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Resonances and oscillations in tunneling in a time-dependent potential

M. Ya Azbel

Raymond and Beverly Sackler Faculty of Exact Sciences, School of Physics and Astronomy, Tel Aviv University, Tel Aviv 69978, Israel

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A quantum particle, tunneling in a stationary potential, passes through a region with a weak slowly changing potential. Its probability density has resonances and oscillations in energy and in time, both in the barrier and beyond it. This is true in any dimensionality, for any type of waves (for example, de Broglie, electromagnetic, acoustic, and hydrodynamic waves).

One of the main manifestations of quantum mechanics is tunneling. Tunneling is involved in the decay of heavy nuclei, field emission from an atom or a solid surface, chemical reactions, paraelectric defects in a solid, metal-insulator-metal, and Josephson junctions, *p-n* diodes, superconducting quantum interference device (SQUID) rings, transport in superlattices, quantum diffusion, absorption, and desorption. Quantum transport (e.g., variable-range-hopping conductivity), as well as wave propagation beyond the geometrical-optics region, is related to tunneling. In all these cases, nonstationary and relaxation phenomena reduce to the problem of solving for the tunneling wave function in a weak time-dependent potential. Its study may also be useful in stationary many-body tunneling, if the latter is reduced to an approximately single-particle problem. Then some of the degrees of freedom adjust to the progress in tunneling and yield an effective time-dependent potential.

Tunneling in a time-dependent potential W was extensively studied.[1-5] However, in this paper W is neither harmonic, nor is a linear approximation used for W. The results prove to be much richer and more interesting than previously anticipated. Consider a particle tunneling in a stationary barrier V which passes through a weak, adiabatic potential $W(t)$. Suppose $W(t)$ monotonically increases from $W(-\infty)=0$ to $W(0)$ and then monotonically decreases back to $W=0$. It will be shown that the particle probability density in the barrier and beyond it oscillates with time and energy. It has maxima long before W has. At certain times and energies it has resonances. This is also true for any waves in a region which is forbidden in geometrical optics. The easiest experimental verification of the oscillations and resonances might be waves in a variable-depth liquid.[6] Suppose an external source generates stationary waves, whose wavelengths do not allow them, according to geometrical optics, to penetrate into a shallow region. Then a very slow and weak local perturbation will lead to oscillations in, and huge amplification of their amplitude at certain times and places.

In this paper I accurately solve the Schrödinger equation for the potential $W(t)=V(x)+\delta(x)F(t)$ [those not interested in mathematics may proceed directly from Eqs. (1)-(3) to Eqs. (8) and (9)], and study its approximations, generalizations, and implications.

Consider a particle tunneling in a potential $V(x)$, which is perturbed by the potential $\delta(x)F(t)$. The Schrödinger equation reads

$$i\dot{\psi}=-\psi''+V\psi+F(t)\delta(x)\psi\,. \tag{1}$$

(The particle mass equals 0.5, and $\hbar=1$.) It is convenient to study the initial conditions at $t=-\infty$ which allows an exact solution. Suppose

$$\lim_{t\to-\infty}[\psi(t,x)\exp(i\Omega t)]=\psi_\Omega^+(x) \tag{2}$$

and

$$\psi_\Omega^{+\prime\prime}+(\Omega-V)\psi_\Omega^+=0;\quad F(-\infty)=\dot{F}(-\infty)=\cdots=0\,. \tag{3}$$

A superscript "+" refers to a stationary wave incoming from $x=-\infty$ (with no reflected waves at $x=+\infty$). Similarly, later on, the superscript "−" refers to a wave incoming from $x=+\infty$. I assume that $V(x)<\Omega$ when $x\geq -x_0$ $(x_0>0)$, but the solution is readily generalized to an arbitrary $V(x)$ (explained later in paper).

The exact solution to Eq. (1) together with the initial conditions Eqs. (2) and (3) is

$$\psi(t,x)=\psi_\Omega^+(x)\exp(-i\Omega t)-\int q_\omega^{-1}H_\omega(x)\exp(-i\omega t)d\omega\int F_{\omega-\omega'}^F a_{\omega'}^F d\omega'\,, \tag{4}$$

$$H_\omega(x)=\psi_\omega^s(x)/\psi_\omega^s(0);\quad q_\omega=[\ln(\psi_\omega^-/\psi_\omega^+)]'_{x=0};\quad s=\text{sign}(x)\,, \tag{5}$$

where F_ω^F is the Fourier component of $F(t)$ and a two-variables dependent $\psi(t,x)$ reduces to a one-variable dependent a_ω^F, which satisfies the equation

$$a_\omega^F+q_\omega^{-1}\int F_{\omega-\omega'}^F a_{\omega'}^F d\omega'=\psi_\Omega^+(0)\delta(\omega-\Omega)\,. \tag{6}$$

[Note that, by Eqs. (4)-(6), $a(t)=\psi(t,0)$]. Indeed, $\psi(t,x)$ satisfies Eq. (1) at $x\neq 0$, since $\psi_\omega^\pm(x)\exp(-i\omega t)$ does. By Eq. (5), $H_\omega(0)=1$, $\delta H'_\omega(0)=-q_\omega$. So, $\delta\psi(t,0)\equiv\psi(t,+0)-\psi(t,0)=0$ and [in virtue of Eq. (6)] $\delta\psi'(t,0)=F(t)\psi(t,0)$ take care of the conditions at $x=0$, imposed by Eq. (1). Finally, Eq. (3) is satisfied, since $\int\phi(\omega+\omega')F_\omega^F\exp(-i\omega t)d\omega=\phi(\omega'+id/dt)F(t)$, $\phi(\omega)=H_\omega/q_\omega$, and, by Eq. (2) $F(-\infty)=\dot{F}(-\infty)=\cdots=0$. Consider an adiabatic $F(t)$, when the energy change is small and $q_\omega \simeq q_\Omega$ in Eqs. (4) and (6). Then the Fourier transformation solves Eq. (6), and Eq. (4) yields an explicit analytical formula for ψ,

$$\psi(t,x)=\exp(-i\Omega t)\left[\psi_\Omega^+(x)-\psi_\Omega^+(0)\int H_{\Omega+\omega}(x)\exp(-i\omega t)[F/(q_\Omega+F)]_\omega^F d\omega\right]. \quad (7)$$

Equation (7) satisfies Eq. (1) for $x\neq 0$, Eq. (2), and $\delta\psi(t,0)=0$. It also satisfies $\delta\psi'(t,0)=F(t)\psi(t,0)$ to order $O(1/\Omega T)$, where T is the characteristic time of $F(t)$.

In an opaque barrier, the WKB approximation for a small energy change yields $H_{\Omega+\omega}(x)/H_\Omega(x)\simeq\exp(s\omega\times t_{\rm BL})$, and thus, by Eq. (7), at $x>0$

$$\psi(t,x)=\exp(-i\Omega t)\psi_\Omega^+(x)[1+F(t+it_{\rm BL})/v_0]^{-1}, \quad (8)$$

where $t_{\rm BL}=0.5\int_0^x(V-\Omega)^{-1/2}dx$ is the Büttiker-Landauer traversal time,[5] $q_\Omega\simeq v_0\equiv v(0)$, $v(x)\equiv 2|\Omega-V(x)|^{1/2}$. Above the barrier ($\Omega>V$) the WKB calculation leads at $x>0$ to $H_{\Omega+\omega}/H_\Omega\simeq\exp(-i\omega t_x)$, and

$$\psi(t,x)=\exp(-i\Omega t)\psi_\Omega^+(x)[1-iF(t-t_x)/v_0]^{-1}. \quad (9)$$

Here $t_x=\int_0^x dx/v(x)$ is the classical retardation time.

Consider the physical implications of Eqs. (8) and (9) when $F(-t)=F(t)$, for the case when $F(t)$ monotonically changes from $F(-\infty)=0$ to $F(0)=F_0$. By Eq. (9), the relative change $\Delta(t,x)$ in the probability density above the barrier equals

$$\Delta=|\psi(t,x)/\psi(-\infty,x)|^2-1 = -[1+v_0^2/F^2(t-t_x)]^{-1}. \quad (10)$$

In this case Δ is related to the classical retardation t_x in the instantaneous value of F. It is always negative. It monotonically decreases from $\Delta(-\infty,x)=0$ to the minimal $\Delta(t_x,x)=-(1+v_0^2/F_0^2)^{-1}$ and then monotonically increases back to $\Delta(\infty,x)=0$. The tunneling case, by Eq. (8), is the analytical continuation of the real-velocity case (9), with t_x being replaced by the imaginary retardation time[7] $(-it_{\rm BL})$:

$$1+\Delta=|\psi(t,x)/\psi(-\infty,x)|^2 = |1+F(t+it_{\rm BL})/v_0|^{-2}. \quad (11)$$

The potential barrier enters only via the absolute local value of the velocity v_0 and via the variable $t_{\rm BL}$, which depends on x (and Ω). When

$$F(t+it_{\rm BL})=-v_0, \quad (12)$$

the relative change manifests giant resonances as then $\Delta=\infty$. For every Ω, Eq. (12) provides quantized values of a complex $(t+it_{\rm BL})$, i.e., of real t and $t_{\rm BL}$. So, at every x, quantized energies yield local resonances at quantized times. In the stationary case Eq. (12) corresponds to a true eigenstate when $F=\text{const}<0$. For a slowly changing $F(t)$, resonances occur for any value and sign of $F(t)$. By Eq. (11), $\Delta(-t)=\Delta(t)$, i.e., Δ is even in t rather than in any retarded time. Assume $F(t)=F_0\exp(-t^2/T^2)$. Then

$$1+\Delta=[1+A^2+2A\cos(2t_{\rm BL}t/T^2)]^{-1}, \quad A=g_0\exp[(t_{\rm BL}^2-t^2)/T^2], \quad (13)$$

where $g_0=F_0/v_0$. Consider the response Δ for a fixed $t_{\rm BL}/T$ as a function of time t/T. Suppose $0<g_0\ll 1$. Start with $x=0$, i.e., $t_{\rm BL}=0$. Then Δ monotonically decreases from $\Delta(-\infty,0)=0$ to $\Delta_{\min}=\Delta(0,0)\simeq -2g_0$ and then monotonically increases to $\Delta(\infty,0)=0$. As one would expect for $x=0$, $\Delta=\Delta_{\min}$ simultaneously with $F=F_{\max}$, i.e., with the maximum of the potential energy at $x=0$. Now consider a nonresonant $t_{\rm BL}/T\neq 0$. Then $\Delta(-t)=\Delta(t)$ oscillates with t/T. In particular, it always has a minimum at $t=0$, at the same time as $|F(t)|$ is at a maximum. Giant resonances occur whenever

$$t/T=\pm\{0.5[(l^2+Q_n^2)^{1/2}+l]\}^{1/2}, \quad t_{\rm BL}/T=\{0.5[(l^2+Q_n^2)^{1/2}-l]\}^{1/2}, \quad (14)$$

where $l=\ln|1/g_0|$, $Q_n=(2n+1)\pi$, and n is an integer. (Note that $t_{\rm BL}\propto T$, and thus x must be sufficiently large. When $g_0<0$, then $Q_n=2n\pi$). The half-width of the resonances exponentially decreases with n. They may occur at very early time $t/T\sim-(n\pi)^{1/2}$, $n\gg 1$, when the original impulse is still exponentially small: $F\sim F_0\exp(-n\pi)$. Similar considerations demonstrate resonances beyond the barrier. Since the value of $t_{\rm BL}/T$ depends on the position x and the energy Ω, one may observe resonance responses at any x (at certain times) at specific resonance values of the energy. Direct verification proves that the accuracy of Eqs. (8) and (9), and thus of Eqs. (10) and (11), is related to $(1+|\Delta|)/\Omega T$ and to $1/\Lambda v$, where Λ and T are the characteristic length scales of $V(x)$ and $F(t)$ correspondingly. When $T\to\infty$, the resonance divergence in Eq. (11) is exact. To determine the resonant Δ for finite T, one must solve Eq. (7) exactly. However, one may speculate that the static case ($\dot F\equiv 0$) formula $|\psi_\Omega(x)/\psi_0^+(-\infty)|\sim 1/[\varphi^2+(\Omega-\Omega_r)^2/\Omega_r^2\varphi^2]$ remains valid in an adiabatic case $\Omega_r=\Omega_r(t)$. Here Ω_r is the resonance energy, and $\varphi\sim\exp[-\int_{x_0}^0(V-\Omega)^{1/2}dx]$ and $V(x_0)=\Omega$.

The obtained results are not restricted to a one-dimensional (1D) δ-function. For an arbitrary adiabatic potential $W(t,x)$ [which replaces $V+F\delta(x)$ in Eq. (1)] one verifies

$$\psi\propto\exp\left[-i\Omega t-\int_0^x dx'[W(t+it_{\rm BL},x')-\Omega]^{1/2}\right], \quad (15)$$

(where $t_{\rm BL}=0.5\int_0^x dx'/[W(t,x')-\Omega]^{1/2}$ and $W(-\infty,0)=\Omega$) is the WKB tunneling wave function. When $|\int_0^x dx'[W(t+it_{\rm BL},x')-\Omega]^{1/2}|\lesssim 1$, the WKB approximation becomes invalid. This is the immediate vicinity of the local eigenvalue of the potential W. There the response blows up for any W.

Stationary WKB tunneling in any dimensionality reduces to tunneling along a 1D line [whose length l should be substituted for x in Eq. (15)] and is determined by $S(l)\equiv\int_0^l[W(l')-\Omega]^{1/2}dl'=\min|\int\mathbf{p}\cdot d\mathbf{r}|$, $\mathbf{p}$ is a classical momentum. Again, one verifies that an adiabatic W yields $\psi\propto\exp\{-i\Omega t-S[t+it_{\rm BL}(l),l]\}$, with the imaginary retardation time and $t_{BL}(l)=0.5\int_0^l dl'/[W(t,l')-\Omega]^{1/2}$. In the general case $t\rightarrow t+it_{BL}-t_x$, where t_{BL} is the total Büttiker-Landauer time in the barrier (where $V>\Omega$) and t_x is the total classical retardation time in the well (where $V<\Omega$).

Similar reasoning is applicable to the WKB tunneling of any waves (electromagnetic, acoustic, hydrodynamic). So the presented picture is very general.

In this section I consider time response in more detail. By Eq. (10), above the barrier, the WKB approximation reduces to the classical retardation. One may formally relate response time t_r and velocity v_r to maximal response at a given place:

$$\max_t|\Delta(t,x)|=|\Delta(t_r,x)|,\ v_r=dx/dt_r\,. \tag{16}$$

Now consider the tunneling response time. While the tunneling rate is well known, the controversy about the tunneling traversal time has not subsided after almost six decades.[1-5] Wigner suggested following the peak of a wave packet.[2] However, the description of any change in the wave-packet shape in a stationary potential[5(a)] (including the decay of a metastable state[8]) reduces to the problem of solving for the time evolution of independent eigenstates. The characteristic period of a harmonic time-dependent potential[5(b)] determines the Büttiker-Landauer traversal time, but does not prove its universality or uniqueness.[9] The persistence of the problem calls for more accurate study. Above the barrier all activated energies participate in the probability-density transfer—see Eq. (4). Activation maybe approximately neglected only in the WKB case of an opaque barrier. The probability of activation above the barrier is related to $F_{V-\Omega}$. The probability of tunneling is related to $\exp[-(V-\Omega)t_{BL}]$. If $F(t)$ has a discontinuity, then F_ω decreases as a power of ω, and activation exponentially wins over tunneling. Tunneling wins when $F(t)$ is an entire function, which changes slowly compared to $1/(V-\Omega)$. Then Eq. (11) relates tunneling response to the imaginary retardation time. The real response time and velocity are introduced in accordance with Eq. (16). Then, by Eq. (13), a single imaginary time yields the whole ensemble of positive (retarded), negative (advanced), and zero (instantaneous) response times—see, e.g., Eq. (14). They scale with the Büttiker-Landauer traversal time, but change from $(-\infty)$ to $(+\infty)$.

Here I discuss the possibility of direct experimental observation of resonances and oscillations in tunneling response. According to the previous section, the necessary condition of tunneling domination is $(V-\Omega)T\gg 1$. The lower is $\omega=2\pi/T$, the larger, and thus experimentally more available, is the characteristic distance $\Lambda\sim vT\sim\lambda\Omega/\omega$, where λ is the de Broglie wavelength. However, the tunneling probability exponentially decays with Ω/ω. That is why realistically ω/Ω should not be too small. Electrons on helium[10] have $\Omega\sim 10^{12}$ $\sec^{-1}$. Variable-range hopping usually also yields $\Omega\sim 10^{12}$ $\sec^{-1}$. This means $\omega\sim 10^{10}$–10^{11} $\sec^{-1}$, and $\Lambda\sim 0.1$–1 μm.

Electromagnetic waves maybe perturbed by time-dependent dielectric constants or magnetic susceptibility. Again, reasonable wavelengths lead to high characteristic ω.

The best experiment may be on waves in a shallow variable-depth liquid.[6] Then $v\sim\sqrt{gh}$ (where $g\sim 10^3$ cm/sec^2 is the gravitational acceleration and h is the depth) implies reasonable frequencies and lengths.

Once resonances and/or oscillations in tunneling responses are observed, one may determine, by Eq. (16), tunneling response times. (On other experimental approaches to the measurement of tunneling traversal time see Refs. 1-3 and especially Ref. 4).

To summarize, WKB tunneling in an adiabatic potential $W(t,\mathbf{x})$ is reduced to the stationary solution. The method: consider t in W as a parameter, find the wave function, shift t in W by the imaginary classical retardation time. The latter is equivalent to the ensemble of real response times.

Tunneling oscillates with energy, space, and time. When $F(t)$ is symmetric in time, resonant maxima and antiresonant minima maybe advanced, retarded, and instantaneous. The corresponding times scale with the Büttiker-Landauer time, but are quantized, depend on interaction, and may change from $-\infty$ to $+\infty$. Specific quantized energies yield local giant resonances. The advanced resonances with a large relative increase in probability density may occur when the original impulse is still exponentially weak. Resonances amplify extremely the relative response at certain times and places. Thus, by using the proper time-dependent potential one may tune the interaction in a barrier. All these features maybe important, in particular, when the potential is time dependent, in virtue of the many-body interactions of tunneling particle.

The theory is applicable to any time-dependent potentials, higher dimensionalities, types of waves (de Broglie, electromagnetic, acoustic, hydrodynamic, and in particular, surface), and systems (in particular, random systems giving rise to Anderson localization).

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