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Unified symmetry-breaking theory of Bose-Einstein condensation in superfluids

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The usual symmetry-breaking procedures for Bose condensed systems, namely, the Bogoliubov prescription, the symmetry-breaking term added to the Hamiltonian, and the canonical shift transformation are unified into a single formalism. By taking into account the condensate reservoir as a source and sink of excited particles, exact Ward identities are solved in the shielded-potential approximation. A relationship between the condensate density n_0 and the superfluid density n_S is obtained in closed form. It is shown that the Bogoliubov prescription yields $n_0 \simeq n_S$ and $nU_0 \ll |\mu|$, where n is the total density, U_0 the interaction constant, and μ the chemical potential. On the other hand, for the canonical shift transformation one has $n_0 \ll n_S$ and $nU_0 \gg |\mu|$. The latter, applied to superfluid ^{4}He at saturated vapor pressure, gives excellent agreement between theory and experiment, without any adjustable parameter. The condensate density turns out to be strongly dependent on pressure as observed experimentally. The formalism provides in a natural way a consistent description of Bose systems in arbitrary D-dimensional space.

I. INTRODUCTION

Three decades have passed since London's proposal[1] and the experimental determination[2–5] Bose-Einstein condensation in superfluid ^{4}He. From the theoretical side, the first estimate of the condensate fraction was due to Penrose and Onsager who found $n_0 \sim 0.08n$ at $T=0$.[6] More recently, Monte Carlo computer simulations at absolute zero[7] and finite temperatures[8] have yielded $n_0 \sim 0.10n$ ($T=0$), a value consistent with high-momentum inelastic-neutron-scattering measurements.[2–5] Hence, it is now generally accepted that a gauge symmetry breaking of the first kind is basic to the field-theoretical analysis of superfluid ^{4}He. In this regard, Griffin[9] has shown that even the phenomenological theories of Landau[10] and Feynman[11] are based on such a broken invariance.

Although the extensive theoretical work on Bose condensed systems has shed light on many fundamental aspects of superfluidity, it has failed to give a quantitative account of the condensate in bulk superfluid ^{4}He. At $T=0$, the weakly interacting dilute Bose system (WIDBS) has long been well understood.[12–14] Substantial progress occurred with studies of the response functions of Bose condensed systems.[15–20] This approach, which has its origin in the work of Gavoret and Nozières,[15] has been used in the investigation of charged[16] and neutral[17] Bose gases at zero temperature. Besides the weak interaction and/or diluteness of the system, these $T=0$ theories[12–17] exhibit a small depletion of the condensate, $n_0 \sim n$. Hence, they bear no resemblance with actual superfluid ^{4}He. A more realistic approach is found in the finite-temperature formalism of Griffin and Cheung[18] and Szépfalusy and Kondor,[19] where the shielded-potential approximation (SPA) in the strong-coupling case allows a large depletion of the condensate. But this also implies a large depletion of the superfluid density as well for $n_S = n_0$ in their treatment.

A source term appears in the continuity equation as a consequence inherent to the nonconservation of the total particle number. In this paper we show that the condensate reservoir, as a source and sink of excited particles, can be responsible for a large depletion of the condensate alone.

In Sec. II we discuss the sources that result from the usual symmetry-breaking procedures, namely, the Bogoliubov prescription,[21] the symmetry-breaking field added to the Hamiltonian,[22] and the shift transformation first introduced by Gross[23] and studied diagrammatically by Popov and Faddeev.[24] We proceed to develop a model that encompasses these three schemes into a single formalism. This unified approach allows us to compare the source effect in each case.

The continuity equation underlies exact relationships among correlation functions involving different combinations of its constituents. In the present case, besides particle density and current, the source enters as a third constituent. These relationships, known as Ward identities, are reviewed in Sec. III. In the hydrodynamic limit they provide a rigorous expression for n_S as an implicit function of n_0.

In Sec. IV the Ward identities are solved in the SPA for the case of a symmetry-breaking term added to the Hamiltonian. This allows one to compare the effect of the condensate reservoir with known source-free result.

In Sec. V the generalized symmetry-breaking model is worked out in the SPA. The Bogoliubov prescription is shown to correspond to a weak-coupling constant, whereas the Gross-Popov-Faddeev approach corresponds to a strong-coupling constant.

Bose-Einstein condensation in arbitrary D-dimensional space is studied in Sec. VI. In Sec. VII the theory is used as a model calculation applied to superfluid ^{4}He. We conclude with a discussion in Sec. VIII.

II. SYMMETRY-BREAKING MODELS AND THE CONDENSATE RESERVOIR

We consider a uniform system of bosons of mass m, enclosed in a volume $V=L^D$, and interacting via a two-body potential $U(x)$ whose Hamiltonian and total particle number are ($\hbar=1$ is set throughout)

$$\hat{H}=\hat{H}_0+\hat{U}\ , \tag{2.1}$$

$$\hat{H}_0=\int d^Dx\ \psi^\dagger(x)K(x)\psi(x),\quad K(x)\equiv-\nabla^2/2m\ , \tag{2.2}$$

$$\hat{U}=\tfrac{1}{2}\int d^Dx\ d^Dx'U(x-x')\psi^\dagger(x)\psi^\dagger(x')\psi(x')\psi(x)\ , \tag{2.3}$$

$$\hat{N}=\int d^Dx\ \psi^\dagger(x)\psi(x)\ . \tag{2.4}$$

The Fourier expansion of the field operator is

$$\psi=\psi_0+\psi_1\ , \tag{2.5}$$

$$\psi_0=V^{-1/2}b_0\ , \tag{2.6}$$

$$\psi_1=V^{-1/2}\sum_{k\neq 0}b_k e^{ikx}\ . \tag{2.7}$$

The momentum-space condensation into the zero mode is characterized by an order parameter defined as the ensemble average of the field operator,

$$\langle\psi\rangle=\langle\psi_0\rangle\neq 0\ . \tag{2.8}$$

The source formal expression depends on how the symmetry is broken. We consider first the well-known Bogoliubov prescription.[21] It amounts to the replacement of the zero-mode amplitudes by c numbers, i.e., $b_0=b_0^\dagger=(N_0)^{1/2}$, N_0 being the number of particles in the condensate. Consequently, it follows the commutation relation

$$[b_0,b_0^\dagger]=0\ , \tag{2.9}$$

and, for the fields,

$$[\psi(x),\psi^\dagger(x')]=\delta^D(x-x')-V^{-1}\ . \tag{2.10}$$

In the finite-temperature formalism, the equation of motion for the particle density $\hat{n}=|\psi|^2$ reads

$$\frac{\partial\hat{n}}{\partial\tau}=[\hat{H},\hat{n}]-\mu[\hat{N},\hat{n}]\ , \tag{2.11}$$

where $\tau=it$ denotes the imaginary time and μ the chemical potential. Due to the extra term in (2.10), the commutators $[\hat{U},\hat{n}]$ and $[\hat{N},\hat{n}]$ no longer vanish. Recalling that $[\hat{H}_0,\hat{n}]=i\nabla\cdot\hat{J}$, where $\hat{J}$ is the usual current operator, Eq. (2.11) yields the continuity equation

$$\frac{\partial\hat{n}}{\partial\tau}-i\nabla\cdot\hat{J}=\hat{S} \tag{2.12}$$

with the source operator defined by

$$\hat{S}=[\hat{U},\hat{n}]-\mu[\hat{N},\hat{n}]\ . \tag{2.13}$$

In momentum space, the continuity equation becomes

$$\frac{\partial\hat{\rho}_k}{\partial\tau}+k\hat{J}_k=\hat{S}_k\ , \tag{2.14}$$

where the density, longitudinal current, and source operators are given, respectively, by

$$\hat{\rho}_k=\sum_p b_p^\dagger b_{p+k}\ , \tag{2.15}$$

$$\hat{J}_k=\frac{\hat{k}}{m}\sum_p(p+\tfrac{1}{2}k)b_p^\dagger b_{p+k}\ , \tag{2.16}$$

$$\hat{S}_k=[\hat{U},\hat{\rho}_k]-\mu[\hat{N},\hat{\rho}_k]\ . \tag{2.17}$$

In (2.16), $\hat{k}$ denotes the unit vector in the direction of k.

An alternative approach that takes into account the dynamics of the condensate consists in removing the gauge group by adding a small symmetry-breaking perturbation to the Hamiltonian (2.1),

$$\hat{H}\to\hat{H}+\hat{H}_{\rm SB}\ . \tag{2.18}$$

Since, in this case, all canonical commutation relations are now preserved, the source operators are defined by

$$\hat{S}=[\hat{H}_{\rm SB},\hat{n}],\quad \hat{S}_k=[\hat{H}_{\rm SB},\hat{\rho}_k]\ . \tag{2.19}$$

Introduced by Bogoliubov,[22] this technique describes the spontaneous symmetry breaking of thermal states. For definiteness, we consider $\hat{H}_{\rm SB}$ as given in momentum-space by

$$\hat{H}_{\rm SB}=-\eta N_0^{1/2}(b_0+b_0^\dagger-2N_0^{1/2})\ , \tag{2.20}$$

where η is a small positive energy. Anomalous averages such as the order parameter may, in principle, be defined by letting $\eta\to 0$ after the thermodynamic limit. The source operator associated with (2.20) is easily determined from (2.19),

$$\hat{S}_k^0=\eta N_0^{1/2}(b_{-k}^\dagger-b_k)\ . \tag{2.21}$$

A more satisfactory procedure is found in the works of Gross[23] and Popov and Faddeev[24] in terms of the shift transformation

$$b_0=a_0+N_0^{1/2},\quad b_k=a_k\quad(k\neq 0) \tag{2.22}$$

with the additional requirement

$$\langle a_0\rangle=0\ . \tag{2.23}$$

At first, Popov and Faddeev chose Eqs. (2.1)–(2.7) as the original Hamiltonian. Hence, there is no source. The latter can only be defined when the Hamiltonian is written as a function of the operators a_k and $a_k^\dagger$. It is then of type (2.18). Popov and Faddeev proceed to develop a diagrammatic calculation where the nonquadratic terms in $a_k,a_k^\dagger$ are taken as perturbations.

In summary, Bose condensed systems are usually dealt with by either a noncanonical commutation relation or a symmetry-breaking perturbation added to the original Hamiltonian. In the latter approach, $\hat{H}_{\rm SB}$ is defined in terms of a fictitious field coupled to b_0 and $b_0^\dagger$ or is a consequence of a transformation on these amplitudes. The Bogoliubov prescription and its Gross-Popov-Faddeev generalization are conceptually more well-defined than the fictitious field scheme. In contrast to the shift transformation, the Bogoliubov prescription neglects the microscopic fluctuation of the condensate occupation number altogether. On physical grounds, one may then expect the Bogoliubov prescription to be a good approximation to WIDBS, whereas the shift transformation is more realistic for dense, strongly interacting systems such as superfluid ^{4}He.

We have seen that the source (2.21) does not depend on the interparticle interaction. In contrast, the sources originated from both the Bogoliubov and shift transformations are interaction dependent. It is then interesting to investigate whether the validity of each symmetry-breaking procedure is related to the interaction strength, specifically, whether the Bogoliubov and the shift transformations imply weak and strong interactions, respectively.

In this view, we now introduce a symmetry-breaking procedure that has the Bogoliubov prescription and the shift transformation as limiting cases. This can be simply attained by the following conditions:

$$b_0=(1-\xi)^{1/2}a_0+N_0^{1/2} , \tag{2.24}$$

$$b_k=a_k \quad (k\neq 0) , \tag{2.25}$$

$$\langle a_0\rangle=\langle a_0^\dagger\rangle=0 , \tag{2.26}$$

where a_k and $a_k^\dagger$, for all k, obey Bose commutation relations and ξ is a real parameter. Clearly, for $\xi\rightarrow 1$ and $\xi\rightarrow 0$, Eq. (2.24) reduces to the Bogoliubov and shift transformations, respectively. Bose-Einstein condensation is expressed by $\langle b_0\rangle=N_0^{1/2}$. From Eqs. (2.24) and (2.26), one has

$$\langle b_0^\dagger b_0\rangle=(1-\xi)\langle a_0^\dagger a_0\rangle+N_0 . \tag{2.27}$$

Due to the condition (2.26), $\langle a_0^\dagger a_0\rangle$ cannot be a finite fraction of the total particle number, hence, the condensate density in the thermodynamic limit equals

$$\langle b_0^\dagger b_0\rangle/V=N_0/V \quad (V\rightarrow\infty) . \tag{2.28}$$

The transformation (2.24) represents a slight yet essential modification of an earlier definition.[25] In that work the c-number part of b_0 was linearly dependent on ξ. In the present formulation, however, as $\xi\rightarrow 0$, Eqs. (2.24)–(2.26) are asymptotic to the shift transformation.

Having Eqs. (2.24)–(2.26) as the underlying assumption, we now introduce a new symmetry-breaking procedure based on the fundamental hypothesis

$$[b_0,b_0^\dagger]=1-\xi , \tag{2.29}$$

$$\langle b_0\rangle=\langle b_0^\dagger\rangle=\sqrt{N_0} . \tag{2.30}$$

The grand canonical Hamiltonian $\hat{H}-\mu\hat{N}$ is that given by Eqs. (2.1)–(2.7) whose amplitudes satisfy

$$[b_k,b_p^\dagger]=(1-\xi\delta_{k,0})\delta_{k,p} , \tag{2.31}$$

$$[b_k,b_p]=[b_k^\dagger,b_p^\dagger]=0 , \tag{2.32}$$

and, in terms of the fields,

$$[\psi(x),\psi^\dagger(x')]=\delta^D(x-x')-\xi V^{-1} , \tag{2.33}$$

$$[\psi(x),\psi(x')]=[\psi^\dagger(x),\psi^\dagger(x')]=0 , \tag{2.34}$$

$$\langle\psi(x)\rangle=\langle\psi^\dagger(x)\rangle=n_0^{1/2} . \tag{2.35}$$

The parameter ξ can be interpreted as a fictitious dimensionless quantity that breaks the symmetry according to the gauge transformation

$$e^{i\theta\hat{N}}\psi e^{-i\theta\hat{N}}=\psi_0 e^{i(1-\xi)\theta}+\psi_1 e^{i\theta} . \tag{2.36}$$

Therefore, the gauge symmetry is broken for values of ξ in the interval (0,1], where the upper bound is dictated by (2.29). We shall refer to the cases $1-\xi\ll 1$ and $\xi\ll 1$ as the Bogoliubov and quasicanonical regions, respectively.

The source operator is of the type (2.13) where the commutators are straightforwardly evaluated by taking into account the basic commutation relations (2.33) and (2.34). In the Schrödinger picture we find

$$\begin{aligned}\hat{S}(x)=&-\xi\mu V^{-1}\int d^Dx'[\psi^\dagger(x)\psi(x')-\psi^\dagger(x')\psi(x)]\\&+\xi V^{-1}\int d^Dx'd^Dx''U(x'-x'')\\&\quad\times[\psi^\dagger(x)|\psi(x'')|^2\psi(x')\\&\quad-\psi^\dagger(x')|\psi(x'')|^2\psi(x)]\end{aligned} \tag{2.37}$$

with the notation $|\psi|^2=\psi^\dagger\psi$. In momentum space one has

$$\begin{aligned}\hat{S}_k=&-\xi\mu(b_{-k}^\dagger b_0-b_0^\dagger b_k)\\&+\xi V^{-1}\sum_{p,q}U_p(b_{-k}^\dagger b_{p+q}^\dagger b_q b_p-b_p^\dagger b_q^\dagger b_{p+q}b_k) ,\end{aligned} \tag{2.38}$$

where U_p is the Fourier transform of $U(x)$. From time-reversal invariance, $\langle\hat{S}_k\rangle=0$ and, consequently, the total particle number is conserved on the average as expected.

An interesting feature of $\hat{S}_k$ concerns noninteracting systems. The equation of motion of the total particle-number operator equals

$$\frac{\partial\hat{N}}{\partial\tau}=[\hat{H},\hat{N}]=\int d^Dx\,\hat{S}(x)=\hat{S}_0 , \tag{2.39}$$

where $\hat{S}_0\equiv\hat{S}_{k=0}$. For the ideal Bose gas (IBG), $\hat{S}_0=0$. Therefore, $\hat{S}_k$ is consistent with the well-known conservation of total particle number in noninteracting systems. On the other hand, for the source operator (2.21), we have

$$\hat{S}_0^0=\eta N_0^{1/2}(b_0^\dagger-b_0) . \tag{2.40}$$

Thus, $\hat{S}_0^0=0$ implies $b_0=b_0^\dagger$, which violates the canonical nature of b_0 and $b_0^\dagger$ assumed in the symmetry-breaking scheme (2.18)–(2.21).

We finally notice that, in the Bogoliubov limit ($\xi=1$, $b_0=b_0^\dagger=N_0^{1/2}$), the IBG version of (2.38) becomes

$$\hat{S}_k=-\mu N_0^{1/2}(b_{-k}^\dagger-b_k) \quad (\text{IBG}, \ \xi=1) . \tag{2.41}$$

In $\hat{S}_k^0$, Eq. (2.21), η is an infinitesimal positive energy, whereas in the IBG μ is an infinitesimal negative energy. In Sec. IV we shall show that, indeed, $\eta=-\mu$ in the SPA.

Consequently, $\hat{S}_k^0$ equals Eq. (2.41). In Eq. (2.38), $-\xi\mu$ can be interpreted as a fictitious energy analogous to η, so that $\hat{S}_k$ is clearly a generalization of $\hat{S}_k^0$ that, in addition, satisfies particle conservation in IBG.

III. WARD IDENTITIES

Ward identities relate irreducible and proper (regular) diagrams of correlation functions. These identities are direct consequences of the continuity equation which is usually imposed as a source-free conservation law.[16–20] The source, however, is an essential constituent of the present formulation.

The determination of Ward identities based on the continuity equation (2.14) has been discussed earlier.[25] For completeness and description of notation, we briefly review the main results.

We define correlation functions of the form

$$\chi_{AB}(k,\tau) \equiv -V^{-1}\langle T_\tau A_k(\tau) B_k^\dagger \rangle \ , \tag{3.1}$$

$$C_\mu^A(k,\tau) \equiv -V^{-1/2}\langle T_\tau A_k(\tau) b_{k\mu}^\dagger \rangle \ , \tag{3.2}$$

where T_τ is the τ-ordering operator, A_k and/or B_k stand for the operators $\hat{\rho}_k$, $m\hat{J}_k$, and $\hat{S}_k$. The amplitude obeys the standard convention

$$b_{k\mu} = \begin{cases} b_k, & \mu = + \\ b_{-k}^\dagger, & \mu = - \end{cases} \ .$$

We denote by $\chi_{AB}(k,\omega)$ the Fourier transform of (3.1), where ω is the analytic continuation of the Bose frequency $2\pi i l/\beta$, with $\beta \equiv 1/k_B T$ the inverse temperature. In analogy with the dielectric formalism, correlation functions can be expressed in terms of irreducible functions to be denoted with overbars. In turn, $\bar{\chi}_{AB}(k,\omega)$ can be further split into proper and improper parts,

$$\bar{\chi}_{AB} = \bar{\chi}_{AB}^R + \bar{\chi}_{AB}^C \ . \tag{3.3}$$

The irreducible and improper correlation function can, in addition, be given in terms of (regular) vertex functions, $\Lambda_\mu^A(k,\omega)$, defined by

$$\bar{C}_\mu^A = \Lambda_\nu^A \bar{G}_{\nu\mu} \ , \tag{3.4}$$

so that

$$\bar{\chi}_{AB}^C = \Lambda_\mu^A \bar{G}_{\mu\nu} \Lambda_\nu^B \ , \tag{3.5}$$

where the summation convention over repeated indices ($\mu,\nu = +,-$) is assumed, and $\bar{G}_{\mu\nu}(k,\omega)$ is the irreducible one-particle Green's function.

Now, from Eqs. (2.9)–(2.12) in the second paper of Ref. 25, one easily obtains the following Ward identities:

$$\omega \Lambda_\mu^n = \frac{k}{m}\Lambda_\mu^J + n_0^{1/2}\beta_\nu \bar{G}_{\nu\mu}^{-1} - \Lambda_\mu^S \ , \tag{3.6}$$

$$\omega^2 \bar{\chi}_{nn}^R = \frac{k^2}{m}\left[\frac{m}{Vk}\langle [\hat{J}_k, \hat{\rho}_k^\dagger] \rangle + m^{-1}\bar{\chi}_{JJ}^R - n_0^{1/2} k^{-1}\beta_\mu \Lambda_\mu^J - k^{-1}\bar{\chi}_{SJ}^R\right] - \omega \bar{\chi}_{Sn}^R - n_0^{1/2}\omega\beta_\mu \Lambda_\mu^n + \frac{\omega}{V}\langle [\hat{\rho}_k, \hat{\rho}_k^\dagger] \rangle \ , \tag{3.7}$$

where $\beta_\mu \equiv$ sgn μ and the sub- or superscripts n (instead of ρ) refer to the particle density. The irreducible one-particle Green's function satisfies the Dyson-Beliaev equation

$$\bar{G}_{\mu\nu}(k,\omega)^{-1} = G_{\mu\nu}^0(k,\omega)^{-1} - \bar{\Sigma}_{\mu\nu}(k,\omega) \ . \tag{3.8}$$

$\Sigma_{\mu\nu}$ is the irreducible self-energy and the noninteracting Green's function equals

$$G_{\mu\nu}^0(k,\omega) = \frac{\delta_{\mu\nu}}{(\text{sgn } \nu)\omega - (\varepsilon_k - \mu)} \quad (k \neq 0) \ , \tag{3.9}$$

where $\varepsilon_k \equiv k^2/2m$ and the chemical potential must not be confused with the indices.

We proceed now to the low-frequency and long-wavelength analysis of Eqs. (3.6) and (3.7). Combining Eqs. (3.6), (3.8), and (3.9), the first Ward identity gives, for $k = 0$ and $\omega = 0$,

$$\mu = \bar{\Sigma}_{++}(0,0) - \bar{\Sigma}_{-+}(0,0) + \tfrac{1}{2} n_0^{-1/2}\beta_\mu \Lambda_\mu^S(0,0) \ . \tag{3.10}$$

This result can be interpreted as a generalized version of the Hugenholtz and Pines theorem, where the new term, the last one in (3.10), explicitly displays the effect of the broken symmetry. It is the analog of the fictitious energy first obtained by Talbot and Griffin in the Hugenholtz and Pines theorem.[26] As these authors have pointed out, this term removes, in a natural way, the infrared divergences that appear in the work of Gavoret and Nozières.[15] This point will be further discussed in Sec. IV.

An explicit expression for the superfluid density can be obtained from the second Ward identity (3.7). We first recall that

$$n = n_S + n_N \ , \tag{3.11}$$

where n_N is the normal fluid density and $n = N/V$ is the total density, the latter given by the hydrodynamic limit of the longitudinal part of the current correlation function

$$n = -\lim_{k \to 0} m^{-1}\chi_{JJ}^l(k,0) \ . \tag{3.12}$$

The normal fluid density is defined as the transverse response from a moving boundary, i.e.,

$$n_N = -\lim_{k \to 0} m^{-1}\chi_{JJ}^t(k,0) \ . \tag{3.13}$$

This result can be shown to be diagrammatically equivalent to[26]

$$n_N = -\lim_{k \to 0} m^{-1}\bar{\chi}_{JJ}^R(k,0) \ . \tag{3.14}$$

We then consider the identity (3.7) in the $k \to 0$, $\omega = 0$ limit. From the commutation relations (2.31) an (2.32), we have

$$V^{-1}\langle [\hat{J}_k, \hat{\rho}_k^\dagger] \rangle = \frac{k}{mV}\langle \hat{N} + \tfrac{1}{2}\xi(b_k^\dagger b_k + b_{-k}^\dagger b_{-k}) \rangle \ . \tag{3.15}$$

The evaluation of the right-hand side of Eq. (3.15) depends on whether the hydrodynamic limit or the thermodynamic limit is taken first. Since n_S and n_N are defined according to their respective linear responses to boundary motion, we assume the following order:

$$\lim_{V\to\infty}\lim_{k\to 0} V^{-1}\langle[\hat{J}_k,\hat{\rho}_k^\dagger]\rangle=\frac{k}{m}(n+\xi n_0)\ . \quad (3.16)$$

The second term on the right-hand side of (3.16) disappears if the order of the limits is inverted. This is also the case of the symmetry-breaking scheme (2.18), where b_0 and $b_0^\dagger$ are canonical, so that $\xi=0$. Likewise, we shall see that, in the quasicanonical region ($\xi \ll 1$) this term can be neglected. It will be relevant though in the Bogoliubov region ($\xi \sim 1$).

Now, combining Eqs. (3.7), (3.11), (3.14), and (3.16), we finally have

$$n_S=\lim_{k\to 0}[n_0^{1/2}k^{-1}\beta_\mu\Lambda_\mu^J(k,0)+k^{-1}\overline{\chi}_{SJ}^R(k,0)]-\xi n_0\ . \quad (3.17)$$

Equations (3.10) and (3.17) are exact to within the Bose broken-symmetry model. They are the basis for the study of the condensate density we take up next.

IV. CONDENSATE DENSITY: $\hat{S}_k^0$ MODEL

It is instructive to first illustrate the formalism of the previous sections with the source $\hat{S}_k^0$ given by Eq. (2.21). The algebra is simpler and the result is nevertheless unexpected. It also serves as a comparison with the $\hat{S}_k$ model to be discussed in the next section.

In order to determine the source vertex function that appears in Eqs. (3.6) and (3.10), we start from the definition

$$C_+^S(k,\tau)=-V^{-1/2}\langle T_\tau\hat{S}_k^0(\tau)b_k^\dagger\rangle\ . \quad (4.1)$$

Substituting Eq. (2.21) into (4.1), one obtains

$$\begin{aligned}C_+^S(k,\tau)&=\eta n_0^{1/2}[\langle T_\tau b_k(\tau)b_k^\dagger\rangle-\langle T_\tau b_{-k}^\dagger(\tau)b_k^\dagger\rangle]\\&=-\eta n_0^{1/2}[G_{++}(k,\tau)-G_{-+}(k,\tau)]\ . \quad (4.2)\end{aligned}$$

The irreducible part of the Fourier transform of Eq. (4.2) yields

$$\overline{C}_+^S(k,\omega)=-\eta n_0^{1/2}[\overline{G}_{++}(k,\omega)-\overline{G}_{-+}(k,\omega)]\ , \quad (4.3)$$

and this equation combined with (3.4) gives

$$\Lambda_\mu^S=-\eta n_0^{1/2}\beta_\mu\ . \quad (4.4)$$

The first Ward identity (3.6) then becomes

$$\omega\Lambda_\mu^n=\frac{k}{m}\Lambda_\mu^J+n_0^{1/2}\beta_\nu\overline{G}_{\nu\mu}^{-1}+\eta n_0^{1/2}\beta_\mu\ . \quad (4.5)$$

Substitution of (4.4) into the generalized Hugenholtz and Pines formula (3.10) gives

$$\mu=\overline{\Sigma}_{++}(0,0)-\overline{\Sigma}_{-+}(0,0)-\eta\ . \quad (4.6)$$

Equations (4.5) and (4.6) are identical to the ones first obtained by Talbot and Griffin.[26] As these authors have pointed out, the last term in (4.6) implies an infinitesimal energy gap in the excitation spectrum. Talbot and Griffin argue that the fictitious energy gap used by Gavoret and Nozières[15] is actually not fictitious, but a consequence of the broken symmetry.

We now consider the second Ward identity (3.17) in the SPA, which amounts to evaluating all regular functions in the noninteracting Bose gas approximation. Accordingly,

$$\overline{G}_{\mu\nu}=G_{\mu\nu}^0,\quad \overline{\Sigma}_{\mu\nu}=0\ , \quad (4.7)$$

and, from previous results,[16–19] one has

$$\Lambda_\mu^n=n_0^{1/2},\quad \Lambda_\mu^J(k)=\tfrac{1}{2}kn_0^{1/2}\beta_\mu\ . \quad (4.8)$$

Similarly to Λ_μ^S, the source-longitudinal-current correlation function is determined from the respective definition. Accordingly, from Eqs. (2.16), (2.21), and (3.1), we obtain

$$\begin{aligned}\chi_{SJ}(k,\tau)&=-mV^{-1}\langle T_\tau\hat{S}_k^0(\tau)\hat{J}_k^\dagger\rangle\\&=\eta\left[\frac{n_0}{V}\right]^{1/2}\hat{k}\sum_p(p+\tfrac{1}{2}k)[\langle T_\tau b_k(\tau)b_{p+k}^\dagger b_p\rangle-\langle T_\tau b_{-k}^\dagger(\tau)b_{p+k}^\dagger b_p\rangle]\ . \quad (4.9)\end{aligned}$$

In the SPA, the ensemble average of a τ-ordered product factorizes in pairs of unperturbed Green's functions. Since the averages in (4.9) contain an odd number of amplitudes, only the nonvanishing averages must contain zero-mode amplitudes. Hence,

$$\begin{aligned}\langle T_\tau b_k(\tau)b_{p+k}^\dagger b_p\rangle&=\delta_{p,0}\langle T_\tau b_k(\tau)b_k^\dagger\rangle\langle b_0\rangle\\&\quad+\delta_{p,-k}\langle T_\tau b_k(\tau)b_{-k}\rangle\langle b_0^\dagger\rangle\ , \quad (4.10)\end{aligned}$$

$$\begin{aligned}\langle T_\tau b_{-k}^\dagger(\tau)b_{p+k}^\dagger b_p\rangle&=\delta_{p,0}\langle T_\tau b_{-k}^\dagger(\tau)b_k^\dagger\rangle\langle b_0\rangle\\&\quad+\delta_{p,-k}\langle T_\tau b_{-k}^\dagger(\tau)b_{-k}\rangle\langle b_0^\dagger\rangle\ . \quad (4.11)\end{aligned}$$

The ensemble averages in Eqs. (4.10) and (4.11) refer to unperturbed states and the anomalous noninteracting Green's functions (3.9) vanish. Thus, Eqs. (4.10) and (4.11) reduce to

$$\langle T_\tau b_k(\tau)b_{p+k}^\dagger b_p\rangle=-\delta_{p,0}N_0^{1/2}G_{++}(k,\tau)\ , \quad (4.12)$$

$$\langle T_\tau b_{-k}^\dagger(\tau)b_{p+k}^\dagger b_p\rangle=-\delta_{p,-k}N_0^{1/2}G_{--}(k,\tau)\ , \quad (4.13)$$

Substituting Eqs. (4.12) and (4.13) in (4.9), and taking the Fourier transform, we find

$$\overline{\chi}_{SJ}^R(k,\omega)=-\tfrac{1}{2}\eta n_0k[G_{++}^0(k,\omega)+G_{--}^0(k,\omega)]\ . \quad (4.14)$$

According to (4.7), the SPA solution of (4.6) simply reads

$$\mu=-\eta\ , \quad (4.15)$$

and combining Eqs. (3.9), (4.14), and (4.15), one has

$$\lim_{k\to 0} k^{-1}\overline{\chi}^{R}_{SJ}(k,0) = -\eta n_0 \mu^{-1} = n_0 \ . \tag{4.16}$$

The source contribution to the superfluid density turns out to be independent of the arbitrary energy η. Substitution of Eqs. (4.8) and (4.16) in (3.17) gives

$$n_0 = \frac{n_S}{2-\xi} \ . \tag{4.17}$$

Since $[b_0, b_0^\dagger]=1$, $\xi=0$ and Eq. (4.17) yields $n_0 = n_S/2$, which means that $n_0 \le n/2$. This can be expected in the strongly interacting dense Bose system (SIDBS), but not in WIDBS, where the depletion of the condensate is known to be small. From the arguments given after Eq. (2.40), the IBG requires $b_0 = b_0^\dagger$ or, equivalently, $\xi=1$. Using this value in (4.17) we would obtain $n_0 = n_S$, as it should be in this case. The source contribution comes from Eq. (4.16) which is finite even in the limit $\eta \to 0$. This indicates how the source operator modifies, in an essential way the relationship between n_0 and n_S. On the other hand, it is also clear that $\hat{S}_k^0$ is inadequate to handle the IBG, as already stated by Eq. (2.40). This argument may be extended to WIDBS since ξ can be viewed as a measure of the c-number nature of b_0 and $b_0^\dagger$. This interpretation will become apparent when the source $\hat{S}_k$ is considered in the next section.

V. CONDENSATE DENSITY: $\hat{S}_k$ MODEL

We now apply the same procedure of the last section to $\hat{S}_k$. The algebra involved in the SPA evaluation of Λ^S_μ and $\overline{\chi}^R_{SJ}$ is described in the Appendix. Since we are only interested in the long-wavelength limit, we have replaced U_k by the interaction constant

$$U_k \to U_0 = 4\pi a m^{-1} \ , \tag{5.1}$$

where a is the s-wave scattering length. The source vertex function determined in the Appendix is

$$\Lambda^S_\mu = \xi n_0^{1/2}(\mu - 2nU_0)\beta_\mu \ . \tag{5.2}$$

Combining Eqs. (3.10), (4.7), and (5.2), we obtain an equation for ξ; that is,

$$\xi = (1-2nU_0\mu^{-1})^{-1} \ . \tag{5.3}$$

As advanced in Sec. II, this result relates the symmetry-breaking parameter with the interaction strength. Since U_0 in non-negative and ξ lies in the interval (0,1], it follows from Eq. (5.3) that $\mu<0$. Therefore, the Bogoliubov region $(1-\xi \ll 1)$ implies $nU_0 \ll |\mu|$, whereas the quasicanonical region $(\xi \ll 1)$ holds for $nU_0 \gg |\mu|$. Also, the Bogoliubov limit $(\xi=1)$ corresponds to the IBG, where $U_0=0$ and μ is an infinitesimal negative energy.

We now turn to the condensate density. In the Appendix we find

$$k^{-1}\overline{\chi}^R_{SJ}(k,\omega) = \frac{\xi n_0(\mu-2nU_0)(\varepsilon_k - \xi\mu)}{\omega^2 - (\varepsilon_k - \xi\mu)^2} \ . \tag{5.4}$$

The contribution to the superfluid density (3.17) is

$$\lim_{k\to 0} k^{-1}\overline{\chi}^R_{SJ}(k,0) = (1-2nU_0\mu^{-1})n_0 \ . \tag{5.5}$$

As in Eq. (4.16), the contribution (5.5) turns out to be independent of the symmetry-breaking parameter and it is by no means negligible. Substituting Eqs. (4.8) and (5.5) into (3.17), we obtain a relationship between the condensate and the superfluid densities,

$$n_0 = \frac{n_S}{2-\xi-2nU_0\mu^{-1}} \ . \tag{5.6}$$

For $U_0=0$, Eqs. (5.5) and (5.6) become identical to Eqs. (4.16) and (4.17), respectively. Hence, $\hat{S}_k^0$ may be viewed as the zero-order approximation to $\hat{S}_k$. Moreover, Eq. (5.3) gives $\xi=1$, a feature that was missing in the discussion of Eq. (4.17).

For convenience we measure the energy nU_0 in terms of $|\mu|/2$ and introduce the dimensionless quantity

$$\nu(T) \equiv 2nU_0|\mu(T)|^{-1} \ . \tag{5.7}$$

Equation (5.3) then becomes

$$\xi = (1+\nu)^{-1} \ , \tag{5.8}$$

and substituting this result into Eq. (5.6) we obtain

$$\frac{n_0}{n_S} = \frac{1+\nu}{(1+\nu)^2+\nu} = \frac{(1+\nu^{-1})\nu^{-1}}{(1+\nu^{-1})^2+\nu^{-1}} \ . \tag{5.9}$$

Equations (5.8) and (5.9) yield the central result. The Bogoliubov limit $(\xi=1)$ describes the IBG $(\nu=0)$ with $n_0=n_S$. The Bogoliubov region $(1-\xi \ll 1)$ corresponds to a system with weak interparticle interaction $(\nu \ll 1)$ and, as a result, $n_0 \simeq n_S$. Finally, the quasicanonical region $(\xi \ll 1)$ holds for the large interaction constant $(\nu^{-1} \ll 1)$ and $n_0 \ll n_S$ follows.

VI. ARBITRARY D-DIMENSIONAL SPACE

In this section we show how the source is a useful concept in the discussion of the Bose-Einstein condensation at finite temperatures in arbitrary D-dimensional space. The Mermin and Wagner theorem[27] states that a global continuous symmetry cannot be spontaneously broken in systems with dimensions $D \le 2$. The theorem is based on a rigorous inequality due to Bogoliubov,[28]

$$\tfrac{1}{2}\langle\{\hat{A},\hat{A}^\dagger\}\rangle \ge k_B T \frac{|\langle[\hat{A},\hat{C}]\rangle|^2}{\langle[\hat{C}^\dagger,[\hat{\mathcal{H}},\hat{C}]]\rangle} \ . \tag{6.1}$$

This inequality still holds if, instead of $\hat{H}$ as in the original formulation, we take for $\hat{\mathcal{H}}$ the grand canonical Hamiltonian, i.e., $\hat{\mathcal{H}} = \hat{H} - \mu\hat{N}$. The operators $\hat{A}$ and $\hat{C}$ are arbitrary provided the ensemble averages exist. If we now let $\hat{A} \equiv b_k^\dagger$ and $\hat{C} \equiv \hat{\rho}_k$, one has

$$\tfrac{1}{2}\langle\{b_k^\dagger, b_k\}\rangle = \tfrac{1}{2} + \langle b_k^\dagger b_k\rangle \ , \tag{6.2}$$

$$\langle[b_k^\dagger, \hat{\rho}_k]\rangle = -\langle b_0^\dagger\rangle = -N_0^{1/2} \ , \tag{6.3}$$

$$[\hat{\mathcal{H}}, \hat{\rho}_k] = \frac{\partial\hat{\rho}_k}{\partial\tau} = -k\hat{J}_k + \hat{S}_k \ . \tag{6.4}$$

In (6.4) use has been made of the continuity equation

(2.14). Substituting Eqs. (6.2)–(6.4) into (6.1), and taking into account Eq. (3.15) with $k\neq 0$, we obtain

$$\frac{1}{V}\sum_{k\neq 0}\langle b_k^\dagger b_k\rangle \geq \int \frac{d^Dk}{(2\pi)^D}\left[\frac{mk_BT(n_0/n)}{k^2+\Theta^2}-\frac{1}{2}\right] \tag{6.5}$$

with

$$\Theta^2\equiv\frac{m}{nV}\langle[\hat{\rho}_k^\dagger,\hat{S}_k]\rangle \ . \tag{6.6}$$

Before discussing the implications of (6.5), we first consider the quantity Θ. For the source operator $\hat{S}_k^0$, Eq. (6.6) is immediately evaluated and we find

$$(\Theta^0)^2=2\eta m(n_0/n) \ . \tag{6.7}$$

$(\Theta^0)^2$ turns out to be k independent and a linear function of the condensate fraction. Except for a complex η used by Chester, Fisher, and Mermin, Eq. (6.7) equals the result found in that work.[29] This is expected since these authors have used a symmetry-breaking procedure similar to that of Eq. (2.20).

In the case of $\hat{S}_k$ [Eq. (2.38)], the evaluation of (6.6) is lengthy, but otherwise straightforward, and we write $\Theta^2=\Theta_0^2+\Theta_k^2$, where

$$\Theta_0^2=-\frac{2\xi m\mu}{nV}\langle b_0^\dagger b_0\rangle+\frac{\xi mU_0}{nV^2}\sum_{p,q}\langle b_0^\dagger b_{p+q}^\dagger b_q b_p+b_p^\dagger b_q^\dagger b_{p+q}b_0\rangle \ , \tag{6.8}$$

$$\begin{aligned}\Theta_k^2=&\frac{(1-\xi)\xi m\mu}{nV}\langle b_k^\dagger b_k+b_{-k}^\dagger b_{-k}\rangle\\&-\frac{\xi mU_0}{nV^2}\sum_p\Bigg[\sum_q\langle b_{p+k}^\dagger b_q^\dagger b_{p+q}b_k+b_{-k}^\dagger b_{p+q}^\dagger b_q b_{p-k}\rangle\\&\qquad-2\xi\langle b_k^\dagger b_p^\dagger b_p b_k+b_{-k}^\dagger b_p^\dagger b_p b_{-k}\rangle+\xi\langle b_k^\dagger b_{-k}^\dagger b_p b_{-p}+b_{-p}^\dagger b_p^\dagger b_k b_{-k}\rangle\Bigg] \ .\end{aligned} \tag{6.9}$$

We see that $\Theta_k^2=O(V^{-1})$, thus vanishing in the thermodynamic limit. Hence, $\Theta=\Theta_0$, which is also k independent. In the SPA, the ensemble averages in Eq. (6.8) factorize in pairs of unperturbed contractions and we immediately obtain

$$\Theta_0^2=2m\xi(2nU_0-\mu)(n_0/n) \ . \tag{6.10}$$

Substituting Eq. (5.3) into (6.10), Θ_0^2 becomes ξ independent and we write

$$\varepsilon_0\equiv\Theta_0^2/2m=-\mu(n_0/n) \ . \tag{6.11}$$

Recalling Eq. (4.15), $\eta=-\mu$, we see that Θ_0^2 is consistent with $(\Theta^0)^2$.

We turn now to the inequality (6.5). The left-hand-side equals $n-n_0$, which implies a nondivergent integral

$$\int\frac{d^Dk}{k^2+\Theta^2}<\infty \ . \tag{6.12}$$

Otherwise, one must have $n_0=0$ in the numerator of (6.5) if $T>0$. Hereafter, we assume that Θ is k independent such that the convergence of (6.12) depends solely on the space dimensionality.

An analogous approach to this problem was taken by Hohenberg,[30] except that, instead of Eq. (6.4), a source-free ($\hat{S}_k=0$) continuity equation was considered. Consequently, condition (6.12) would have to be satisfied with $\Theta=0$.

The integral (6.12) can be set in a more general context by the dimensional regularization method introduced independently by Bollini and Giambiagi[31] and 't Hooft and Veltman.[32] Accordingly, this integral can be regarded as a function of D by the process of analytic continuation in D. It is then a simple exercise to show that[33]

$$\int_0^\infty k^\alpha dk=0 \tag{6.13}$$

for all values of α, including $\alpha=-1$. Therefore, in the case $\Theta=0$, $T>0$, Bose-Einstein condensation occurs whatever the space dimension. For $\Theta\neq 0$, on the other hand, it is easy to show that[31,34]

$$\int\frac{d^Dk}{k^2+\Theta^2}=\pi^{D/2}\Theta^{D-2}\Gamma(1-\tfrac{1}{2}D) \ . \tag{6.14}$$

Due to the Γ function this integral diverges as a simple pole at $D=2$. According to (6.5) and (6.12), one must then have $n_0=0$ in two-dimensional (2D) systems. As $\Theta^2\rightarrow 0$ in the neighborhood of this dimension, i.e., $D=2-\varepsilon$, the integral (6.14) depends on whether $\varepsilon>0$ or $\varepsilon<0$. Therefore, the properties of Bose systems are fundamentally different below and above the critical dimension $D_c=2$.

From Eq. (6.13) we see that the source-free ($\Theta=0$) treatment of Hohenberg[30] does not lead to a critical dimensionality. Chester, Fisher, and Mermin[29] have modified Hohenberg's arguments by considering the symmetry-breaking procedure (2.18) with $\hat{H}_{SB}$ equivalent to (2.20). But this is really an implicit way of taking into account the source (2.19) in Eq. (6.4). According to the anomalous average rule, these authors set $\eta\rightarrow 0$ at the end of the calculation, which obviously corresponds to $(\Theta^0)^2\rightarrow 0$. Consequently, the critical dimension emerges.

We finally note that the SPA result (6.11) is ξ independent and therefore Θ_0 does not vanish when $\xi\rightarrow 0$ (or $\xi\rightarrow 1$). However, the limit $\Theta_0\rightarrow 0$ is to be taken as $D_c=2$

is approached from above because $n_0 \to 0$ is required in order to keep the integral in (6.5) bounded.

VII. APPLICATION TO SUPERFLUID ^{4}He

In the so-called strong-coupling regime ($nU_0 \gg k_B T$),[19] response functions of Bose systems that exhibit a large depletion of the condensate have been analyzed in the SPA.[18,19] In those works the contribution of the source term is not taken into account and consequently $n_0 = n_S$. On the other hand, we have shown in Sec. V that the condensate reservoir disentangles n_0 from n_S. In particular, $nU_0 \gg |\mu|$ implies $n_0 \ll n_S$.

Strong interactions, $n_0 \ll n$, and $n_S \leq n$ are well-known properties of bulk superfluid ^{4}He. One is naturally tempted to use superfluid ^{4}He data in Eq. (5.9) in order to extract the condensate fraction. There remains, however, the conceptual difficulty of applying the low-order SPA to a dense, strongly interacting system.

The next step past the SPA consists in calculating the regular functions in the one-loop approximation. The latter, however, cannot be worked out explicitly and, as pointed out by Talbot and Griffin,[20] the SPA contains some of the same physics as the one-loop approximation. In fact, for a dilute Bose gas, Payne and Griffin[35] have shown that the one-loop approximation gives only small corrections to the SPA.

By the same token we recall Anderson's basic principle of continuation.[36] This means that such systems as Fermi liquid and superfluid ^{4}He can be referred back to simpler problems involving only noninteracting particles. Without crossing past a symmetry boundary, one can follow some adiabatic path from the simple state to the one including more or less strong interactions.

We thus regard the present SPA formalism not as a rigorous approximation, but simply as a model calculation to gain physical insight into the phenomenon of Bose-Einstein condensation in superfluid ^{4}He. Accordingly, we now use bulk superfluid ^{4}He experimental data in Eq. (5.9) and extract the corresponding condensate density. Maynard[37] has calculated self-consistent tables of $n(T,P)$, $n_S(T,P)$, $\mu(T,P)$, and the phonon velocity $c(T,P)$ for temperatures $T \geq 1.2$ K from saturated vapor pressure (SVP) to 25 bar. The interaction constant U_0 can be obtained from the excitation spectrum. In the SPA, Szépfalusy and Kondor[19] have shown that the $k \to 0$ excitation spectrum in the strong-coupling regime are phonons with isothermal sound velocity given by

$$c = (nU_0/m)^{1/2} . \tag{7.1}$$

Similarly, from sum-rule arguments in the $k \to 0$ limit, Aldrich and Pines use Eq. (7.1) as the scalar polarization potential to deduce the phonon-roton spectrum in excellent quantitative agreement with that observed in neutron-scattering experiments.[38]

At SVP we take, for the isothermal sound velocity,[39]

$$c(T=0, \text{SVP}) = 238.3 \text{ m sec}^{-1} .$$

For this value and m the He-atom mass Eq. (7.1) gives $nU_0 = 27.32$ K consistently with Aldrich and Pines. By using the Gibbs free energy tabulated by Brooks and Donnelly,[40] we infer from Maynard's value of the chemical potential at $T = 1.2$ K that

$$\mu(T=0, \text{SVP}) = -14.882 \text{ J g}^{-1} (= -7.1617 \text{ K})$$

and, therefore,

$$\nu^{-1}(T=0, \text{SVP}) = 0.1311 .$$

In Fig. 1 we compare the calculated values, furnished by Eq. (5.9), with the more recent data analysis performed by Mook on neutron-scattering experiments at SVP.[5,41] In particular, one finds $n_0(T=0) = 0.1051n$. We remark that the works of Maynard,[37] Aldrich and Pines,[38] and Brooks and Donnelly,[40] make no reference to the condensate density. The curve shown in Fig. 1 does not involve any adjustable parameter, hence, the agreement between theory and experiment is surprisingly good.

We now apply Eq. (5.9) to the condensate density under pressure. In order to find out $n_0(T=0,P)$, we determine the variation of the input quantities between $T=0$ and 1.2 K according to Brooks and Donnelly and normalize them with respect to Maynard's $T=1.2$-K data. The predicted condensate densities, relative to

$$n_0(T=0, \text{SVP}) = 0.1051n ,$$

are displayed in Fig. 2. About $P \sim 23.8$ bar, the condensate vanishes with the chemical potential. For still higher pressures, μ becomes positive and an unphysical negative n_0 would result. Actually, Eq. (5.9) does not apply in this case, for ξ would be negative. One is usually more inclined to expect that n_0 falls to zero at the superfluid-

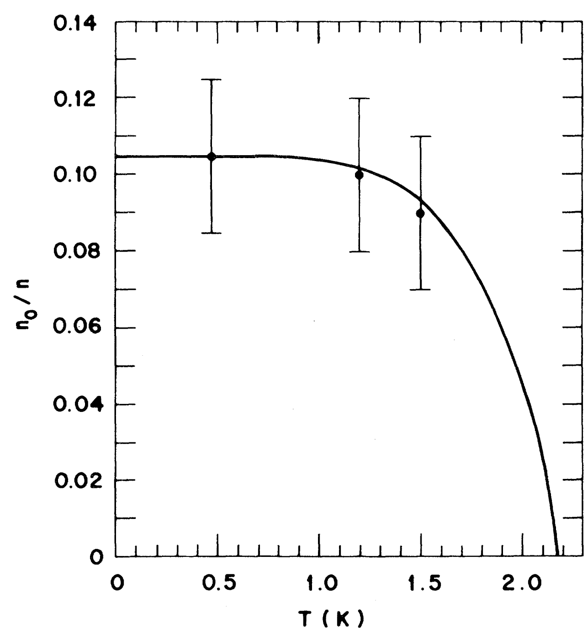

FIG. 1. Condensate fraction as function of temperature at saturated vapor pressure. Curve: as calculated by Eq. (5.9). Circles: experimental values from Ref. 5.

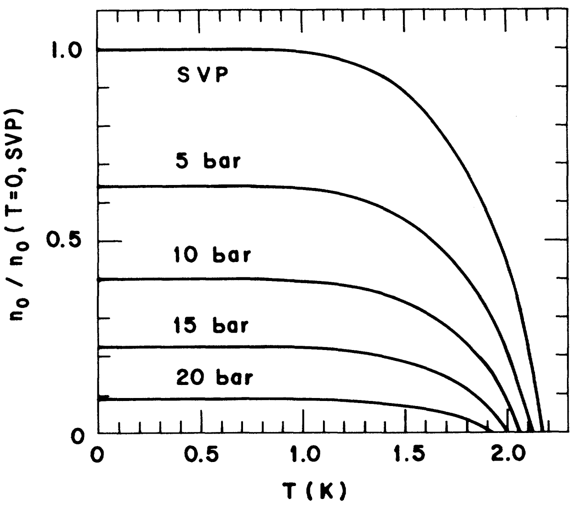

FIG. 2. Condensate density as function of temperature at various pressures calculated by Eq. (5.9) and normalized with respect to $n_0(T=0, \mathrm{SVP})=0.1051n$.

solid transition pressure $P \sim 25$ bar. Unfortunately, Mook's investigation of n_0 under pressure is limited to about 15 bar. In any case, we must await an accurate determination of the condensate fraction as a function of density. Nevertheless, the strong dependence of the condensate fraction on pressure implies that n_0 is also strongly dependent on density. This behavior is consistent with the experimental trend observed by Mook.[4]

As a final application of the foregoing theory to superfluid ^{4}He, we consider the infrared cutoff discussed in Sec. VI. It has been conjectured that such a cutoff might be attributed to the zero-point motion.[42] Thus in Eq. (6.11) we use the quantities found above for superfluid ^{4}He at $T=0$ and SVP to obtain

$$\varepsilon_0=0.753\ \mathrm{K}, \quad \Theta_0^{-1}=2.84\ \text{Å} . \tag{7.2}$$

The inverse of the infrared cutoff lies between $a \simeq 2.5$ Å and $n^{-1/3} \simeq 3.5$ Å. It is indeed reasonable to associate ε_0 with the zero-point energy. In this case, the number of condensed particles in the ground state is thus balanced by the overall zero-point energy according to $-\mu N_0 = \varepsilon_0 N$.

VIII. CONCLUDING REMARKS

The key features of the foregoing theory are twofold: (i) the symmetry-breaking scheme based on $[b_0, b_0^\dagger]=1-\xi$, and (ii) the corresponding source operator $\hat{S}_k$. In the absence of interactions, $\xi=1$ and $n_0=n_S$. A small interaction is characterized by $\xi \simeq 1$ and $n_0 \simeq n_S$, while a strong one by $\xi \ll 1$ and $n_0 \ll n_S$. The physical picture is then straightforward: the stronger the interparticle potential, the larger are both the depletion of the condensate and the operator nature of b_0 and $b_0^\dagger$.

It has been a controversial matter whether the IBG is superfluid or not. According to Landau's criterion of a critical velocity, it is not. This criterion is, however, a sufficient, but not necessary, one. Fisher, Barber, and Jasnow[43] have studied superfluidity in the Bose system in terms of a helicity modulus that measures the incremental free energy due to a twist of the order parameter. These authors have found $n_0=n_S$ for the IBG. Incidentally, one may interpret $-\xi\theta$ in (2.36) as a twisting phase angle with respect to the overall phase.[25] Fisher *et al.*'s[43] microscopic definition of n_S requires specified boundary conditions, a procedure that corroborates the arguments leading to Eq. (3.16). In the IBG, $n_0=n_S$ is also strengthened by Anderson's continuation principle mentioned in Sec. VII.

The good agreement without adjustments between the present theory and superfluid ^{4}He leads us to the conjecture that the inclusion of $\hat{S}_k$ into the formalism is, by itself, more important than the approximations involved in the solution of the exact Ward identities. This was, in fact, already suggested in Sec. IV where $\hat{S}_k^0$ reduces, by half, the ratio n_0/n_S. Moreover, the source operator (2.38) depends linearly on ξU_0, and from Eq. (5.3), we have

$$\lim_{U_0 \to \infty} \xi U_0 = -\tfrac{1}{2}\mu n^{-1} . \tag{8.1}$$

Thus, $\hat{S}_k$ remains finite even in the case of an infinite hard core. The interaction strength in $\hat{S}_k$ is therefore attenuated by the symmetry-breaking parameter, which might indicate that the condensate reservoir gives sensible results in the SPA.

ACKNOWLEDGMENTS

I would like to thank Professor Kerson Huang for interesting comments on my earlier work related to this subject and for bringing Ref. 24 to my attention. I have also benefited greatly from discussions with Professor J. J. Giambiagi on dimensional regularization.

APPENDIX

In this appendix we derive Eqs. (5.2) and (5.4) in the SPA. In Ref. 25, the μ-dependent term of Eq. (2.17) was incorrectly left out in the determination of $\hat{S}_k$, Eq. (2.38), on the argument that this term does not contribute to the equation of motion (2.39). The $[\hat{U}, \hat{\rho}_k]$ contribution to $\hat{S}_k$ has been discussed in the second paper of Ref. 25. For completeness, however, we take for $\hat{S}_k$ the entire formula (2.38).

To calculate Λ_μ^S it is first convenient to split (2.38) as follows:

$$\hat{S}_k = \hat{R}_{-k}^\dagger - \hat{R}_k ,$$

$$\hat{R}_k = -\xi\mu b_0^\dagger b_k + \xi V^{-1} \sum_{p,q} U_p b_p^\dagger b_q^\dagger b_{p+q} b_k .$$

From Eq. (3.2) C_μ^S is then given by

$$\begin{aligned} C_+^S(k,\tau) &= -V^{-1/2}\langle T_\tau \hat{S}_k(\tau) b_k^\dagger \rangle \\ &= V^{-1/2}\langle T_\tau \hat{R}_k(\tau) b_k^\dagger \rangle \\ &\quad - V^{-1/2}\langle T_\tau \hat{R}_{-k}^\dagger(\tau) b_k^\dagger \rangle . \end{aligned} \tag{A1}$$

The first term in (A1) reads

$$V^{-1/2}\langle T_\tau \hat{R}_k(\tau) b_k^\dagger \rangle = -\xi\mu V^{-1/2}\langle T_\tau b_0^\dagger(\tau) b_k(\tau) b_k^\dagger\rangle + \xi U_0 V^{-3/2}\sum_{p,q}\langle T_\tau b_p^\dagger(\tau) b_q^\dagger(\tau) b_{p+q}(\tau) b_k(\tau) b_k^\dagger\rangle \ , \tag{A2}$$

where we have made the substitution $U_k \to U_0$. In the SPA, the ensemble average of a τ-ordered product factorizes in pairs of unperturbed Green's functions. The factorization of an odd number of amplitudes, such as in (A2), is made possible by the anomalous average (2.30). Accordingly, the average of the first term on the right-hand side of (A2) becomes

$$\begin{aligned}\langle T_\tau b_0^\dagger(\tau) b_k(\tau) b_k^\dagger\rangle &= \langle b_0^\dagger\rangle\langle T_\tau b_k(\tau) b_k^\dagger\rangle \\ &= -N_0^{1/2} G_{++}^0(k,\tau) \ .\end{aligned} \tag{A3}$$

Likewise, the average of the second term yields

$$\begin{aligned}\sum_{p,q}\langle T_\tau b_p^\dagger(\tau) b_q^\dagger(\tau) b_{p+q}(\tau) b_k(\tau) b_k^\dagger\rangle &= 2\langle b_0^\dagger\rangle\langle b_k^\dagger b_k\rangle\langle T_\tau b_k(\tau) b_k^\dagger\rangle + 2\langle b_0^\dagger\rangle\langle T_\tau b_k(\tau) b_k^\dagger\rangle\sum_p\langle b_p^\dagger b_p\rangle \\ &= -2N_0^{1/2}(N + \langle b_k^\dagger b_k\rangle) G_{++}^0(k,\tau) \ .\end{aligned} \tag{A4}$$

The factor 2 in (A4) is due to the symmetry of p and q, and the diagonal property of the unperturbed Green's functions greatly simplifies the factorization. Substituting Eqs. (A3) and (A4) into (A2), one has

$$V^{-1/2}\langle T_\tau \hat{R}_k(\tau) b_k^\dagger\rangle = \xi n_0^{1/2}[\mu - 2U_0(n + V^{-1}\langle b_k^\dagger b_k\rangle)] G_{++}^0(k,\tau) \ . \tag{A5}$$

Similar steps show that

$$V^{-1/2}\langle T_\tau \hat{R}_{-k}^\dagger(\tau) b_k^\dagger\rangle = 2\xi n_0^{1/2} U_0 V^{-1}\langle b_{-k}^\dagger b_{-k}\rangle G_{++}^0(k,\tau) \ . \tag{A6}$$

Combining Eqs. (A1), (A5), and (A6), we have

$$\bar{C}_+^S(k,\tau) = \xi n_0^{1/2}\{\mu - 2U_0[n + V^{-1}(\langle b_k^\dagger b_k\rangle - \langle b_{-k}^\dagger b_{-k}\rangle)]\} G_{++}^0(k,\tau) \ . \tag{A7}$$

where the overbar denotes the irreducible character of the correlation function. By taking the τ Fourier transform and comparing the results with Eq. (3.4), we immediately obtain

$$\Lambda_+^S(k) = \xi n_0^{1/2}\{\mu - 2U_0[n + V^{-1}(\langle b_k^\dagger b_k\rangle - \langle b_{-k}^\dagger b_{-k}\rangle)]\} \ . \tag{A8}$$

In a uniform system, the ensemble averages in (A8) cancel out and Λ_+^S becomes k independent, i.e.,

$$\Lambda_+^S = \xi n_0^{1/2}(\mu - 2nU_0) \ . \tag{A9}$$

It is straightforward to see that the evaluation of $C_-^S(k,\tau)$ leads to a vertex function $\Lambda_-^S = -\Lambda_+^S$. Hence, Eq. (5.2) follows.

We next consider the correlation function $\bar{\chi}_{SJ}^R$. Although this case involves some more algebra, the technique is basically the same as in the evaluation of Λ_μ^S. To begin with, the definition of χ_{SJ} is

$$\begin{aligned}\chi_{SJ}(k,\tau) &= -mV^{-1}\langle T_\tau \hat{S}_k(\tau)\hat{J}_k^\dagger\rangle \\ &= mV^{-1}(\langle T_\tau \hat{R}_k(\tau)\hat{J}_k^\dagger\rangle - \langle T_\tau \hat{R}_{-k}^\dagger(\tau)\hat{J}_k^\dagger\rangle) \ .\end{aligned} \tag{A10}$$

From Eqs. (2.16) and (2.38) one has

$$\begin{aligned}\chi_{SJ}(k,\tau) &= mV^{-1}\langle T_\tau \hat{R}_k(\tau)\hat{J}_k^\dagger\rangle \\ &= -\xi\mu V^{-1}\hat{k}\sum_l (l + \tfrac{1}{2}k)\langle T_\tau b_0^\dagger(\tau) b_k(\tau) b_{l+k}^\dagger b_l\rangle \\ &\quad + \xi U_0 V^{-1}\hat{k}\sum_{lpq}(l + \tfrac{1}{2}k)\langle T_\tau b_p^\dagger(\tau) b_q^\dagger(\tau) b_{p+q}(\tau) b_k(\tau) b_{l+k}^\dagger b_l\rangle \ .\end{aligned} \tag{A11}$$

To make the notation less cumbersome, we adopt the following convention:

$$\begin{aligned}\langle T_\tau A(\tau) B\rangle &\equiv \langle A|B\rangle \ , \\ \langle T_\tau A(\tau) B(\tau)\rangle &\equiv \langle AB\rangle \ .\end{aligned} \tag{A12}$$

The summation of the first term on the right-hand side of (A11) becomes

$$\sum_l (l+\tfrac{1}{2}k)\langle b_0^\dagger b_k | b_{l+k}^\dagger b_l\rangle = \sum_l (l+\tfrac{1}{2}k)\langle b_k | b_{l+k}^\dagger\rangle\langle b_0^\dagger | b_l\rangle$$
$$= \tfrac{1}{2}kG_{++}^0(k,\tau)G_{--}^0(0,\tau) \; . \tag{A13}$$

The summation of the second term in (A11) reads

$$\sum_{lpq}(l+\tfrac{1}{2}k)\langle b_p^\dagger b_q^\dagger b_{p+q} b_k | b_{l+k}^\dagger b_l\rangle = 2\sum_{lpq}(l+\tfrac{1}{2}k)(\langle b_k | b_{l+k}^\dagger\rangle\langle b_p^\dagger | b_l\rangle\langle b_q^\dagger b_{p+q}\rangle + \langle b_{p+q} | b_{l+k}^\dagger\rangle\langle b_p^\dagger | b_l\rangle\langle b_q^\dagger b_k\rangle)$$
$$= NkG_{++}^0(k,\tau)G_{--}^0(0,\tau) + 2\langle b_k^\dagger b_k\rangle\sum_l (l+\tfrac{1}{2}k)G_{++}^0(l+k,\tau)G_{++}^0(l,\tau) \; . \tag{A14}$$

Substituting Eqs. (A13) and (A14) into (A11), one has

$$mV^{-1}\langle T_\tau \hat{R}_k(\tau)\hat{J}_k^\dagger\rangle = -\tfrac{1}{2}\xi k V^{-1}(\mu - 2nU_0)G_{++}^0(k,\tau)G_{--}^0(0,\tau)$$
$$+\xi U_0 V^{-2}\langle b_k^\dagger b_k\rangle \hat{k}\sum_l (2l+k)G_{++}^0(l+k,\tau)G_{--}^0(l,\tau) \; . \tag{A15}$$

The same procedure applied to the second term of (A10) gives

$$mV^{-1}\langle T_\tau \hat{R}_{-k}^\dagger(\tau)\hat{J}_k^\dagger\rangle = \tfrac{1}{2}\xi k V^{-1}(\mu - 2nU_0)G_{--}^0(k,\tau)G_{++}^0(0,\tau)$$
$$+\xi U_0 V^{-2}\langle b_{-k}^\dagger b_{-k}\rangle \hat{k}\sum_l (2l+k)G_{++}^0(l+k,\tau)G_{--}^0(l,\tau) \; . \tag{A16}$$

From Eqs. (A10), (A15), and (A16), we obtain a proper and irreducible correlation function

$$\bar{\chi}_{SJ}^R(k,\tau) = -\tfrac{1}{2}\xi k V^{-1}(\mu - 2nU_0)[G_{++}^0(k,\tau)G_{--}^0(0,\tau) + G_{--}^0(k,\tau)G_{++}^0(0,\tau)]$$
$$+\xi U_0 V^{-2}(\langle b_k^\dagger b_k\rangle - \langle b_{-k}^\dagger b_{-k}\rangle)\hat{k}\sum_l (2l+k)G_{++}^0(l+k,\tau)G_{--}^0(0,\tau) \; . \tag{A17}$$

Again, translational and rotational invariance eliminates the second term of (A17) whose Fourier transform then reads

$$\bar{\chi}_{SJ}^R(k,\omega) = -\tfrac{1}{2}\xi k(\mu - 2nU_0)(V\beta)^{-1} \times \sum_{\omega'}[G_{++}^0(k,\omega'+\omega)G_{++}^0(0,\omega') + G_{++}^0(k,\omega')G_{++}^0(0,\omega'+\omega)] \; . \tag{A18}$$

By using Eq. (3.9) in (A18) and performing the frequency summations by standard techniques, we find

$$\bar{\chi}_{SJ}^R(k,\omega) = (1-\xi)\xi k(\mu - 2nU_0) \times V^{-1}[N^0(-(1-\xi)\mu) - N^0(\varepsilon_k - \mu)] \times \frac{\varepsilon_k - \xi\mu}{\omega^2 - (\varepsilon_k - \xi\mu)^2} \; , \tag{A19}$$

where ω was analytic continued to the real axis and

$$N^0(x) = (e^{\beta x} - 1)^{-1}$$

is the Bose distribution function. In the ξ formalism and the SPA, $N^0(\varepsilon_k - \mu)$ is the average occupation of the $k \neq 0$ state, which does not result in a condensed state as $k \to 0$. Therefore, $V^{-1}N^0(\varepsilon_k - \mu)$ can be neglected in (A19). On the other hand,

$$N^0(-(1-\xi)\mu) = (1-\xi)^{-1}\langle b_0^\dagger b_0\rangle \; , \tag{A20}$$

hence,

$$V^{-1}[N^0(-(1-\xi)\mu) - N^0(\varepsilon_k - \mu)] = (1-\xi)^{-1}n_0 \quad (V \to \infty) \; . \tag{A21}$$

The substitution of (A21) in (A19) gives

$$\bar{\chi}_{SJ}^R(k,\omega) = \frac{\xi k(\mu - 2nU_0)n_0(\varepsilon_k - \xi\mu)}{\omega^2 - (\varepsilon_k - \xi\mu)^2} \; , \tag{A22}$$

which is identical to Eq. (5.4).

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