were to use our general solutions [(A9) of I] in (7) and (8), one would obtain the same leading order terms of (11) and (12) for the above stated reason. Also observe that $G_{xx}(-t)=G_{xx}(t)$. (c) For our system, which is of scalar interaction, we should require $G_{ji}(t)=G_{ij}(t)$, where $i,j=x,y$ or z. To satisfy this requirement, we need to symmetrize the definition of the SAF [Eq. (6)] as follows: $G_{ij}(t)=2\langle [s^i(t)s^j(0)+s^j(t)s^i(0)]\rangle$. The diagonal elements [Eqs. (7) and (8)] are unaffected by the symmetried definition. All the nondiagonal elements, however, vanish, to leading order in N.

[20]In this paper a solution obtained when $T\rightarrow\infty$ will be called an *asymptotic* solution. It should not be confused with the asymptotic solution of I, by which we earlier meant a solution of $N\rightarrow\infty$. In the present paper, $N\rightarrow\infty$ is always implicitly assumed. This limit is taken first before any other limits. Strictly speaking, the time t in a mean-field Hamiltonian such as ours is not independent of N. One can show that the frequency Ω_z in (18a), for example, is proportional to $1/\sqrt{N}$ if $T>T_c$. See Ref. 19(a). Hence, time-dependent behavior exists if and only if t itself is proportional to $\sqrt{N}$. It is thus necessary to introduce a new scaled time $\mathbf{t}=t/\sqrt{N}$ as pointed out by us in Ref. 19(a) and also noted earlier by W. Thirring [Commun. Math. Phys. **7**, 181 (1968)] in another mean-field model. Also see P. Résibois and M. De Leener, Phys. Rev. **178**, 806 (1969), especially Eq. (5.7). By long times, we then mean $\mathbf{t}\rightarrow\infty$. If $T<T_c$, the frequency Ω_z is independent of N. Hence, it is not necessary to scale the time. Matching of the two different time scales as $T\rightarrow T_c\pm$ is a source of the critical anomaly in this model. See Ref. 14. The N dependence in t arises from the structure of mean-field Hamiltonians, i.e., being made extensive.

[21]That is to say, if $R>2$ (i.e., $|\Omega_z|>\Omega$), the condition requiring $\phi\rightarrow 0$ is attained through replacing ϕ by $-\Omega_z$ and then $\Omega_z\rightarrow 0$. This discontinuous behavior of ϕ pertains to the slow-decay part of the exchange term only. There are various Gaussian-decay components which are unaffected by these arguments, but they are, however, not germane.

[22](a) One can write Eq. (22a) as

$$G_{xx}^{(1)}(t)=(-b^2/2)A^{-3/2}\int_0^{\sqrt{A}}\frac{du\,u^2(1-2u^2)e^{-u^2}}{1-\alpha u^2/A}\ .$$

Now Eq. (22b) is an asymptotic expansion of the above in powers of A^{-1}, valid if $A\gg 1$. The upper limit may be increased to ∞, introducing errors proportional to $A^k\exp(-A)$, $k<\infty$. The errors are smaller than any higher order expansion terms of Eq. (22b) if $A\gg 1$. This asymptotic condition directly gives $t\gg\sqrt{N}/(abJ|1-R|)\equiv t_0$. Thus, if $R\neq 1$, the asymptotic condition can be satisfied since $t_0<\infty$. (Also, see Ref. 20 for our comment about t being proportional to $\sqrt{N}$). By $t\rightarrow\infty$ or by large or long times we shall thus mean that $t\gg t_0$. Note, however, that t_0 is R dependent. If $R=1$, one may not use this asymptotic approach. But in this case one can solve the SAF exactly for any t. See Eq. (43). Here one can define large times by $B\gg 1$ or $t\gg\sqrt{N}/aJ$. Now observe that as $A\rightarrow\infty$, the long-time tail part emerges merely as a scale factor of the original integral. Apropos of this point, see Ref. 11. The existence of a finite upper limit, however, is crucial for the analytic behavior of $G_{xx}(t)$ about $t=0$, i.e., $G_{xx}(t)=1+O(t^2)$. The short-time behavior is illustrated in Fig. 2. Also, see Sec. VI. In certain Brownian-motion models (e.g., see Widom, Ref. 8), the VAF is not analytic about $t=0$, i.e, moments do not exist. It arises presumably from the fact that a B particle is suddenly given a velocity at $t=0$. In these models, the integral expressions for the VAF are not of finite interval. (b) As shown in Ref. 22(a) for the direct term, Eq. (26) can also be expressed in an asymptotic expansion form. Then one obtains the following asymptotic conditions for the exchange term: $|z_1|A^{1/2}\gg 1$ and $|z_2|A^{1/2}\gg 1$ if z_1, $z_2\neq 0$ (i.e., $R\neq 0,2$). Hence, $t\gg\sqrt{N}/(abJR)$ and $t\gg\sqrt{N}/(abJ|R-2|)$, $R\neq 0,2$. Both conditions are nearly the same. If z_1 or $z_2=0$, only one of the limits is applicable. The asymptotic conditions for the exchange term are essentially identical to that given by the direct term.

[23]This is in contrast to static symmetry, defined by $R<1$ and $R>1$, established earlier. It was shown that the critical temperature of the spin van der Waals model depends on J only if $R<1$ and on J_z only if $R>1$. See Ref. 19(a). We shall see that dynamic behavior (especially long time behavior) is differently characterized, given by the inequality [Eq. (25)]. To distinguish it from the static case, we have introduced this

term "dynamic symmetry." The static symmetry, as applied to static universality classes, for example, derives from the oblate (XY-like) or prolate (Ising-like) form of the Heisenberg interaction, in which $R=1$ represents a dividing or singular point of the interaction. There are no such obvious changes seen in the neighborhood of e.g., $R=2$. Yet it represents a singular point in the dynamic behavior of this model. Apparently, dynamic universality classes even in a simple model like the spin van der Waals depend on the interaction symmetry much more subtly than do the static ones. Equation (25) suggests that there are other properties—beyond those determined by symmetry and dimensionality alone—which play a role in dynamic behavior.

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[25]From a numerical log-log analysis of data, the result $\kappa=3.0000\pm0.0001$ for both $R=0.5$ and 1.4 is easily obtained even when the interval $0\leq u\leq 1$ is divided into only 1000 steps. For $R=0$, however, the interval must be divided into at least 10 000 parts just to obtain $\kappa=2.00\pm0.01$.

[26]Whether asymptotic regions of the time have been attained in computer simulations and experimental measurements is of crucial importance. R. F. Fox [Phys. Rev. A **27**, 3217 (1983)] has analyzed some numerical results on fluids. The largest times seem to be $t\sim 30$, where the time is given in units of the mean free time. In Ferry's work (Ref. 2), times measured approximately in units of the momentum relaxation time are also in this range. In Müller's work (Ref. 3), times measured in units of the exchange energy J have these values. When compared with our numerical work shown in Fig. 2, they seem to be near the threshold of an asymptotic region of the time.

[27]Résibois and De Leener, *Classical Kinetic Theory of Fluids* (Ref. 7), especially pp. 57–63.

[28]See, e.g., A. M. Yaglom, *Introduction to the Theory of Stationary Random Functions,* translated by R. A. Silverman (Dover, New York, 1962), pp. 42–43.

[29]It follows from the fact that $\langle S_x^2\cos(\Omega_z t)\rangle=\langle S_x^2\rangle\langle\cos(\Omega_z t)\rangle$, i.e., fluctuations of S_x^2 and S_x^2 decouple. See Ref. 19(a) and also Appendix D.

[30]Note that $\boldsymbol{\tau}$ [see Eq. (43) of I] may be written as $\boldsymbol{\tau}=2g(\mathbf{S}-(1-R)S_z\hat{\mathbf{z}})$. Now Eq. (15) gives $\phi=2g(S-(1-R)S_z)$, which is thus the eigenvalue of $\boldsymbol{\tau}$. The slow-decay condition $\phi=0$ means zero eigenvalue of $\boldsymbol{\tau}$. In terms of $\boldsymbol{\tau}$, the energy H may be expressed as follows: $H=-(\mathbf{S}/2+\mathbf{s})\cdot\boldsymbol{\tau}$. If $\boldsymbol{\tau}$ were to be regarded as an effective field, the energy has the structure of an effective field model. Indeed, the equations of motion are those of an effective field model, $\dot{\mathbf{s}}=\mathbf{s}\times\boldsymbol{\tau}$ and $\dot{S}=\mathbf{S}\times\boldsymbol{\tau}$, respectively, for the small and large spins. These equations imply that $\mathbf{s}(t)\cdot\mathbf{S}(t)=\mathbf{s}\cdot\mathbf{S}$. Also, see R. Dekeyser and M. H. Lee, in *Rigorous Results in Quantum Dynamics,* edited by J. Dittrich (World Scientific, Singapore, 1991).