Variable-range-hopping magnetoresistance

Mark Ya Azbel
School of Physics and Astronomy, Tel Aviv University, Tel Aviv 69978, Israel
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The hopping magnetoresistance R of a two-dimensional insulator with metallic impurities is considered. In sufficiently weak magnetic fields it increases or decreases depending on the impurity density n: It decreases if n is low and increases if n is high. In high magnetic fields B, it always exponentially increases with $\sqrt{B}$. Such fields yield a one-dimensional temperature dependence: $\ln R \propto 1/\sqrt{T}$. The calculation provides an accurate leading approximation for small impurities with one eigenstate in their potential well. In the limit of infinitesimally small impurities, an impurity potential is described by a generalized function. This function, similar to a δ function, is localized at a point, but, contrary to a δ function in the dimensionality above 1, it has finite eigenenergies. Such functions may be helpful in the study of scattering and localization of any waves.

I. INTRODUCTION

Magnetoresistance (MR) is an intensively studied property of a conductor. In metals[1,2] it always increases with magnetic field B ("positive MR"). In semiconductors[3-5] it may increase or decrease ("negative MR") with B. Theory does allow for both positive[5-9] and negative[10-12] MR R. However, e.g., in high magnetic fields different authors suggest the saturation of a two dimensional (2D) R (Ref. 12); $\ln R \propto (B/T)^{-1/3}$ (Ref. 7; this dependence agrees with experiment[4]); $\ln R \propto T^{-1/3}$ (Ref. 8); and $\ln R \propto T^{-1/2}$ with a not quite certain dependence on B (Ref. 9).

In this paper, I introduce a model that in strong fields allows for an accurate 2D solution and always (even in metals) yields[13] $\ln R \propto (B/T)^{1/2}$. In lower fields, MR behavior is more specific. I predict that at high metal impurity concentration c_i, negative MR has a minimum in very strong B, and then becomes positive, yielding the Ono law.[7] At low c_i, MR is always positive, switching from the Li-Thouless-type law[8] in strong B to the Ono law[7] in very strong B.

The presented model considers metallic impurities, whose size ρ is small compared to their Bohr radius r_0, and determines the leading approximation in $\rho/r_0 \to 0$.

In 1D the potential, which has an eigenstate with finite r_0 when $\rho \to 0$, is a δ function. However, this is accidental. A δ function is adjusted to solving an inhomogeneous equation via the Green function. It meets this goal in any dimensionality, whereas it yields no eigenstates and no scattering in any dimensionality above 1. The approximation $\rho/r_0 \to 0$ yields a generalized function ("Impurity D function"—IDF), which, similar to a δ function, is localized at a point ($r \to 0$), but, different from a δ function, has an eigenstate (with finite r_0) in 2D.

The definition $\rho \to 0, r_0$ is finite, allows one to introduce an IDF in any eigenstate problem (in electrodynamics, elasticity, etc.). Then for an arbitrary set of N IDF's the problem reduces to eigenstates of an $N \times N$ matrix.

In the next section I introduce an impurity D function. In Sec. III I compare δ and D functions and their properties. Both sections have a mathematical flavor and those not interested in it may proceed directly from Eqs. (2.1)–(2.4) to Eqs. (2.10)–(2.14) and then to Sec. IV, which accurately reduces the Schrödinger equation in magnetic field with N arbitrarily situated IDF's to N linear algebraic equations with explicitly presented coefficients. Section V considers physical implications of these equations. I conclude with the summary.

II. IMPURITY D FUNCTION

In this section I introduce the central idea of this paper—an impurity D function (IDF) $D(r)$. Consider a single impurity with the negative (i.e., attractive) potential

$$U(r) \equiv -(\hbar^2/2M)D(r) , \tag{2.1}$$

$$D(r) \equiv Wv(r/\rho) , \tag{2.2}$$

where M is the electron mass and $v(x)$ decays at $x \sim 1$, e.g.,

$$v(x) = \exp(-x^2) . \tag{2.3}$$

Then the Schrödinger equation in magnetic field $\mathbf{B} = B\hat{z}$ reads:

$$(\nabla - \mathbf{b} \times \mathbf{r})^2 \Psi + [(2ME/\hbar^2) + D(r)]\Psi = 0 , \tag{2.4}$$

$$\mathbf{b} = \hat{z}/2l_H^2 = \hat{z}eB/2c\hbar . \tag{2.5}$$

Here, l_H is the magnetic length, and the vector potential $\mathbf{A}$ is in the symmetric gauge: $\mathbf{A} = \mathbf{B} \times \mathbf{r}/2$. I find such $W(\rho)$, that even when $\rho \to 0$ the potential well (2.1)–(2.3) has an eigenstate, while $D(r)\Psi(\mathbf{r})$ in Eq. (2.4) may be replaced by $D(r)\Psi(0)$. Suppose such a W exists. Then the s state $\Psi(r)$, by Eq. (2.4), yields[14(a)]

$$\Psi'' + \Psi'/r + [(2ME/\hbar^2) - b^2r^2]\Psi = -D(r)\Psi(0) . \tag{2.6}$$

The solution to Eq. (2.6) is straightforward:

$$\Psi(r)=\exp(-\xi/2)\Psi(0)\int D(r')\exp(-\xi'/2)r'dr'M(\xi,\xi') , \tag{2.7}$$

$$M(\xi,\xi')=\tfrac{1}{2}\sum_{n=0}^{\infty} L_n(\xi)L_n(\xi')/(n+a) ,$$

where

$$\xi=r^2/2l_H^2, \quad a=0.5-MEc/2eB\hbar . \tag{2.8}$$

Equation (2.7) with $r=0$ determines E. When $\rho\to 0$, boring calculations yield

$$\rho^2 W\left[\ln\frac{l_H}{\rho}+\frac{1}{2a}-\frac{a}{2}\sum_{n=1}^{\infty}\frac{1}{n(a+n_2)}\right]=1 . \tag{2.9}$$

Thus, if $(\rho^2 W)^{-1}=\ln(d/\rho)$, i.e., by Eqs. (2.1)–(2.3),

$$U=-\hbar^2\exp(-r^2/\rho^2)/M\rho^2\ln(d/\rho), \quad \rho\to 0 , \tag{2.10}$$

then the eigenvalue a satisfies the equation

$$\ln(d/l_h)=(1/2a)-(a/2)\sum_{n=1}^{\infty}1/n(n+a) . \tag{2.11}$$

If magnetic field $B\to 0$, and thus $l_H\to\infty$, $a\to\infty$, then, by Eqs. (2.8) and (2.11),

$$E\to-\hbar^2/2Md^2 . \tag{2.12}$$

When B increases, a decreases. When $a<\frac{1}{2}$, then, by Eq. (2.8), $E>0$ and the level is above the vacuum. When $B\to\infty$, then $a\to 0$, and $E\to$ to the ground-state energy in a vacuum. When $\rho\to 0$ and $r\gg\rho$, then in Eq. (2.7) $M(\xi,\xi')\simeq M(\xi,0)$, where[15(a),16(b)]

$$M(\xi,0)=\Gamma(a)\psi(a,1,\xi)/2\equiv G(a,\xi) \tag{2.13}$$

and

$$\Psi(r)\propto\exp(-\xi/2)G(a,\xi) . \tag{2.14}$$

Here, Γ is the Γ function and ψ is the degenerate hypergeometrical function.

When $l_H\to\infty$, then [16(a)]

$$G(a,\xi)\to K_0(r/d)/2 , \tag{2.15}$$

where K_0 is the Bessel function. In a general case[15(b)]

$$\Psi(r)\propto\exp[-(r/d)-(r^2/4l_H^2)]ln[r(d^{-1}+l_H^{-1})] . \tag{2.16}$$

(Note that when $r\to\infty$, then $\ln G\propto -r^2$ rather than $\propto -r$ as in conventional localization.) When $B\to 0$, then Ψ decays at $r\simeq d$, which, in agreement with Eq. (2.12), plays the role of an impurity Bohr radius a_B. Thus, although d vanishes in the leading approximation of Eq. (2.10) when $\rho\to 0$, it is crucial for the eigenstate.

III. δ AND D FUNCTIONS

The difference between a 2D IDF $D(r)$ and a 2D δ function $\delta(\mathbf{r})$ is clear from Eqs. (2.10) and (2.11): $\int D(r)r\,dr\,d\theta=\pi/|\ln\rho|\to 0$, while $\int\delta(\mathbf{r})r\,dr\,d\theta=1$. A 1D D function may be derived similar to the derivation in the preceding section, and yields

$$D(x)=Wv(x/\rho) , \tag{3.1}$$

where

$$W=w/\rho . \tag{3.2}$$

When $\rho\to 0$, this 1D IDF is proportional to a 1D δ function:

$$D(x)\to w\int_{-\infty}^{\infty} v(x_1)dx_1\delta(x) . \tag{3.3}$$

Note that the strength w enters as a factor, and that only the leading approximation in $\rho\to 0$ matters. Since

$$W\rho^2\propto\rho\to 0 , \tag{3.2'}$$

a δ-function potential (which is $\propto 1/\rho$ and $\to\infty$) is infinitely shallow [14(b)] compared to the characteristic energy $\hbar^2/2M\rho^2$. That is why its eigenfunction changes infinitely slowly compared to its width[14(b)] ρ, and

$$\int_{-\infty}^{\infty}\delta(x)\Psi(x)dx=\Psi(0) . \tag{3.4}$$

In virtue of $W\propto w$ in Eq. (3.2), one may introduce a potential

$$U\propto\delta(x) \tag{3.5}$$

and the shorthand $\delta(x)\Psi(x)=\delta(x)\Psi(0)$, in the sense of Eq. (3.4). According to Eq. (2.10), in 2D,

$$W\rho^2\propto 1/|\ln\rho|\to 0 , \tag{3.6}$$

i.e., the well is still infinitely shallow,[14(b)] although only logarithmically. Again, by Eq. (2.14), the wave function changes infinitely slowly compared to ρ.

Until now I considered a circularly symmetric IDF. However, one may study the extreme anisotropy, choosing

$$D(\mathbf{r})=W\delta(x)v(y/\rho), \quad v(-y)=v(y) . \tag{3.7}$$

Then it is convenient to use the Landau gauge for the vector potential: $A=A_y=-Bx$. The Schrödinger equation for, e.g., $E=-\hbar^2K^2/2M<0$, reads

$$\frac{\partial^2\Psi}{\partial x^2}+\left[\frac{\partial}{\partial y}+2ibx\right]^2\Psi^2-K^2\Psi=-Wv(y/\rho)\delta(x)\Psi(0,y) . \tag{3.8}$$

The further reasoning is similar to the case of a symmetric IDF. In higher dimensionalities a shallow well has no eigenstates,[14(b)] $D\propto 1/\rho^2$, and even in the leading approximation $D(r)$, $\Psi(r)$ may be replaced by $D(r)\Psi(0)$ only qualitatively—cf. the Lifshitz model.[17] Then the IDF approach becomes a convenient model.

IV. EIGENSTATES OF AN IDF SET

Consider a 2D insulator (with zero potential energy) with N metallic impurities. Suppose the sth impurity has a potential U_s [cf. Eq. (2.10)]:

$$U_s=-(\hbar^2/2M)D_s(|\mathbf{r}-\mathbf{r}_s|) , \tag{4.1}$$

$$D_s(r)=2\exp(-r^2/\rho^2)/\rho^2\ln(d_s/\rho). \tag{4.1a}$$

Then, according to Sec. II, for $\rho\to 0$ the Schrödinger equation may be presented as [cf. Eqs. (2.4) and (2.6)]

$$(\nabla - i\mathbf{b}\times\mathbf{r})^2\Psi + (2ME/\hbar^2)\Psi = -\sum_s D_s(|\mathbf{r}-\mathbf{r}_s|)\Psi_s \ , \tag{4.2}$$

$$\Psi_s = \Psi(\mathbf{r}_s) \ . \tag{4.3}$$

The transformation

$$\Psi = \sum_s \exp[i(\mathbf{b}\times\mathbf{r}_s)\cdot\mathbf{r}]\Psi_s\Phi_s(|\mathbf{r}-\mathbf{r}_s|) \tag{4.4}$$

yields,[18] similar to Eq. (2.7),

$$\Phi_s(r) = \exp(-\xi/2)\int D_s(r')r'dr'M(\xi,\xi')\exp(-\xi'1/2) \ . \tag{4.5}$$

When $\mathbf{r}=\mathbf{r}_\sigma$, Eq. (4.4) yields, by Eq. (4.3),

$$\Psi_\sigma = \sum_s \exp[i(\mathbf{r}_\sigma\times\mathbf{r}_s)\cdot\mathbf{b}]\Phi_s(|\mathbf{r}_\sigma-\mathbf{r}_s|)\Psi_s \ . \tag{4.6}$$

Boring transformations, similar to those in Sec. II, lead to the equations independent of $\rho\to 0$:

$$\varepsilon_\sigma\Psi_\sigma = \sum_{S\neq\sigma} A(\mathbf{r}_\sigma-\mathbf{r}_s,\mathbf{r}_\sigma)\Psi_s \ . \tag{4.7}$$

Here,

$$A(\mathbf{r},\mathbf{r}_s) = \exp[i(\mathbf{r}_\sigma\times\mathbf{r})\mathbf{b} - (r^2/2l_H^2)]G(a,r^2/2l_H^2) \ , \tag{4.8}$$

$$\varepsilon_\sigma = \ln(d_\sigma/l_H) - (1/2a) + \sum_{\nu=1}^{\infty} a/2\nu(\nu+a) \ . \tag{4.9}$$

a is presented by Eq. (2.8); $G(a,\xi)$ is related to Eq. (2.13).

Equations (4.7)–(4.9) are the main ones in this paper. They are *accurate* for any IDF configuration. The diagonal terms depend on the IDF strength d_σ only; the nondiagonal terms depend on the IDF positions only. They include the magnetic flux phase *and* the decrease of the coupling constants. The order of the Hermitian $\hat{\mathbf{A}}$ is the total number of N of IDF. The knowledge of Ψ_s provides (for $\rho\to 0$ and $r\gg\rho$), by Eq. (4.4),

$$\Psi(\mathbf{r}) = \sum_s A(\mathbf{r}-\mathbf{r}_s,\mathbf{r}_s)\Psi_s, \tag{4.10}$$

where the summation includes *all* IDF's.

V. PHYSICAL IMPLICATIONS

Now let us consider the physical implications of Eq. (4.7) for $a<0$.

(i) A Bloch electron in zero field: $\mathbf{r}_\sigma \equiv \hat{\mathbf{x}}md_x + \hat{\mathbf{y}}nd_y$, m and n are integers; $\varepsilon_\sigma = \varepsilon$ is independent of σ, when $B\to 0$; then $\xi\to 0$, $a\to\infty$, and Eq. (2.15) is valid. The Fourier transformation of Eq. (4.7) yields the analytical formula for the whole Bloch band $E=E(\mathbf{q})<0$ ($\mathbf{q}$ is a quasi-wave-vector)

$$\varepsilon = \sum_{mn\neq 0} 0.5K_0[d^{-1}(m^2d_x^2+n^2d_y^2)^{1/2}] \times\cos(mq_xd_x)\cos(nq_yd_y) \ . \tag{5.1}$$

(ii) A Bloch electron in magnetic field. Then the main difference in Eq. (4.7) from the conventional Harper equation is $\exp(-r_{\sigma s}^2/4l_H^2)$, $\mathbf{r}_{\sigma s}^2 = |\mathbf{r}_\sigma - \mathbf{r}_s|^2$. Since $2l_H = 2(c\hbar/eB)^{1/2} \sim 500B_T^{1/2}A$ (B_T is the magnetic field in teslas), in a common crystal lattice $d_x, d_y \ll 2l_H$, and with high accuracy, Eq. (4.7) reduces to the Harper equation. This may be different in superlattices, certainly in fabricated ones. The study of $r_{\sigma s} \lesssim l_H$ may be useful for the understanding of variable range hopping (VRH) with the Mott hopping length $\lesssim 2l_H$. Consider the Wentzel-Kramers-Brillouin (WKB) approximation. Then $\Psi\propto\exp(iS)$ yields, by Eq. (4.7), the dispersion relation for a kinematic wave vector $\mathbf{q}$:

$$\varepsilon = \sum_{\mu,\nu\neq 0} \cos(\mu q_x)\cos(\nu q_y)G(a,\xi_\tau)\exp(-\xi_\tau/2) \ . \tag{5.2}$$

Here, $q_x = \partial S/\partial m - \phi n$, $q_y = \partial S/\partial n + \phi m$, $\phi = d_xd_y/2l_H^2$, $\xi_\tau \equiv (\mu^2d_x^2+\nu^2d_y^2)/2l_H^2$. When $\xi\ll 1$, then[16] $G\to K_0(d^{-1}r)$; when $\xi\ll 1$, then $G\to\xi^{-1}\Gamma(a)/2$. The effective mass[1] is $m^* = (\hbar^2/2\pi d_xd_y)(\partial S_F/\partial E)$, where S_F is the Fermi area of $E=E(\mathbf{q})<0$ from Eq. (5.2). It increases with B. The cyclotron frequency $\Omega\propto B/m^*$ sublinearly increases with B in weak fields, and exponentially decreases in strong fields ($d_x, d_y > 2l_H$). Thus, according to conventional formulas[1] for MR in metals, in intermediate fields ($d_x, d_y \sim 2l_H$) MR may have a minimum; in strong fields it increases exponentially with B.

(iii) The simplest non-Bloch situation is demonstrated by three IDF's ($s=0,1,2$) with the strengths d_0 and $d_1=d_2$ at the sites of an equilateral triangle with the side d. Then, by Eq. (4.8), $A_{01}=A_{02}=\overline{A}_{20}\equiv A$; $A_{12}=\overline{A}_{21}=A\exp(-\beta - i\beta\sqrt{3}/2)$, where $\beta = d^2/4l_H^2$. Introducing $\eta=\varepsilon_0/A$, $\gamma=(\varepsilon_1-\varepsilon_0)/A$, one obtains the dispersion relation

$$(\eta+1)[(\eta+\gamma)(\eta-1)-2] = -4\sin^2(\beta\sqrt{3}/8) \ . \tag{5.3}$$

Consider the eigenstate at $s=0$, i.e., $\chi = |\Psi_1/\Psi_0| = |\Psi_2/\Psi_0| < 1$. If different eigenstates overlap very little ($|\gamma|\gg 1$), then $\chi\sim 1/|\gamma|$ and the eigenstate tail $\chi\propto\exp(-\beta)$ decreases with magnetic field. If different eigenstates significantly overlap ($|\gamma|\ll 1$; when $A\ll 1$, this may happen only if ε_1 is very close to ε_0), the tail increases with B. For instance, when $\beta^2\ll|\gamma|$, then $\chi\simeq 0.5+27\beta^2w^2/4A^2$. Thus, the magnetic field enhances the existing "insulatory" ($|\gamma|>>1$) or "metallic" ($|\gamma|\ll 1$) state.

(iv) The nearest-neighbor approximation of Eq. (4.7) on a square lattice $d_x=d_y=d$ reads

$$\left[\cos\left[i^{-1}\frac{\partial}{\partial m} - n\phi\right] + \cos(i^{-1}\partial n + m\phi)\right]\Psi_{mn} = \varepsilon_{mn}Q \ , \tag{5.4}$$

$$Q = \exp(\phi/2)/G(a,\phi); \phi = d^2/2l_H^2 \ , \tag{5.5}$$

cf. Eq. (5.2). It reduces to the Anderson model with the Lorentz magnetic flux phase and the effective diagonal disorder $\varepsilon_{mn}\to\varepsilon_{mn}Q$. In the Bloch case, $\varepsilon_{mn}=\varepsilon$ is independent of m,n. Then, Eq. (5.4) is the Harper equation with the effective energy $\varepsilon Q \equiv \varepsilon^*(\phi)$. Since the Harper

$\varepsilon^*(\phi)$ was determined in Ref. 19, this equation determines the Bloch energy E as a function of ϕ.

(v) Suppose there is any number of randomly scattered IDF's. First consider $[Kd_i+(d_i^2/4l_H^2)]\gg 1$, where $d_i=n^{-1/2}$, n is the impurity density. If a state is localized at $\sigma=\tau$, then the leading approximation in Eq. (4.7) yields $\varepsilon_\tau=0, \Psi_\sigma=\delta_{\sigma\tau}$. The next approximations are $(\sigma\neq\tau)$

$$\Psi_\sigma=A_{\sigma\tau}/\varepsilon_\sigma,\quad \varepsilon_\tau=|A_{\kappa\tau}|^2/\varepsilon_\kappa\ , \tag{5.6}$$

where $|A_{\kappa\tau}|=\max_\sigma|A_{\sigma\tau}|$. By Eqs. (5.6) and (4.8), low concentration of random IDF always leads to the tunneling, which decreases with B.

Now suppose $Kd_i\ll 1$ and l_H is the largest characteristic length. Then many terms contribute to the Eq. (4.7) sum. If the sum in Eq. (4.7) converges when $\xi_{\sigma s}=r_{\sigma s}^2/2l_H^2\ll 1$, then $\exp(-\xi_{\sigma s}/2)\simeq 1$. So, Eq. (4.7) leads to the Nguyen-Spivak-Shklovskii model[10] (NSS) and to the tunneling increase in magnetic field.

Some speculations on physics. Suppose the Fermi ε is between $\min\varepsilon_{mn}$ and $\max\varepsilon_{mn}$. In the nearest-neighbor approximation on a square lattice, similar to (ii), the effective energy is $\varepsilon^*_{mn}=\varepsilon_{mn}\exp[(\phi/2)/G(a,\phi)]$. The exponent $\exp(\phi/2)$ makes "metallic" sites ($\varepsilon_{mn}>0$) more metallic (increasing the effective energy distance to the barrier) and makes "insulating" sites ($\varepsilon_{m,n}<0$) more "insulatotory"—cf. point (iii). For a given energy and disorder, one may imagine a sample, consisting of "metallic" and "insulatory" regions. The magnetic field yields negative magnetoresistance of the former and positive magnetoresistance of the latter.

Furthermore, one may refine the Miller-Abrahams random resistor network model.[20] Introduce metallic wires with the zero potential energy inside and infinity outside them. The wires have varying shape and width (of order of localization length), and may not percolate through the whole sample. The varying width leads to localization in the vicinity of a local width maxima,[21] while magnetic-field-induced edge states yield delocalization in magnetic field. The less the curvature is, the quicker it develops. Its competition with magnetic-field-enhanced localization in the insulator regions [cf. Eq. (5.6)] may lead to oscillations with B—cf. the experiments in Ref. 22.

(vi) Finally, consider variable range hopping MR R:

$$1/R\sim(e^2/h)\sum_{s,\sigma}|\Psi_s(E^\sigma)|^2\exp(-|E^s-E^\sigma|/T)\ . \tag{5.7}$$

Here, E^σ is the eigenvalue of Eq. (4.7), where $\max_\sigma|\Psi_s(E^\sigma)|=|\Psi_\sigma(E^\sigma)|$; T is the temperature; the summation is over all paths between electrodes. If $na_B^2\ll 1$ and/or $nl_H^2\ll 1$, then, by Eqs. (5.7) and (4.8), the conventional Mott reasoning yields positive MR: $R\propto\max_r\exp(r/a_B+r^2/4l_H^2+\pi^2\hbar^2/2Ma_B^2r^2nT)$, where a_B is the characteristic IDF Bohr radius and n is an IDF density. In strong magnetic field, MR is exponential:

$$\ln R\propto B,\ T^{-1/3}\ \text{ if } \sqrt{r_Ba_B}<l_H<r_M\ , \tag{5.7a}$$

$$\ln R\propto(B/T)^{1/2}\ \text{ if } l_H<\sqrt{r_Ma_B}\ , \tag{5.7b}$$

where $r_M\sim(\pi^2\hbar^2/2Ma_BnT)^{1/3}$ is the Mott hopping distance in $B=0$ [cf. Eq. (5.7a) and Ref. 8]. By Eq. (5.7b), very strong magnetic field "restores" the 1D temperature dependence and yields $\ln|\Psi_s(\varepsilon^\sigma)|\propto -r_{\sigma s}^2B$ rather than conventional $\propto -r_{\sigma s}$.

If $na_B^2\gg 1$ and $l_H\gg r_M$, then, by (v), NSS (Ref. 10) is valid and, by Refs. 10–12, MR is negative. Since, by Eq. (5.7), $nl_H^2\ll 1$ yields exponentially increasing R, there must be a minimum in $R(B)$ if impurity concentration is high.

(vii) Further progress may be achieved in numerical experiments according to Eqs. (4.7)–(4.9). Start with the study of eigenstates. Suppose for a fixed B one finds the eigenvalues of Eqs. (4.7)–(4.9):

$$E^\sigma=E^\sigma(B),\quad \sigma=1,2,\ldots,N \tag{5.8}$$

and the eigenfunctions

$$\Psi_s^\sigma=\Psi_s(E^\sigma,B),\quad s=1,2,\ldots,N\ , \tag{5.9}$$

where

$$\max_s|\Psi_s(E^\sigma,B)|=|\Psi_\sigma(E^\sigma,B)\ . \tag{5.10}$$

The number of branches σ, by Eq. (5.8), equals the number of impurities N. Eigenenergies E are finite, the distance between them is $\sim E/N$. So, when $N\to\infty$, then σ/N and E are continuous, and Eq. (5.8) provides $\sigma=\sigma(E,B)$. By Eq. (5.10), E^σ is localized at $\mathbf{r}_\sigma=\mathbf{r}(E,B)$. Thus, one may express an eigenfunction (5.9) as

$$\Psi_s=\Psi(E,B;\mathbf{r}_s-\mathbf{r}(E,B))\ . \tag{5.11}$$

Obviously, when N is finite, Eq. (5.11) is valid for discrete values E^σ. The function f,

$$\ln|\Psi_s|=-f(E,B;r_E), r_E=|\mathbf{r}_s-\mathbf{r}(E,B)|\ , \tag{5.12}$$

strongly fluctuates. However, its average f_{av} (over N^* states, where $1\ll N^*\ll N$) is a smooth function. Note that, by Eqs. (5.6), (4.8), and (2.16), in strong magnetic field one *does not* expect $f_{\text{av}}\propto -r_E$, as in a conventional localization in a weak magnetic field. Rather, in a strong magnetic field $f_{\text{av}}\propto -r_E^2$.

In principle, the knowledge of eigenstates allows one to calculate the Mott variable-range-hopping magnetoresistance R by Eq. (5.7). However, the total number of terms in Eq. (5.7) is the number of all possible paths between IDF's. It is exponential with $N\ln N$. Therefore, to perform a numerical calculation, one must consider only the major contributions into Eq. (5.7). One may do it according to the following qualitative reasoning (cf. Ref. 23).

Interpolate f_{av} as

$$f_{\text{av}}=(r_E/L_0)^\delta,\quad L_0=L_0(E,B)\ . \tag{5.13}$$

Consider, e.g., the hopping between the states i (for "initial") and f (for "finite"), which are on the opposite sides of the Fermi energy E_F. Then,

$$R^{-1}\propto\sum_{i,f}\exp[-(\Delta r/L_0)^\delta-(\Delta E/T)]\ , \tag{5.14}$$

where Δr is the space and $\Delta E=|E_f-E_i|=\Delta E_f+\Delta E_i$, $\Delta E_f=|E_f-E_F|$, $\Delta E_i=|E_i-E_F|$ is the energy distance

between such states. Estimate the exponent in Eq. (5.14) as

$$(\Delta r/L_0)^{\delta}+(\Delta E/T) \simeq \{(\Delta x/L_0)^2+(\Delta y/L_0)^2 +[(\Delta E_i/T)^{1/\delta}+(\Delta E_f/T)^{1/\delta}]^2\}^{\delta/2} . \quad (5.15)$$

The right-hand side of Eq. (5.15) is the distance between the points in the space (x_σ/L_0), y_σ/L_0, and $(|E_\sigma-E_F|/T)^{1/\delta}\mathrm{sgn}(E_\sigma-E_F)$, where $\mathrm{sgn}x=x/|x|$. It is natural to assume that the minimum of the left-hand side of Eq. (5.15) is reached on the points, which are close (although not necessarily nearest neighbors) in the above space, and the distance between which, according to the Mott reasoning, is ~ 1. The true Mott "f" state for a given "i" state is determined by the minimum of the accurate Mott term in Eq. (5.7). Once the best "f" is established, it serves as an "i" for the next Mott hop, etc. This way one calculates all Mott paths between electrodes and determines R, as well as its fluctuations with B, E_F, and n.

VI. SUMMARY

(i) I introduced an impurity D function (IDF) $D(r)$, which is as useful in solving the Schrödinger (and in general a partial differential) equation in higher dimensionalities as a δ function is in 1D. In 2D, e.g.,

$$D(r)=2[\rho^2\ln(d/\rho)]^{-1}\exp(-r^2/\rho^2) . \quad (6.1)$$

If an impurity potential is $U(r)=(\hbar^2/2M)D(r)$, then its eigenenergy is

$$E=-\hbar^2/2Md^2 ; \quad (6.2)$$

its eigenfunction is

$$\Psi \propto K_0(r/d) , \quad (6.3)$$

where K_0 is the Bessel function and d is the impurity Bohr radius. In magnetic field the eigenenergy is determined by Eqs. (2.11) and (2.8); the eigenfunction is determined by Eq. (2.14).

(ii) The Schrödinger equation in magnetic field for an arbitrary configuration of N metallic IDF's with the potentials $-(\hbar^2/2M)D_s(\mathbf{r}-\mathbf{r}_s)$ accurately reduces to the eigenstates of (explicitly presented) the $N\times N$ matrix in Eqs. (4.7)–(4.9) and (2.13). Magnetic flux determines its (NSS-type) phase *and* the exponential factor.

(iii) Bloch dispersion (for $B=0$) is presented by the explicit analytical formula (5.1).

(iv) Bloch electrons in very high magnetic fields yield magnetoresistance R, which increases exponentially with B. In intermediate fields, R may have a minimum.

(v) A low concentration ($a_B^2 n<1$) of randomly situated metallic IDF's leads to positive magnetoresistance $R(B)$. It is exponential in B and the Li-Thouless[8] $T^{-1/3}$ in high magnetic fields ($\sqrt{r_M a_B}<l_H<r_M$). High IDF concentration ($a_B^2 n>1$) leads to a magnetoresistance minimum (or minima). In very high magnetic fields ($l_H^2 n<1$) in both cases probability density decreases exponentially with r^2B (rather than with B), and $\ln R \propto (B/T)^{1/2}$.

Positive and negative $R(B)$ were observed experimentally and are consistent with the theory. The main predictions of this paper are $\ln R \propto \sqrt{B/T}$ in very high magnetic fields and the magnetoresistance minimum at high metal concentrations.

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