Transmission through a Fibonacci chain

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We study transmission $|t_N|$ and reflection $|r_N|$ of a plane wave (with wave number $k>0$) through a one-dimensional array of N δ-function potentials with equal strengths v located on the Fibonacci chain sequence $x_n=n+u[n/\tau]$, $n=1,2,..,N$ (where u is an irrational number, $\tau=(1+\sqrt{5})/2$, and $[\cdots]$ denotes the integer part thereof) in the limit $N\to\infty$. Using analytical and number-theoretical methods, we arrive at the following results. (i) For any k, if v is large enough, the sequence of reflection coefficients $|r_N|$ has a subsequence that tends to unity. (ii) If k is an integer multiple of π/u, then there is a threshold value v_0 for v such that, if $v\ge v_0$, then $|r_N|\to 1$ as $N\to\infty$, whereas if $v<v_0$, then $|r_N|\not\to 1$ (and moreover, $\overline{\lim}|r_N|<1$ and $\underline{\lim}|r_N|=0$). (iii) For other values of k, we present theoretical considerations indicating (though not proving) that $|r_N|$ has a subsequence converging to unity for any $v>0$. (iv) Numerical simulations seem to hint that if a subsequence converges to unity, this holds, in fact, for the whole sequence $|r_N|$. Consequently, for almost every k, $|r_N|\to 1$ as $N\to\infty$.

Experimental advances in submicrometer physics, which allow the fabrication of nearly ideal one-dimensional wires,[1] naturally leads to an increasing interest in their physical properties, especially those related to transport phenomena. The quantum-mechanical relation between electrical conductance at zero temperature and the transmission probability[2] indicates that some measurable physical quantities can be accurately explained on the microscopic level.

One-dimensional lattices of infinite extent are, of course, extensively studied in the literature in connection with Bloch theory[3] (if they are periodic), Anderson localization[4] (if they are completely disordered), quasicrystals[5] (if they are not periodic but maintain long-range order), commensurate-incommensurate structures,[6] etc. On the other hand, the theory of scattering from a semi-infinite one-dimensional array of potentials is much less familiar.[7,8] A useful technique is to express the transmission and reflection amplitudes through $N+1$ scatterers in terms of the amplitudes for N scatterers (a combination of Möbius transformation and multiplication by a phase) and to let $N\to\infty$ (the so-called thermodynamic limit). If the system of scatterers is arranged in an arithmetic progression (a perfectly ordered crystal) the limit can be easily obtained and a band structure of the transmission can be deduced, namely, the transmission as a function of the energy is zero on some segments and greater than zero on others. However, if the position of scatterers is completely random, a closed-form expression for the transmission cannot be found, but an ensemble average of the transmission over many samples can sometimes be carried out and the results show that the transmission decays exponentially with N (namely, with length) with some characteristic localization length.

The intermediate case, where the scatterers are located on an arbitrary sequence, is interesting in itself. In an earlier publication[9] we concentrated on scattering from an infinite system of δ-function potentials located on the Fibonacci numbers. Admittedly, this exercise was not related to any experimentally accessible system, but served as a theoretical model for the use of certain mathematical tools, basically analytical and number-theoretical techniques. Here we use the same mathematical framework and report the results of our study on scattering from a one-dimensional quasicrystal. Specifically, we consider one-dimensional quasicrystals which are characterized by a system of scatterers located on the sequence $x_n=n+u[n/\tau]$, $n=1,2,\ldots,N$, as $N\to\infty$, where u is an irrational number, $\tau=(1+\sqrt{5})/2$, and $[\cdots]$ denotes the integer part thereof. The special case $u=1/\tau$ (termed a Fibonacci chain) is of particular importance.

The central question which will be addressed in this study is whether a one-dimensional quasicrystal is a conductor ($|r_N\not\to 1$) or an insulator ($|r_N|\to 1$). To be more specific, is there a curve in the (v,k) plane that separates the conductor and insulator "phases?" The finer details pertaining to the behavior of $|r_N|$ as a function of N will not be discussed here. These and other questions have been investigated by several authors,[10] and the results presented in the abstract (and proved below) seem to corroborate their findings. In this respect, the present work should be regarded as a substantiation of earlier conjectures.

We believe, however, that the mathematical framework developed here is capable of solving the scattering problems encountered in other interesting one-dimensional arrays. It has in fact been recognized that in this kind of problem, mathematical rigor is essential. That is the reason the presentation below is mathematically oriented. Yet, the reader should be aware of the physical basis for the present study. For the sake of smooth reading, we relegate the proofs of all the statements to the Appendix.

Consider a one-dimensional array of N δ-function potentials

$$V_N(x)=v\sum_{n=1}^{N}\delta(x-x_n)\ , \tag{1}$$

where $v>0$ and x_n is given by

$$x_n=n+u[n/\tau],\quad n=1,2,\ldots,N\ . \tag{2a}$$

We will mainly employ the sequence of differences

$$y_n=x_{n+1}-x_n=1+u([(n+1)/\tau]-[n/\tau])\ . \tag{2b}$$

A plane wave with momentum k, e^{-ikx}, coming from the right will have reflection and transmission amplitudes r_N and t_N, respectively. For $N=1$, these amplitudes are given by

$$r=\frac{v}{2ik-v},\quad t=\frac{2ik}{2ik-v}\ , \tag{3}$$

which satisfy unitarity

$$|r|^2+|t|^2=1,\quad tr^*+t^*r=0 \tag{4a}$$

and continuity at the point $x=x_0$,

$$t=1+r\ . \tag{4b}$$

The unitarity relation (4a) is valid of course for any N. For $N>1$ scattering centers, the reflection and transmission amplitudes can be determined from a recursion relation as follows. We define

$$a_n=\frac{1}{t_n},\quad b_n=\frac{r_n}{t_n},\quad \underline{A}=\begin{bmatrix}\frac{1}{t} & -\frac{r}{t}\\ \frac{r}{t} & \frac{t^2-r^2}{t}\end{bmatrix}$$
$$\underline{\Lambda}_n=\begin{bmatrix}e^{-iky_n} & 0\\ 0 & e^{iky_n}\end{bmatrix},\quad \underline{D}_n=\underline{A}\,\underline{\Lambda}_n\ , \tag{5}$$

with

$$\det(\underline{A})=\det(\underline{\Lambda}_n)=\det(\underline{D}_n)=1\ . \tag{6}$$

Then we have

$$\begin{bmatrix}a_{n+1}\\ b_{n+1}\end{bmatrix}=\underline{A}\,\underline{\Lambda}_n\begin{bmatrix}a_n\\ b_n\end{bmatrix}\ . \tag{7}$$

The conductance (at zero temperature) of this system is given by the limit of $|t_N|^2=1/|a_N|^2$ as $N\to\infty$. Equivalently, we may inspect the limit of $|r_N|^2=|b_N/a_N|^2$ and use unitarity. If $|t_N|\to 0$ (equivalently $|r_N|\to 1$) we say that the system is an insulator. If $|t_N|$ does not tend to 0 the system may conduct. Our aim is to find out for what values of the momentum k and the strength v the system is an insulator or a conductor. It is also of interest to know when the sequence $|r_N|$ has a subsequence converging to 1 (or a subsequence converging to 0).

Now, we point out that in general (independently of the sequence x_n), the matrices by which we multiply [$\underline{A}$ and $\underline{\Lambda}_n$, see Eq. (5)], belong to the multiplicative group of 2×2 matrices of the form

$$\begin{bmatrix}\alpha & \beta\\ \beta^* & \alpha^*\end{bmatrix}$$

with $|\alpha|^2-|\beta|^2=1$. This group is SU(1,1), a noncompact group of real dimension 3, which preserves the form $|v_1|^2-|v_2|^2$, namely,

$$T\in \mathrm{SU}(1,1)\ ,$$
$$T\begin{bmatrix}v_1\\ v_2\end{bmatrix}=\begin{bmatrix}w_1\\ w_2\end{bmatrix}\Longrightarrow |w_1|^2-|w_2|^2=|v_1|^2-|v_2|^2\ . \tag{8}$$

A linear transformation T of $\mathbb{C}^2$ gives rise to a Möbius transformation T'; if the line with slope ρ_1 through the origin is mapped by T to the line with slope ρ_2, then $T'(\rho_1)=\rho_2$. From (8) it is obvious that, if $T\in\mathrm{SU}(1,1)$, then the interior of the unit circle, its circumference, and its exterior are all T' invariant. It is also evident that for every n,

$$|b_n|^2-|a_n|^2=|b_0|^2-|a_0|^2=\left|\frac{r}{t}\right|^2-\left|\frac{1}{t}\right|^2=-1\ . \tag{9}$$

Therefore, if $\lim_{N\to\infty}|a_N|=\infty$, then $\lim_{N\to\infty}|b_N|=\infty$, and therefore $\lim_{N\to\infty}|r_N|=1$. Conversely, if $|a_N|\not\to\infty$, then $|b_N|\not\to\infty$, and therefore $|r_N|\not\to 1$ as $N\to\infty$.

We denote by $\underline{M}_n$ the sequence of finite products of the matrices $\underline{D}_n=\underline{A}\,\underline{\Lambda}_n$ [Eq. (5)]. Thus

$$\underline{M}_n=\underline{A}\,\underline{\Lambda}_n\underline{A}\,\underline{\Lambda}_{n-1}\cdots\underline{A}\,\underline{\Lambda}_1\ . \tag{10a}$$

It is then evident that if $\{\underline{M}_n\}$ has a bounded subsequence $\underline{M}_{n_k}$ then the corresponding sequence of transformed vectors $\binom{a_{n_k}}{b_{n_k}}$ is bounded and hence $|r_N|\not\to 1$. On the other hand, if the sequence of matrices $\underline{M}_n$ does not have a bounded subsequence then this holds also for the sequence $\{\underline{M}_n\underline{A}\}$ which is a sequence in SU(1,1). Consequently, the sequence of vectors obtained by taking the first column in each matrix $\underline{M}_n\underline{A}$ does not have a bounded subsequence. This sequence of vectors is exactly the sequence $\binom{a_n}{b_n}$, therefore we arrive at the following.

Proposition 1: $\lim_{N\to\infty}|r_N|=1$ if and only if the sequence of matrices $\{\underline{M}_N\}$ does not have a bounded subsequence.

For further discussion it will be more convenient to start with the initial vector $\binom{1}{0}$ instead of $\binom{a_0}{b_0}$. It is easily verified that

$$\underline{A}\begin{bmatrix}1\\ 0\end{bmatrix}=\begin{bmatrix}\frac{1}{t} & -\frac{r}{t}\\ \frac{r}{t} & \frac{t^2-r^2}{t}\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix}=\begin{bmatrix}a_0\\ b_0\end{bmatrix}\ .$$

Therefore we can assume now that we start from the vector $\binom{1}{0}$ and not $\binom{a_0}{b_0}$, but multiply first with $\underline{A}$ and then with $\underline{\Delta}_n$. Regarding the sequence $\{\underline{M}_n\}$, these manipulations imply its replacement by $\{\underline{A}^{-1}\underline{M}_n\underline{A}\}$, but clearly, the existence of a bounded subsequence in either one of $\{\underline{M}_n\}$ or $\{\underline{A}^{-1}\underline{M}_n\underline{A}\}$ implies its existence in the other one. From now on we therefore redefine $\underline{M}_n$ as

$$\underline{M}_n=\underline{\Delta}_n\underline{A}\underline{\Delta}_{n-1}\cdots\underline{A}\underline{\Delta}_1\underline{A}\ . \tag{10b}$$

Our task is thus to find out under what conditions (on the strength of the interaction v and the momentum k) the sequence of matrices $\{\underline{M}_n\}$ defined in Eq. (10b) does not have a bounded subsequence. Define a norm on the ring of matrices of the form $\begin{pmatrix}\alpha & \beta\\ \beta^* & \alpha^*\end{pmatrix}$ by

$$\left\|\begin{bmatrix}\alpha & \beta\\ \beta^* & \alpha^*\end{bmatrix}\right\|=|\alpha|+|\beta|\ . \tag{11}$$

If we regard this ring as an algebra over $\mathbb{R}$, then the definition (11) is indeed a (submultiplicative) norm. A sequence of matrices is then bounded if and only if the corresponding sequence of norms is bounded.

We now return to our original problem for which x_n is the Fibonacci chain. First, there is a trivial case which can be easily handled as we demonstrate below.

The special case $k=m\pi/u$, m is a positive integer. In this case ky_n is either $m\pi/u$ or $m\pi/u+m\pi$. These two cases are in fact identical since as a quotient, the result will not be affected if the matrix $\underline{A}$ is multiplied by a scalar matrix ($-\underline{I}$ in this particular case). Denote $q=v/2k$ (note a sign change in the definition with respect to Ref. 9), $\mu=e^{ik}$, and using $r=t-1$ the matrix $\underline{D}_n=\underline{A}\underline{\Delta}_n$ can be replaced by the constant matrix

$$\underline{D}=\begin{bmatrix}\mu^* & 0\\ 0 & \mu\end{bmatrix}\begin{bmatrix}1+iq & iq\\ -iq & 1-iq\end{bmatrix}=\begin{bmatrix}\mu^*(1+iq) & i\mu^*q\\ -i\mu q & \mu(1-iq)\end{bmatrix}\ . \tag{12}$$

One shows easily (cf. Ref. 9) that $|r_N|\to 1$ if and only if $|\mathrm{tr}(\underline{D})|\geq 2$. Notice, however, that $\mathrm{tr}(\underline{D})=\mu^*+\mu+iq(\mu^*-\mu)=2(\cos k+q\sin k)$. Hence we obtain

$$|r_N|\to 1 \Longleftrightarrow q\geq q_0(\geq 0)\ ,$$

$$q_0=\begin{cases}\dfrac{1-\cos k}{\sin k}, & \sin k>0\\[2ex] -\dfrac{1+\cos k}{\sin k}, & \sin k<0\ .\end{cases} \tag{13a}$$

Equivalently,

$$|r_N|\to 1 \Longleftrightarrow v\geq v_0(\geq 0)\ ,$$

$$v_0=\begin{cases}2k\tan(k/2), & \sin k>0\\ -2k\cot(k/2), & \sin k<0\ .\end{cases} \tag{13b}$$

It is worth mentioning here that if $|r_N|\to 1$ then the sequence r_N itself converges, while if the sequence $|r_N|$ does not tend to unity then the set of accumulation points of the sequence r_N is located on a circle which passes through the origin. In fact, denoting by s_1 and s_2 the eigenvalues of $\underline{D}$ one easily shows that

$$r_N=-iq\mu\frac{(s_1/s_2)^N-1}{[(1+iq)\mu^*-s_2](s_1/s_2)^N+[s_1-(1+iq)\mu^*]}\ . \tag{14}$$

In the case where r_N does not converge, s_1 and s_2 lie on the unit circle and are complex conjugates. The sequence r_N lies on the image of the unit circle under the Möbius transformation

$$z\to -iq\mu\frac{z-1}{[(1+iq)\mu^*-s_2]z+[s_1-(1+iq)\mu^*]}\ . \tag{15}$$

This image is itself a circle passing through $z=0$. Denote this circle by e. If (s_1/s_2) is a root of unity (which happens if and only if both s_1 and s_2 are such), then r_N is a periodic sequence. If (s_1/s_2) is not a root of unity, then the sequence r_N is dense in e. We can now use known results in complex-function theory[11] and check that

$$\max\{|w|:\ w\in \mathrm{e}\}=\frac{q}{|\sin k-q\cos k|}\ . \tag{16}$$

Hence, for $v<v_0$ we have

$$\limsup_{N\to\infty}|r_N|\leq\frac{q}{|\sin k-q\cos k|}\ , \tag{17}$$

where this is actually an equality unless s_1/s_2 is a root of unity of odd order. On the other hand,

$$\liminf_{N\to\infty}|r_N|=0 \tag{18}$$

holds whether or not (s_1/s_2) is a root of unity.

The case of general k. Now we turn to the case of general $k>0$. The results will be true for any k, but some of the considerations will apply only to k's which are not multiples of π/u. The basic ideas on which the following discussion are based are related to the trace map formalism suggested in the work of Sutherland and Kohmoto.[10] Beside some technical differences (such as working with complex transfer matrices in the present work), there are some more profound differences. First, the stress here is not on the growth of the resistance with length, but rather, the behavior of the resistance in the (v,k) plane as the length of the system tends to infinity. Second, the mathematical apparatuses here are different, and rely strongly on concepts from ergodic theory. This allows us to treat all orbits [see discussion following Eq. (32) below] and not only the periodic ones.

Lemma 1. (See also Ref. 10, third equation in Sec. III.) Denote $\underline{A}_n=\underline{M}_{F_n}$, where $\{F_n\}_{n=1}^{\infty}$ is the Fibonacci sequence with $F_1=1$ and $F_2=2$. Then

$$\underline{A}_{n+2}=\underline{A}_n\underline{A}_{n+1}\quad\text{for } n\geq 1\ . \tag{19}$$

Lemma 2. (See also Ref. 10, fourth equation in Sec. III.) Let $\{\underline{A}_n\}_{n=1}^{\infty}$ be a sequence of matrices in SU(1,1) satisfying (19) and let $\chi_n=\mathrm{tr}(\underline{A}_n)$ denote the trace of $\underline{A}_n$. Then

$$\chi_{n+3}=\chi_{n+1}\chi_{n+2}-\chi_n \ . \tag{20}$$

From the lemma it is clear that we should study the solutions of the recurrence equation (20). One family of solutions is obtained by starting with two commuting 2×2 matrices $\underline{A}_1$ and $\underline{A}_2$ of determinant 1, defining a sequence $\{\underline{A}_n\}_{n=1}^{\infty}$ by (19), and taking the sequence $\{\mathrm{tr}(\underline{A}_n)\}_{n=1}^{\infty}$. We then easily get that, for any nonzero complex numbers a and b, the sequence $\{\chi_n\}_{n=1}^{\infty}$ given by

$$\chi_n=a^{F_{n+1}}b^{F_n}+a^{-F_{n+1}}b^{-F_n} \quad \text{for } n\geq 1 \tag{21}$$

forms a solution of (20). This sequence consists of real numbers in two cases: (i) Both a and b are real and (ii) a and b lie on the unit circle. In general, to study solutions of (20) it is natural to consider the mapping H: $\mathbb{R}^3\to\mathbb{R}^3$ defined by

$$H(x,y,z)=(y,z,yz-x), \quad (x,y,z)\in\mathbb{R}^3 \ . \tag{22}$$

The following lemma will be very useful for the study of the transformation H.

Lemma 3. The function f: $\mathbb{R}^3\to\mathbb{R}$ defined by

$$f(x,y,z)=xyz-x^2-y^2-z^2 \tag{23}$$

is H invariant. In other words, every level surface of the function f forms an H-invariant set.

Lemma 4. The solutions of (20) given in (21) lie on L_{-4}, the (-4)-level surface of f. Solving for z in terms of x and y we find that

$$L_{-4}=\{(x,y,z):\ z=\tfrac{1}{2}xy\pm\tfrac{1}{2}[(4-x^2)(4-y^2)]^{1/2}\} \ . \tag{24}$$

It is thus clear that the projection of L_{-4} on the (x,y) plane is $[-2,2]^2\cup(\mathbb{R}\setminus(-2,2))^2$. All points of this projection have two points of L_{-4} lying above them, except for the points on the boundary, namely, those on either of the four lines $x=\pm 2$, $y=\pm 2$, having only one point of L_{-4} above each. We may then write

$$L_{-4}=S\cup P_{1,1}\cup P_{1,-1}\cup P_{-1,1}\cup P_{-1,-1} \ , \tag{25}$$

where S is the portion of L_{-4} lying above the square $[-2,2]^2$, $P_{1,1}$ is the portion lying above $[2,\infty]^2$, $P_{1,-1}$ is the portion lying above $[2,\infty]\times[-\infty,-2]$, etc.

Lemma 5. (i) The portion of L_{-4} lying above the square $[-2,2]^2$ is given by

$$S=\{(2\cos\alpha,2\cos\beta,2\cos(\alpha+\beta)):\ \alpha,\beta\in\mathbb{R}\} \ . \tag{26}$$

(ii) The domains $P_{\sigma\rho}$ $(\sigma,\rho=\pm 1)$ are given by

$$P_{\sigma\rho}=\{(2\sigma\cosh\alpha,2\rho\cosh\beta,2\sigma\rho\cosh(\alpha+\beta)):\ \alpha,\beta\in\mathbb{R}\} \ . \tag{27}$$

From the foregoing description it is clear that S is topologically homeomorphic to a two-dimensional sphere. The importance of this set for our purposes stems from the following.

Lemma 6. S is both H and H^{-1} invariant. The complement of S in $\mathbb{R}^3$ consists of two connected components, a compact one and a noncompact one. The former, along with its boundary S, forms a three-dimensional body

$$B=\{(x,y,z):\ -2\leq x,y\leq 2,\tfrac{1}{2}xy-\tfrac{1}{2}[(4-x^2)(4-y^2)]^{1/2}\leq z\leq\tfrac{1}{2}xy+\tfrac{1}{2}[(4-x^2)(4-y^2)]^{1/2}\} \tag{28}$$

which is topologically a ball. From Lemma 6 we immediately obtain Lemma 7.

Lemma 7. The set B is H invariant.

Remark. It seems interesting, from an ergodic-theoretical point of view, to analyze the action of H on B. (The notions and results which we mention subsequently can be found in any standard book on ergodic theory.) Evidently, by Lemma 3, the intersection with B of each level surface of f is H invariant. What does the action of H on any level look like? On $S=L_{-4}\cap B$, this question is easy to answer: Denote by $\mathsf{T}=\mathbb{R}/2\pi\mathbb{Z}$ the circle group. T^2 is the two-dimensional torus, and Lemma 5(i) provides a mapping R: $\mathsf{T}^2\to S$ given by

$$R(\alpha,\beta)=(2\cos\alpha,2\cos\beta,2\cos(\alpha+\beta)) \ . \tag{29}$$

This mapping is onto, and is two-to-one except for four points on S having a single inverse image each. Now consider the following map σ: $\mathsf{T}^2\to\mathsf{T}^2$:

$$\sigma(\alpha,\beta)=(\beta,\alpha+\beta), \quad (\alpha,\beta)\in\mathsf{T}^2 \ . \tag{30}$$

This map forms a continuous algebraic automorphism and, since the matrix $\left(\begin{smallmatrix}0 & 1\\ 1 & 1\end{smallmatrix}\right)$ corresponding to it has no roots of unity among its eigenvalues, it is strongly mixing. Now, the diagram

$$\begin{tikzcd} \mathsf{T}^2 \arrow[r, "\sigma"] \arrow[d, "R"'] & \mathsf{T}^2 \arrow[d, "R"'] \\ S \arrow[r, "H"] & S \end{tikzcd} \tag{31}$$

commutes, so that the flow (S,H) is a factor of the flow (T^2,T), whence it is also strongly mixing and has a dense set of H-periodic points. It seems interesting, though not for the purposes of the current paper, to investigate the flow determined by the restriction of H to other level curves of f in B.

It may seem at first sight that the points in B are the only ones having bounded H orbits in $\mathbb{R}^3$. This is, however, not the case. In fact, there are at least several one-dimensional curves the points on which share the same property. First, on $P_{1,1}$ consider the set

$$J=\{(\cosh\alpha,\cosh(-\tau\alpha),\cosh(\alpha-\tau\alpha)):\ \alpha\in\mathbb{R}\} \ . \tag{32}$$

It is readily verified that, since $F_{n+1}-\tau F_n \to 0$ as $n\to\infty$ the H orbit of each point of J converges to the fixed point (2,2,2). Analogous curves are straightforwardly constructed on the other points $P_{\sigma\rho}$. Also, all points on the coordinate axis are periodic with period 6, except for the origin, which is a fixed point. (Note that the curve J as well as the similar curves on the other $P_{\sigma\rho}$'s are contained in L_{-4}, but the points on the coordinate axis represent level curves of any nonpositive level of f.)

We now introduce some notations

$$\mu=e^{ik},\quad \nu=e^{ik(1+u)},\quad \underline{C}=\begin{bmatrix}1+iq & iq\\ -iq & 1-iq\end{bmatrix},$$
$$\underline{A}=\begin{bmatrix}\mu^* & 0\\ 0 & \mu\end{bmatrix}\underline{C},\quad \underline{D}=\begin{bmatrix}\nu^* & 0\\ 0 & \nu\end{bmatrix}\underline{C}\ . \tag{33}$$

Then it is easy to see that

$$\underline{A}_1=\underline{M}_1=\underline{D},\quad \underline{A}_2=\underline{M}_2=\underline{A}\underline{D}\ ,$$
$$\underline{A}_3=\underline{D}\ \underline{A}\ \underline{D},\quad \underline{A}_4=\underline{A}\ \underline{D}\ \underline{D}\ \underline{A}\ \underline{D},\ldots\ . \tag{34}$$

The recursion relation (19) for the matrices $\underline{A}_n$ will not be affected if we define

$$\underline{A}_0=\underline{A},\quad \underline{A}_{-1}=\begin{bmatrix}e^{-iku} & 0\\ 0 & e^{iku}\end{bmatrix}. \tag{35}$$

Then with the definition $l=k(1+u)$ we computed the traces

$$\mathrm{tr}(\underline{A}_{-1})=2\cos(ku),\quad \mathrm{tr}(A_0)=2(\cos k+q\sin k)\ ,$$
$$\mathrm{tr}(A_1)=2(\cos l+q\sin l)\ . \tag{36}$$

Consider now the point

$$\zeta=(\mathrm{tr}(\underline{A}_{-1}),\mathrm{tr}(\underline{A}_0),\mathrm{tr}(\underline{A}_1))\in\mathbb{R}^3\ . \tag{37}$$

Lemma 8. If k is not an integer multiple of π/u, then the point $\zeta=\zeta(q)$ does not belong to B for any $q>0$.

Remark. It is easily verified that ζ also cannot lie on J or any of the other curves consisting of points with bounded H orbits described after Eq. (32).

The above results suggest that, if k is not a multiple of π/u, then the H orbit of ζ escapes to infinity (in each coordinate) for any potential strength v. We have only been able to verify this statement for sufficiently large v.

Lemma 9. For every large enough v we have $\lim_{n\to\infty}|\mathrm{tr}(\underline{A}_n)|=\infty$, and in particular

$$\lim_{n\to\infty}\|\underline{A}_n\|=\infty\ . \tag{38}$$

As a consequence we now arrive at our main result.

Theorem 1. For every large enough v the sequence $|r_{F_n}|$ tends to unity as $n\to\infty$. In other words, the sequence $|r_N|$ has at least one subsequence which, for large enough v, tends to unity.

The obvious question that now arises is, of course, what happens to the whole sequence and not only to $|r_{F_n}|$? Here we could not arrive at rigorous results, and had to rely on numerical simulations. Fortunately, the iteration procedure implied by Eq. (7) is simple enough and free from round-off errors even after 10^6 steps. All our numerical simulations indicate that the convergence property of the subsequence $|r_{F_n}|$ is shared also by the whole sequence $|r_N|$. In some cases when v is small, the convergence is extremely slow, and does not show up after many terms (although we *know* that $|r_{F_n}|$ should converge to unity). Nevertheless, for any finite value of N up to 10^6, the subsequence $|r_{F_n}|$ behaves exactly as the whole sequence $|r_N|$. The physical implication of our results is simply this: For any value of the interaction strength v, and for almost every k, the one-dimensional Fibonacci chain is an insulator in the sense that $|r_N|\to 1$ as $N\to\infty$. Only for the discrete sequence $k=m\pi/u$ is there a transition from a conductor to an insulator (in the sense explained above) at finite value of v.

While the relation between this work and some earlier works (notably that of Sutherland and Kohmoto[10]) has already been explained, we should remind the reader that in this paper we have restricted the discussion to a set of δ-function potentials. Many aspects of the spectral and transmission properties of this type of one-dimensional quasiperiodic structures are independent on the nature of the scatterers and depend on the way the scatterers are sequenced. In principle, this holds also in the present case, but since the main theorems are stated in terms of $q=v/2k$ which is special for the δ-function case, an appropriate quantity (apparently $|r/t|$) should be suggested in the general case. Notice, however, that the statement about the tendency of $|r_N|$ to 1 for almost every k remains valid also in the general case.

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APPENDIX

Proof of Lemma 1. As is well known, the continued-fraction expansion of $1/\tau$ is $[0;1,1,\ldots]$. Therefore the sequence of convergents to $1/\tau$ is $(F_{n-1}/F_n)_{n=1}^{\infty}$, where $(F_n)_{n=0}^{\infty}$ is the Fibonacci sequence with $F_0=F_1=1$. Denote by $\{x\}$ the fractional part of a real number x and by $\|x\|$ its distance from the nearest integer. We now claim that if $1\le j\le F_n$ then

$$\left[\frac{F_{n+1}+j+1}{\tau}\right]-\left[\frac{F_{n+1}+j}{\tau}\right]=\left[\frac{j+1}{\tau}\right]-\left[\frac{j}{\tau}\right], \tag{A1}$$

or, equivalently

$$\left\{\frac{F_{n+1}+j+1}{\tau}\right\}-\left\{\frac{F_{n+1}+j}{\tau}\right\}=\left\{\frac{j+1}{\tau}\right\}-\left\{\frac{j}{\tau}\right\}. \tag{A2}$$

The sequence $(\{F_{n-1}/\tau\})_{n=1}^{\infty}$ tends to zero (mod 1). Therefore, if the equality (A2) does not hold, then one of the two numbers $\|(j+1)/\tau\|$, $\|j/\tau\|$ is very close to zero (mod 1). More precisely, its distance from zero (mod 1) is smaller than $\|F_{n+1}/\tau\|$. But the sequence of convergents to $1/\tau$ is $(F_{n-1}/F_n)_{n=1}^{\infty}$, and hence each $\|F_n/\tau\|$ is smaller than all the numbers $\|j/\tau\|$with $j<F_n$, which brings about a contradiction. Now let

$$s_n=\left[\frac{n+1}{\tau}\right]-\left[\frac{n}{\tau}\right] , \tag{A3}$$

and for $n\le m$ let $s_{n,m}$ be the finite sequence s_n, $s_{n+1},\ldots,s_m$. Then

$$s_{1,F_{n+2}}=s_{1,F_{n+1}}s_{1,F_n} \quad \text{for } n\ge 1 , \tag{A4}$$

where on the right-hand side of Eq. (A4) we have a concatenation of sequences. If $(\underline{M}_n)_{n=1}^{\infty}$ is the pertinent sequence of matrices then from (A4) we get immediately

$$\underline{M}_{F_{n+2}}=\underline{M}_{F_n}\underline{M}_{F_{n+1}} \quad \text{for } n\ge 1 \tag{A5}$$

[note that the order on the right-hand side of Eq. (A5) is different from that in Eq. (A4) since the additional matrices to the product at any stage are multiplied from the left]. This proves the lemma.

Proof of Lemma 2. We write

$$\begin{aligned}\chi_{n+3}&=\mathrm{tr}(\underline{A}_{n+3})=\mathrm{tr}(\underline{A}_{n+1}\underline{A}_{n+2})=\mathrm{tr}(\underline{A}_{n+1}\underline{A}_n\underline{A}_{n+1})-\mathrm{tr}(\underline{A}_n\underline{A}_{n+1}^2)\\&=\mathrm{tr}[\underline{A}_n(\chi_{n+1}\underline{A}_{n+1}-I)=\chi_{n+1}\mathrm{tr}(\underline{A}_n\underline{A}_{n+1})-\mathrm{tr}(\underline{A}_n)=\chi_{n+1}\mathrm{tr}(\underline{A}_{n+2})-\chi_n=\chi_{n+1}\chi_{n+2}-\chi_n ,\end{aligned} \tag{A6}$$

which proves the lemma.

Proof of Lemma 3. We write

$$\begin{aligned}(f\cdot H)(x,y,z)&=f(y,z,yz-x)=yz(yz-x)-y^2-z^2-(yz-x)^2\\&=y^2z^2-xyz-y^2-z^2-y^2z^2+2xyz-x^2=xyz-x^2-y^2-{}^2=f(x,y,z) ,\end{aligned} \tag{A7}$$

which proves the lemma.

Proof of Lemma 4. The three initial terms in any sequence given by (16) are z_1+1/z_1, z_2+1/z_2, and z_1z_2+1/z_1z_2. Now

$$\begin{aligned}&f(z_1+1/z_1,z_2+1/z_2,z_1z_2+1/z_1z_2)\\&\quad=(z_1+1/z_1)(z_2+1/z_2)(z_1z_2+1/z_1z_2)-(z_1+1/z_1)^2-(z_2+1/z_2)^2-(z_1z_2+1/z_1z_2)^2\\&\quad=z_1^2z_2^2+1+z_1^2+1/z_2^2+z_2^2+1/z_1^2+1+1/z_1^2z_2^2-z_1^2-2-1/z_1^2-z_2^2-2-1/z_2^2-z_1^2z_2^2-2-1/z_1^2z_2^2=-4 ,\end{aligned} \tag{A8}$$

which proves the lemma.

Proof of Lemma 5. Let us prove, say, (i). Denote the set on the left-hand side of Eq. (26) by S'. Employing trigonometric identities we get

$$f(2\cos\alpha,2\cos\beta,2\cos(\alpha+\beta))=2\cos\alpha 2\cos\beta 2\cos(\alpha+\beta)-(2\cos\alpha)^2-(2\cos\beta)^2-[2\cos(\alpha+\beta)]^2=-4 , \tag{A9}$$

whence $(2\cos\alpha,2\cos\beta,2\cos(\alpha+\beta))\in L_{-4}$ for any $\alpha,\beta\in\mathbb{R}$. Hence $S\supseteq S'$. Now suppose $(x,y,z)\in S$. Since $|x|,|y|\le 2$ we may write $x=2\cos\alpha$, and $y=2\cos\beta$ for suitable α,β. Suppose, say, that $z=\frac{1}{2}xy+\frac{1}{2}[(4-x^2)(4-y^2)]^{1/2}$. Replacing α by $-\alpha$ if necessary we may assume that $\sin\alpha>0$, and similarly we may assume that $\sin\beta<0$. Then

$$\begin{aligned}z&=\tfrac{1}{2}2\cos\alpha\, 2\cos\beta+\tfrac{1}{2}[(4-4\cos^2\alpha)(4-4\cos^2\beta)]^{1/2}\\&=2\cos(\alpha+\beta) ,\end{aligned} \tag{A10}$$

which yields the first part. The other part is similarly proved.

Proof of Lemma 6. For any $\alpha,\beta\in\mathbb{R}$ we readily compute

$$\begin{aligned}&H(2\cos\alpha,2\cos\beta,2\cos(\alpha+\beta))\\&\qquad=(2\cos\beta,2\cos(\alpha+\beta),2\cos(\alpha+2\beta)) ,\\&H^{-1}(2\cos\alpha,2\cos\beta,2\cos(\alpha+\beta))\\&\qquad=(2\cos(\beta-\alpha),2\cos\alpha,2\cos\beta) ,\end{aligned} \tag{A11}$$

which proves the lemma.

Proof of Lemma 8. Assume, to the contrary, that

$\zeta \in B$. Then

$$|2(\cos l - q \sin l) - \tfrac{1}{2} 2 \cos(ku)\, 2(\cos k - q \sin k)| \le \tfrac{1}{2}\{[4 - 4\cos^2(ku)][4 - 4(\cos k - q \sin k)^2]\}^{1/2} , \tag{A12}$$

and therefore

$$|\sin(ku)\sin k + q \sin(ku)\cos k| \le |\sin(ku)|[1 - (\cos k - q \sin k)^2]^{1/2} . \tag{A13}$$

Dividing by $|\sin(ku)|$, which may be done since k is not an integer multiple of π/u, and squaring we obtain

$$(\sin k + q \cos k)^2 \le 1 - (\cos k - q \sin k)^2 . \tag{A14}$$

Further multiplication yields $1 + q^2 \le 1$, which is a contradiction. This proves the lemma.

Proof of Lemma 9. Set $\chi_n = \mathrm{tr}\underline{A}_n$, $n = -1, 0, 1, \ldots$. By Lemmas 1 and 2, $\chi_{n+3} = \chi_{n+1}\chi_{n+2} - \chi_n$. Assume first that $\sin k \neq 0$ and $\sin l \neq 0$. Then, clearly, if v is large enough, namely, the parameter $q = v/2k$ is large, then $|\chi_0|, |\chi_1| > 2$, while $|\chi_{-1}| \le 2$. Now we need the following.

Lemma 9(a). Let a, b, and c be three real numbers satisfying $|c| \ge |a|$ and $|b| \ge 2$. Then

$$|bc - a| \ge |c| . \tag{A15}$$

Proof.

$$\begin{aligned} |bc - a| &\ge |b|\,|c| - |c| \\ &\ge |b|\,|c| - |a| = |c|(|b| - 1) \ge |c| . \end{aligned} \tag{A16}$$

This proves the lemma. Now we return to the proof of Lemma 9. From Lemma 9(a) we have

$$|\chi_2| \ge \max\{|\chi_0|, |\chi_1|\} , \tag{A17}$$

$$|\chi_3| \ge |\chi_2\chi_1 - \chi_0| > |\chi_2| . \tag{A18}$$

By induction we conclude that the sequence $(|\chi_n|)_{n=1}^{\infty}$ is increasing. Assume this sequence has a finite limit $\chi < \infty$. Passing to the limit in the recursion relation $\chi_{n+3} = \chi_{n+1}\chi_{n+2} - \chi_n$ we find

$$\pm\chi = \pm\chi^2 \pm \chi , \tag{A19}$$

with possible solutions 0 and ± 2. However, evidently $|\chi| > 2$, a contradiction.

Let us turn to the other cases. If $\sin k = \sin l = 0$, then $\sin(ku) = \sin(l - k) = 0$, which is impossible. Next, we consider the case $\sin k = 0$ but $\sin l \neq 0$. Then using Eq. (31) for the first few traces, and the trace recursion relation (20) we find

$$\chi_2 = \pm 4(\cos l - q \sin l) - 2\cos(ku) , \tag{A20}$$

and using arguments similar to those employed in the first case we get again $\lim_{n\to\infty}|\chi_n| = \infty$ for sufficiently large q. Finally, we consider the case $\sin k \neq 0$, $\sin l = 0$. Then

$$\begin{aligned} \chi_2 &= \pm 4(\cos k - q \sin k) - 2\cos(ku) , \\ \chi_3 &= \pm 8(\cos k - q \sin k) - 4\cos(ku) \\ &\quad - 2(\cos k - q \sin k) , \end{aligned} \tag{A21}$$

which again lead to $\lim_{n\to\infty}|\chi_n| = \infty$ for sufficiently large q. Hence Lemma 9 is proved.

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