PHYSICAL REVIEW B VOLUME 43, NUMBER 8 15 MARCH 1991-I

High-field transport in semiconductors. III. Wave-function renormalization

Pavel Lipavský
Department of Physics, Ohio State University, Columbus, Ohio 43210

Furrukh S. Khan
Department of Electrical Engineering, Ohio State University, Columbus, Ohio 43210

John W. Wilkins
Department of Physics, Ohio State University, Columbus, Ohio 43210

(Received 21 September 1990)

The asymptotic form of the single-particle retarded Green function is discussed in the presence of the homogeneous stationary electrical field. It is shown that different definitions of the instantaneous approximation of the self-energy lead to different values of the field effect on the scattering. This inconsistency is removed if the wave-function renormalization is included. Finally, the choice of the instantaneous approximation that removes the linear dependence on the field is discussed.

I. INTRODUCTION

During the past two decades many studies have been devoted to the influence of the homogeneous electrostatic field on the scattering rates in semiconductor devices.[1–4] As far as we know, in all of these studies the wave-function renormalization was neglected. There are two reasons for this neglect. First, for simplicity a natural assumption was to treat the field effect on the scattering and the wave-function renormalization effects as independent and hence to view the wave-function renormalization as an unimportant complication. Second, the asymptotic formulas of the single-particle Green function that include the wave-function renormalization in the presence of the external field were not available in print. The only asymptotic formulas of the Green function with the renormalization included was derived within the gradient expansion[5] (the gradient expansion of the entire Green function, not the gradient expansion of the self-energy), which is not acceptable in the high-field theory. In this paper we show that the field effect on the scattering rates has to be studied together with the wave-function renormalization, and we derive the asymptotic formula of the single-particle Green function.

A simple illustration of the connection between the wave-function renormalization and the field effect in the total scattering rate is provided by a straightforward evaluation of the lowest-order correction in the field. We start with this example to introduce the problem on an intuitive level.

A. Pole approximation in equilibrium

To introduce the wave-function renormalization we briefly review the pole approximation of the single-particle retarded Green function in equilibrium.[5] We will not use the superscript R because in this paper we are dealing exclusively with retarded Green functions. In the momentum and energy representation the Green function reads

$$G(\omega;\mathbf{k})=\frac{1}{\omega-\varepsilon(\mathbf{k})-\Sigma(\omega)}\ . \tag{1.1}$$

The pole contribution of this Green function can be written

$$G(\omega;\mathbf{k})=z(\mathbf{k})\frac{1}{\omega-\varepsilon(\mathbf{k})-\sigma(\mathbf{k})}\ , \tag{1.2}$$

where the pole value of the self-energy is given by the self-consistent equation

$$\sigma(\mathbf{k})=\Sigma(\varepsilon(\mathbf{k})+\sigma(\mathbf{k})) \tag{1.3}$$

and the wave-function renormalization $z(\mathbf{k})$ is

$$z^{-1}(\mathbf{k})=1-\left.\frac{\partial\Sigma(\omega)}{\partial\omega}\right|_{\omega=\varepsilon(\mathbf{k})+\sigma(\mathbf{k})}\ . \tag{1.4}$$

In the time representation one can express the pole contribution as

$$G(t_1,t_2;\mathbf{k})=-i\theta(t_1-t_2)z(\mathbf{k})\times\exp\left[-i\int_{t_2}^{t_1}dt(\varepsilon(\mathbf{k})+\sigma(\mathbf{k}))\right]\ . \tag{1.5}$$

This representation, with the integral over time in the exponent, will facilitate the connection to the nonequilibrium case.

B. Pole approximation in the presence of the field

Here we momentarily neglect the wave-function renormalization and first generalize the pole approximation by the presence of the electrical field $\mathbf{F}$. With the vector gauge $\mathbf{A}=-\mathbf{F}t$ the relation of the wave vector $\mathbf{k}$ and the canonical momentum $\mathbf{p}$ becomes time dependent: $\mathbf{p}=\mathbf{k}-e\mathbf{F}t$.[1,6] Using the replacement of the canonical momentum $\mathbf{p}$ by its time-dependent value one finds from (1.5) that the pole contribution to the Green function in the presence of the electric field is (remember z has been set to unity for simplicity)

$$G(t_1,t_2;\mathbf{k})=-i\theta(t_1-t_2)\times\exp\left[-i\int_{t_2}^{t_1}dt(\varepsilon(\mathbf{k}-e\mathbf{F}t)+\sigma(\mathbf{k}-e\mathbf{F}t))\right] . \quad (1.6)$$

The formula (1.6) involves the pole approximation of the self-energy $\Sigma(\omega;\mathbf{k})\approx\sigma(\mathbf{k})$. That this approximation may be problematic can be seen by expressing it in the time representation

$$\sigma(\mathbf{k})=\int_{-\infty}^{t}dt'\Sigma(t,t')\exp\left[-i\int_{t'}^{t}d\tilde{t}(\varepsilon(\mathbf{k})+\sigma(\mathbf{k}))\right] . \quad (1.7)$$

The formula (1.7) was derived in equilibrium[7] and there is no external field influence built into it. There are three ways in which the external field can enter a possible generalization of the formula (1.7). First, $\Sigma(t,t')$ depends on the field. In Appendix A it is shown that Σ is field independent to a linear order in the field and thus we can neglect this effect here. Second, the momentum $\mathbf{k}$ in the argument of the kinetic energy has to be time dependent, i.e., replaced by $\mathbf{p}=\mathbf{k}-e\mathbf{F}\tilde{t}$. This correction is the essential one. Third, the pole approximation of the self-energy $\sigma(\mathbf{k})$ that enters the argument of the exponential has to be replaced by a field-dependent value $\sigma_F(\mathbf{p})$. This correction is important if the wave-function renormalization is discussed. Below we take these corrections into account and discuss (1.6) with the field-dependent approximation of the self-energy.

In addition, we note that there is a convergence problem with (1.7) and in fact the integration limit $-\infty$ has to be replaced by some finite value that is crudely $t-3\tau(\mathbf{k})$, where $\tau(\mathbf{k})$ the quasiparticle lifetime. This point is discussed in Appendix A.

C. Field-induced correction to the pole approximation

Taking into account the presence of the electric field, one should modify (1.6) by replacing $\sigma(\mathbf{k}-e\mathbf{F}t)$ by $\sigma_F(\mathbf{k}-e\mathbf{F}t)$, which is obtained from the field-dependent generalization of (1.7) as

$$\sigma_F(\mathbf{k}-e\mathbf{F}t)=\int_{-\infty}^{t}dt'\Sigma(t,t')\times\exp\left[-i\int_{t'}^{t}d\tilde{t}(\varepsilon(\mathbf{k}-e\mathbf{F}\tilde{t})+\sigma_F(\mathbf{k}-e\mathbf{F}\tilde{t}))\right] . \quad (1.8)$$

It was formula (1.8) that was used in Ref. 4 for the discussion of the field effect on the scattering rate. At this point formula (1.8) is used only as an intuitive tool to demonstrate the relation between the wave-function renormalization and the field effect on the scattering rate. There are alternative ways to generalize the equilibrium, pole approximation (1.7), and these alternative generalizations leads to different results. We will clarify this point in Sec. II.

The leading correction term in the strength of the field can be obtained by the gradient expansion of the exponent of (1.8):

$$\sigma_F(\mathbf{k}-e\mathbf{F}t)=\int_{-\infty}^{t}dt'\Sigma(t,t')e^{-i[\varepsilon(\mathbf{k}-e\mathbf{F}t)+\sigma_F(\mathbf{k}-e\mathbf{F}t)](t-t')}\left[1+ie\mathbf{F}\frac{\partial[\varepsilon(\mathbf{k}-e\mathbf{F}t)+\sigma_F(\mathbf{k}-e\mathbf{F}t)]}{\partial\mathbf{k}}\frac{(t'-t)^2}{2}\right] . \quad (1.9)$$

Expressing the time variable with help of the energy derivatives in the field we can simplify expression (1.9) to

$$\sigma_F(\mathbf{k}-e\mathbf{F}t)=\Sigma[\varepsilon(\mathbf{k}-e\mathbf{F}t)+\sigma_F(\mathbf{k}-e\mathbf{F}t)]-ie\mathbf{F}\frac{\partial[\varepsilon(\mathbf{k}-e\mathbf{F}t)+\sigma_F(\mathbf{k}-e\mathbf{F}t)]}{\partial\mathbf{k}}\frac{1}{2}\frac{\partial^2\Sigma(\omega)}{\partial\omega^2}\bigg|_{\omega=\varepsilon(\mathbf{k}-e\mathbf{F}t)+\sigma_F(\mathbf{k}-e\mathbf{F}t)} . \quad (1.10)$$

Since the field-induced correction is supposed to be small, we expand Σ in the vicinity of the equilibrium pole value given by (1.3). Formally it is an expansion in powers of $\sigma_F-\sigma$ that gives

$$\sigma_F(\mathbf{k}-e\mathbf{F}t)-\sigma(\mathbf{k}-e\mathbf{F}t)=-z(\mathbf{k}-e\mathbf{F}t)ie\mathbf{F}\frac{\partial[\varepsilon(\mathbf{k}-e\mathbf{F}t)+\sigma(\mathbf{k}-e\mathbf{F}t)]}{\partial\mathbf{k}}\frac{1}{2}\frac{\partial^2\Sigma(\omega)}{\partial\omega^2}\bigg|_{\omega=\varepsilon(\mathbf{k}-e\mathbf{F}t)+\sigma_F(\mathbf{k}-e\mathbf{F}t)} . \quad (1.11)$$

Here the wave-function renormalization resulted from the expansion of Σ; see (1.4). The left-hand side of (1.11) has the form of the total derivative with respect to time. Thus Eq. (1.11) can be reorganized as

$$\sigma_F(\mathbf{k}-e\mathbf{F}t)-\sigma(\mathbf{k}-e\mathbf{F}t)=i\tfrac{1}{2}z(\mathbf{k}-e\mathbf{F}t)\frac{dz^{-1}(\mathbf{k}-e\mathbf{F}t)}{dt} . \quad (1.12)$$

Now we can evaluate the correction to the propagator that is caused by the field effect on the scattering rate:

$$\begin{aligned}G(t_1,t_2;\mathbf{k})&=-i\theta(t_1-t_2)\exp\left[-i\int_{t_2}^{t_1}dt(\varepsilon(\mathbf{k}-e\mathbf{F}t)+\sigma_F(\mathbf{k}-e\mathbf{F}t))\right]\\&=-i\theta(t_1-t_2)\exp\left[-i\int_{t_2}^{t_1}dt(\varepsilon(\mathbf{k}-e\mathbf{F}t)+\sigma(\mathbf{k}-e\mathbf{F}t))\right]\exp\left[\int_{t_2}^{t_1}dt\tfrac{1}{2}z(\mathbf{k}-e\mathbf{F}t)\frac{dz^{-1}(\mathbf{k}-e\mathbf{F}t)}{dt}\right]\\&=-i\theta(t_1-t_2)\exp\left[-i\int_{t_2}^{t_1}dt(\varepsilon(\mathbf{k}-e\mathbf{F}t)+\sigma(\mathbf{k}-e\mathbf{F}t))\right]\left[\frac{z(\mathbf{k}-e\mathbf{F}t_2)}{z(\mathbf{k}-e\mathbf{F}t_1)}\right]^{1/2} . \quad (1.13)\end{aligned}$$

The last line of (1.13) is expressed exclusively in terms of equilibrium functions: effective self-energy and wave-function renormalization. The first part of the formula is identical to the Green function (1.6) with the field effect on the scattering neglected, the second part includes the entire field effect on the scattering expressed in terms of the time dependence of the wave-function renormalization. Note that we have neglected the wave-function renormalization when deriving the formula (1.13), but the result (1.13) makes this neglect suspicious.

Higher orders of the field effect on the scattering cannot be reduced to the time dependence of the wave-function renormalization. For instance, there is the field effect on the local density of states[8,9] that provides the effect on the scattering rate already in the double-time formulation and thus it does not depend on the artificial procedure by which the instantaneous approximation is obtained. However, to separate this intrinsic field effect from the time dependence of the wave-function renormalization one needs a more systematic procedure to develop the instantaneous approximation.

D. Green function with wave-function renormalization

The formula (1.13) is not symmetric with respect to the times t_1 and t_2. The symmetry can be renewed if one multiplies the formula by the wave-function renormalization $z(\mathbf{k}-e\mathbf{F}t_1)$. Then the Green function reads

$$G(t_1,t_2;\mathbf{k})=-i\theta(t_1-t_2)\exp\left[-i\int_{t_2}^{t_1}dt\,(\varepsilon(\mathbf{k}-e\mathbf{F}t)+\sigma(\mathbf{k}-e\mathbf{F}t))\right]\times\sqrt{z(\mathbf{k}-e\mathbf{F}t_1)}\sqrt{z(\mathbf{k}-e\mathbf{F}t_2)}\;. \quad (1.14)$$

This form of the Green function is intuitively appealing and in this paper we will prove it is correct. If one interprets the pole part of the single-particle Green function as the quasiparticle propagation projected into the single-particle space, the pole value of the self-energy plus the free-particle kinetic energy, $\varepsilon(\mathbf{k})+\sigma(\mathbf{k})$, represents the quasiparticle dispersion relation, while the wave-function renormalization represents an overlap of the quasiparticle and free-particle wave functions. Thus one can read the formula (1.14) in the following manner. At time t_2 the free-electron wave function is projected onto the quasiparticle wave function with the momentum $\mathbf{k}$; the evolution of the quasiparticle state is evaluated up to the time t_1 and the quasiparticle state is projected back on the free-particle state. We note that a formula similar to (1.14), although for a different physical situation, was derived from the WKB solution of the Dyson equation.[10] We will call expression (1.14) the WKB approximation.

The gradient expansion of the Green function[5] provides an expression that is similar to (1.14), but instead of the geometric average of the wave-function renormalization at the initial and final times, $\sqrt{z(\mathbf{k}-e\mathbf{F}t_1)z(\mathbf{k}-e\mathbf{F}t_2)}$, the value at the center of the time interval appears, $z[\mathbf{k}-e\mathbf{F}(t_1+t_2)/2]$. To lowest order in the field in an expansion of the entire Green function these two values are identical.

Of course, the above approach is not a derivation of the formula (1.14). We have used two *ad hoc* steps: (i) the formula for the field effect on the self-energy, Eq. (1.8); and (ii) the multiplication by a wave-function renormalization associated with the time t_1. The main aim of this paper is to derive both steps in a very systematic manner. We will show that these steps are not independent: the choice of the formula for the field effect on the self-energy determines the time argument of the wave-function renormalization. However, formula (1.14) will always be the final result.

E. Content

The structure of the paper is as follows. In the first part of Sec. II we derive formulas that describe the asymptotic behavior of the Green function under quite general conditions. The main assumption used in Sec. II is an existence of a semiasymptotic region that was shown to exist in equilibrium in Ref. 7. A detailed discussion of this assumption for a homogeneous electrostatic field in GaAs is provided in Appendix A, where we adapt the approach of Ref. 7 to the nonequilibrium case. In Appendix B we restrict our attention to the equilibrium case and reorganize new formulas derived in Sec. II into more familiar forms. Section III includes a discussion of the field effect on the scattering. It is shown that one can define the instant approximation of the self-energy in a way that minimizes the field effect on the scattering. In particular, the self-energy has no linear term in the electric field.

II. ASYMPTOTIC FORM OF THE GREEN FUNCTION

In this section we study asymptotic properties of the Green function $G(t_1,t_2)$ for $\tau_{\mathrm{div}} \gg t_1-t_2 \gg \tau_{\mathrm{QF}}$, where τ_{QF} is a quasiparticle formation time and τ_{div} is the characteristic time after which the power-law decay of the Green function takes over the exponential decay.[7] (The existence of such times is demonstrated from the model in Appendix A.) Our theory works if $\tau_{\mathrm{QF}} \ll \tau \ll \tau_{\mathrm{div}}$, where τ is a quasiparticle lifetime. These asymptotic properties of the Green function provide both the formula for the instantaneous approximation of the self-energy with the field effect included and the wave-function renormalization.

In this section we construct as general an argument as possible, even if we are not able to maintain this generality in Sec. III where we explicitly continue the argument in the single-band limit in a static, uniform electric field. We do this in part because we hope others will see how to extend the arguments and in part because we think the formulation is transparent. There are aspects of the formalism that need to be commented upon. (i) For a general band structure the operators (e.g., G, Σ, etc.) are matrices in the band indices. Accordingly, the order of the operators matters and we strictly observe the correct order in this section. (ii) For a general electric field, i.e., one that varies in time, the operators [e.g., $G(t_1,t_2)$] are not a function of time differences. (Even in the simple case where we use a static, uniform field in the momentum representation, the Hamiltonian is time dependent. So the choice of the gauge can by itself produce nonstationary Hamiltonians.) Accordingly, we explicitly retain the two-time Green-function formalism. (iii) A fortuitous accident resulting from (ii) is that the time arguments indicate the matrix formalism of (i). For example,

$$\Sigma(t_1,t)G(t,t_2)\rightarrow \sum_n \langle m|\Sigma(t_1,t)|n\rangle\langle n|G(t,t_2)|m'\rangle \ .$$

Accordingly, in the formulas that follow we completely suppress the band indices and matrix character. Nonetheless, it can be readily reconstructed since the order of the operators is already clear from the time arguments.

It is shown that starting from two strictly equivalent forms of the Dyson equation,

$$G(t_1,t_2)=G_0(t_1,t_2)+\int_{t_2}^{t_1}dt_3\int_{t_2}^{t_3}dt_4G_0(t_1,t_3)\Sigma(t_3,t_4)G(t_4,t_2) \ , \tag{2.1}$$

$$G(t_1,t_2)=G_0(t_1,t_2)+\int_{t_2}^{t_1}dt_3\int_{t_2}^{t_3}dt_4G(t_1,t_3)\Sigma(t_3,t_4)G_0(t_4,t_2) \tag{2.2}$$

one arrives at two different formulations that are consistent only if the wave-function renormalization is taken into account. There are two ways to proceed in reading this paper. If the reader prefers to learn general arguments before coming to the details of the model, we advise reading this section. If the reader prefers to understand properties of the model which stimulated the individual steps made in this section and which guarantee the validity of assumptions made here, we recommend reading Appendix A first.

Our main assumption used in this section is that the self-energy decays with time t_3-t_4 at least as fast as a power law. As it is shown in Ref. 7, the power-law decay of the self-energy results in an independence of the t_4 integration in (2.1) on its lower limit t_2, assuming $\tau_{\text{QF}}<t_3-t_2<\tau_{\text{div}}$. This will allow us to make the instantaneous approximation by replacing the lower integration limit t_2 by a more convenient semiasymptotic value that is independent of t_2. In Appendix A we follow the approach introduced in Ref. 7 and prove the validity of this assumption for the electron in the central valley of the conductivity band in GaAs under the influence of the electric field. A similar argument applies for the upper limit of the t_3 integration in (2.2).

A. Effective self-energies

The quasiparticle nature of the problem is essentially the same as dealing with a non-Hermitian Hamiltonian, hence one needs to define two sets of basis functions—one conjugated to the other. This is handled by defining time-development operators—which we call effective propagators—that describe the time-development of the "left-" and "right-" hand states (the function and its conjugate). Accordingly, we introduce effective propagator–self-energy pairs for the right and left states. For example, in the case of the right states we define

$$\sigma_{\text{right}}(t_3)=\int_{t_3-3\tau}^{t_3}dt_4\Sigma(t_3,t_4)U_{\text{right}}(t_4,t_3) \ , \tag{2.3}$$

where the effective propagator $U_{\text{right}}(t_4,t_3)$ is defined as

$$\left[i\frac{\partial}{\partial t_4}-H_0(t_4)-\sigma_{\text{right}}(t_4)\right]U_{\text{right}}(t_4,t_3)=0 \ , \tag{2.4}$$

with a boundary condition

$$U_{\text{right}}(t_3,t_3)=1 \ . \tag{2.5}$$

Equations (2.3)–(2.5) form a closed system providing the self-energy Σ is known.

There are two important points to note. (i) The effective self-energy σ_{right} does not have a double-time structure. As a result the effective propagator U_{right} satisfies a group property $U_{\text{right}}(t_4,t_3)=U_{\text{right}}(t_4,t)U_{\text{right}}(t,t_3)$ for any t. (ii) There is a certain arbitrariness in the definition (2.3); why, for example, was $\sigma_{\text{right}}(t)$ evaluated at one end point of $\Sigma(t,t')$ and not the other or not the mean time? This choice, together with the one for $\sigma_{\text{left}}(t)$, is one that simplifies the derivation and allows us to simply include renormalization effects in an electric field. One has to keep this specific choice of the time-integration region in mind because it is this choice that determines the field effect on the scattering and the time argument of the wave-function renormalization.

There is a problem with the lower limit of the integration in (2.3). The value $t_3-3\tau$ provides the best semiasymptotic value that can be obtained in the equilibrium.[7] The definition (2.3) is based on our experience that the integral in (2.3) as a function of the lower limit has a plateau in which the integral does not depend on the lower limit. This question is discussed in detail in Ref. 7, and some comments can be found in Appendix A.

B. Asymptotic time development of the Green function

We have denoted the effective propagator $U_{\text{right}}(t_4,t_3)$ by the subscript "right," which denotes that the Green function decouples to the right-hand side in the sense

$$G(t_1,t_2)=U_{\text{right}}(t_1,t_5)G(t_5,t_2) \tag{2.6}$$

for $\tau_{\text{div}}\gg t_{1,5}-t_2\gg\tau_{\text{QF}}$. The validity of Eq. (2.6) can be proved as follows. From (2.4) one finds

$$\left[i\frac{\partial}{\partial t_4}-H_0(t_4)-\sigma_{\text{right}}(t_4)\right]\times(-i)\theta(t_4-t_3)U_{\text{right}}(t_4,t_3)=\delta(t_4-t_3) \ .$$

From the differential form of (2.1), $G_0^{-1}=G^{-1}+\Sigma$, where $G_0^{-1}(t_1,t_2)=[i\partial/\partial t_1-H_0(t_1)]\delta(t_1-t_2)$, one can eliminate G_0^{-1} and express the Green function $G(t_1,t_2)$ as the perturbation to the effective propagator $U_{\text{right}}(t_1,t_2)$ as

$$G(t_1,t_2)=-i\theta(t_1-t_2)U_{\text{right}}(t_1,t_2)-i\int_{t_2}^{t_1}dt_3\int_{t_2}^{t_3}dt_4 G(t_1,t_3)[\Sigma(t_3,t_4)-\sigma_{\text{right}}(t_3)\delta(t_3-t_4)]U_{\text{right}}(t_4,t_2)\ . \tag{2.7}$$

The lower limit t_2 of the t_4 integral prevents a cancellation of the second term, thus we add and subtract the integral from $t_3-3\tau$ to t_2 as

$$\begin{aligned}G(t_1,t_2)=-i\theta(t_1-t_2)U_{\text{right}}(t_1,t_2)&-i\int_{t_2}^{t_1}dt_3\int_{t_3-3\tau}^{t_3}dt_4 G(t_1,t_3)[\Sigma(t_3,t_4)-\sigma_{\text{right}}(t_3)\delta(t_3-t_4)]U_{\text{right}}(t_4,t_2)\\&+i\int_{t_2}^{t_1}dt_3\int_{t_3-3\tau}^{t_2}dt_4 G(t_1,t_3)[\Sigma(t_3,t_4)-\sigma_{\text{right}}(t_3)\delta(t_3-t_4)]U_{\text{right}}(t_4,t_2)\ .\end{aligned} \tag{2.8}$$

Then the first integral term is zero according to the definition (2.3). In the second term $\sigma_{\text{right}}(t_3)\delta(t_3-t_4)$ does not contribute due to the absence of the overlap of the integration region of t_3 and t_4 and one finds

$$G(t_1,t_2)=-i\theta(t_1-t_2)U_{\text{right}}(t_1,t_2)+i\int_{t_2}^{t_1}dt_3\int_{t_2-3\tau}^{t_2}dt_4 G(t_1,t_3)\Sigma(t_3,t_4)U_{\text{right}}(t_4,t_2)\ , \tag{2.9}$$

where we have replaced the lower integration limit $t_3-3\tau$ by the more convenient $t_2-3\tau$. This approximation is possible because the integral in the definition (2.3) does not depend on the particular choice of the lower limit of the integration unless the limit is too close to the time t_3 (closer than the quasiparticle formation time[7]) or too far from t_3 (if the length of the integrated region exceeds τ_{div}).[7]

Equation (2.9) is useful by itself, namely it is used below to derive the wave-function renormalization. In addition (2.9) can be used immediately to prove the asymptotic decoupling (2.6). To prove that (2.6) is a solution of (2.9) we substitute (2.6) in the right-hand side of (2.9)

$$G(t_1,t_2)=-i\theta(t_1-t_2)U_{\text{right}}(t_1,t_5)U_{\text{right}}(t_5,t_2)+U_{\text{right}}(t_1,t_5)i\int_{t_2}^{\min(t_1,t_5)}dt_3\int_{t_2-3\tau}^{t_2}dt_4 G(t_5,t_3)\Sigma(t_3,t_4)U_{\text{right}}(t_4,t_2)\ , \tag{2.10}$$

where we have used the semigroup property of the effective propagator to rearrange the first term. The right-hand side of (2.10) is a product of the effective propagator $U_{\text{right}}(t_1,t_5)$ and two terms which resemble (2.9) except for unessential θ function (we are in the region $t_{1,5}>0$) and the upper limit of the t_3 integration which is the lesser of t_1 and t_5 (according to θ functions implicit in the Green functions). Since the t_3 integration has the property that the integrand combines the power-law decay of the self-energy with the oscillating function from the Green function $G(t_5,t_3)$, the upper limit does not matter if $\tau_{\text{div}}\gg t_{1,5}-t_3\gg\tau_{\text{QF}}$. With t_5 as the upper limit we can use (2.9) to express both terms except via $G(t_5,t_2)$ and the decoupling (2.6) is proved.

C. Wave-function renormalization

The asymptotic form of the Green function $G(t_1,t_2)$ can be separated into a wave-function renormalization which depends only on the time t_2 and the effective propagation given by $U_{\text{right}}(t_1,t_2)$. We start from (2.9), but we need an explicit left-hand effective propagator $U_{\text{left}}(t_1,t_2)$

$$\left[i\frac{\partial}{\partial t_4}-H_0(t_4)-\sigma_{\text{left}}(t_4)\right]U_{\text{left}}(t_4,t_3)=0\ , \tag{2.11}$$

where (note the different region of the time integral)

$$\sigma_{\text{left}}(t_3)=\int_{t_3}^{t_3+3\tau}dt_4 U_{\text{left}}(t_3,t_4)\Sigma(t_4,t_3)\ , \tag{2.12}$$

and the boundary condition is

$$U_{\text{left}}(t_3,t_3)=1\ . \tag{2.13}$$

The asymptotic behavior of the Green function $G(t_1,t_3)$ for $t_1-t_3\gg\tau_{\text{QF}}$ is similar to that of (2.6)

$$G(t_1,t_3)=G(t_1,t_2)U_{\text{left}}(t_2,t_3)\ , \tag{2.14}$$

where $t_1-t_2\gg\tau_{\text{QF}}$.

Now we are ready to derive the wave-function renormalization. Using asymptotic form (2.14) in Eq. (2.9) one finds

$$\begin{aligned}G(t_1,t_2)\left[1-i\int_{t_2}^{t_1}dt_3\int_{t_2-3\tau}^{t_2}dt_4 U_{\text{left}}(t_2,t_3)\right.&\\ \left.\times\Sigma(t_3,t_4)U_{\text{right}}(t_4,t_2)\right]&\\ =-i\theta(t_1-t_2)U_{\text{right}}(t_1,t_2)\ .&\end{aligned} \tag{2.15}$$

The term in large parentheses does not depend on t_1 except via the upper integration limit of the t_3 integration. This dependence is negligible for $t_1-t_2\gg\tau_{\text{QF}}$ and we can use a more convenient value $t_2+3\tau$. After this approximation the term in large parentheses depends exclusively on t_2.

From the asymptotic formula (2.15) we can define the function $z(t_2)$,

$$z^{-1}(t_2)=1-i\int_{t_2}^{t_2+3\tau}dt_3\int_{t_2-3\tau}^{t_2}dt_4 U_{\text{left}}(t_2,t_3)\Sigma(t_3,t_4)U_{\text{right}}(t_4,t_2)\ . \tag{2.16}$$

In Appendix B it is shown in *equilibrium* that z is the standard wave-function renormalization. We will call z the wave-function renormalization even with the external field present.

With the help of the defined wave-function renormalization, the asymptotic form of the Green function reads

$$G(t_1,t_2)=-i\theta(t_1-t_2)U_{\text{right}}(t_1,t_2)z(t_2)\ . \tag{2.17}$$

One can repeat the entire procedure leading to (2.17), but starting with left states. Then one recovers

$$G(t_1,t_2)=-iz(t_1)\theta(t_1-t_2)U_{\text{left}}(t_1,t_2)\ . \tag{2.18}$$

Note that the wave-function renormalizations in (2.17) and (2.18) have different time arguments. Thus we have shown that the time argument of the wave-function renormalization is determined by the choice of the definition of the instantaneous approximation.

Having the Green function expressed in two equivalent forms (2.17) and (2.18) provides us with a useful identity

$$U_{\text{right}}(t_1,t_2)z(t_2)=z(t_1)U_{\text{left}}(t_1,t_2)\ . \tag{2.19}$$

This allows one to set renormalization at any convenient time, for example,

$$G(t_1,t_2)=-i\theta(t_1-t_2)U_{\text{right}}(t_1,t_3)z(t_3)U_{\text{left}}(t_3,t_2)\ . \tag{2.20}$$

To resume the result of this section, we have assumed that the integral of the self-energy multiplied by the oscillating Green function reaches some constant value in a characteristic time τ_{QF} which is much smaller than the quasiparticle lifetime τ. Based on this assumption, we have shown that for $t_1-t_2 \gg \tau_{\text{QF}}$ the Green function $G(t_1,t_2)$ can be expressed as the product of the quasiparticle propagator $U_{\text{right}}(t_1,t_2)$ [or $U_{\text{left}}(t_1,t_2)$] and the wave-function renormalization $z(t_2)$ [or $z(t_1)$]. The quasiparticle propagators have the desirable property that they are solutions of the true differential equation, i.e., the effective self-energy is the instantaneous operator.

The relation of the definitions (2.3), (2.12), and (2.16) to the pole approximation used in equilibrium is discussed in Appendix B.

III. RELATION OF THE FIELD EFFECT ON THE SCATTERING RATE AND THE WAVE-FUNCTION RENORMALIZATION

In this section the asymptotic formulas for the Green functions developed in the preceding section are used to discuss the field effect on the scattering. We assume a homogeneous electrostatic field described by a vector potential $\mathbf{A}(t)=-\mathbf{F}t$. The vector gauge has an advantage that all operators are diagonal in the momentum representation. We keep the representation explicit throughout the section.

A. Gauge invariance

To start we simplify the notation by eliminating the time argument of the self-energy and the wave-function renormalization. This simplification is possible due to the gauge invariance. Since the gauge invariance allows us to add any constant potential, we can add $\mathbf{A}_0=\mathbf{F}t_0$. This change of the gauge appears in the shift of the canonical momentum $\mathbf{p}\rightarrow\mathbf{p}-\mathbf{F}t_0$. If we shift simultaneously the origin of the time axis to t_0, any equation is identical with its form before both changes. One finds that all operators satisfy the invariance

$$\sigma_{\text{right}}(t_0;\mathbf{k})=\sigma_{\text{right}}(0;\mathbf{k}-e\mathbf{F}t_0)\equiv\sigma_{\text{right}}(\mathbf{k}-e\mathbf{F}t_0)\ . \tag{3.1}$$

The last part of the equation is a *definition* of the reduced notation. Briefly, the reduced functions can be recognized by the absence of the time argument. The notation used in Sec. I is identical with this reduction notation.

B. WKB approximation of the Green function

First we confirm the WKB formula (1.14) as a linear approximation of the field effect on the self-energy. The lowest order field effect on σ_{right} results from a linearization of (2.3). However, we do not need to perform the algebra since comparing (1.8) with (2.3) one can see that $\sigma_F(\mathbf{k}-e\mathbf{F}t)=\sigma_{\text{right}}(t;\mathbf{k})$. Thus the field effect on σ_{right} is given by Eq. (1.11). The lowest order of the field effect on σ_{left}, results from a linearization of (2.12),

$$\sigma_{\text{left}}(\mathbf{k}-e\mathbf{F}t)=\sigma(\mathbf{k}-e\mathbf{F}t)+z(\mathbf{k}-e\mathbf{F}t)ie\mathbf{F}\frac{\partial[\varepsilon(\mathbf{k}-e\mathbf{F}t)+\sigma(\mathbf{k}-e\mathbf{F}t)]}{\partial\mathbf{k}}\frac{1}{2}\frac{\partial^2\Sigma(\omega)}{\partial\omega^2}\bigg|_{\omega=\varepsilon(\mathbf{k}-e\mathbf{F}t)+\sigma(\mathbf{k}-e\mathbf{F}t)}\ . \tag{3.2}$$

Note the opposite signs of the field effect in (1.11) and (3.2). These opposite signs have resulted from the different integral regions in the time integral in definitions (2.3) and (2.12). From the different signs it is clear that one should not interpret the linear corrections in (1.11) and (3.2) as the field effect on the scattering. Indeed, we have already shown that it is the time dependence of the wave-function renormalization that is described by these linear terms.

Now we can evaluate the effective propagators $U_{\text{right,left}}$. Solving (2.4) [or (2.11)], one obtains

$$U_{\text{right,left}}(t_1,t_2;\mathbf{k})=\exp\left[-i\int_{t_2}^{t_1}dt(\varepsilon(\mathbf{k}-e\mathbf{F}t)+\sigma_{\text{right,left}}(\mathbf{k}-e\mathbf{F}t))\right]\ .$$

Again we express the terms linear in the field via the derivative of the wave-function renormalization, which yields

$$U_{\text{right}}(t_1,t_2;\mathbf{k})=\exp\left[-i\int_{t_2}^{t_1}dt(\varepsilon(\mathbf{k}-e\mathbf{F}t)+\sigma(\mathbf{k}-e\mathbf{F}t))\right]\times\left[\frac{z(\mathbf{k}-e\mathbf{F}t_2)}{z(\mathbf{k}-e\mathbf{F}t_1)}\right]^{1/2} \tag{3.3}$$

and

$$U_{\mathrm{left}}(t_1,t_2;\mathbf{k})=\exp\left[-i\int_{t_2}^{t_1}dt(\varepsilon(\mathbf{k}-e\mathbf{F}t)+\sigma(\mathbf{k}-e\mathbf{F}t))\right]\times\left[\frac{z(\mathbf{k}-e\mathbf{F}t_1)}{z(\mathbf{k}-e\mathbf{F}t_2)}\right]^{1/2} . \qquad (3.4)$$

Note that the denominator and the numerator are interchanged in (3.4) compared to (3.3). While it was inconsistent to discuss the field effect on the scattering with wave-function renormalization neglected, it is easy to see that the field effect on the scattering is consistent with the formula (2.19). From both (2.17) and (2.18) one recovers the formula (1.14).

C. The asymptotic form of the Green function beyond the WKB approximation

In fact in the homogeneous electric field a formula identical to (1.14),

$$G(t_1,t_2;\mathbf{k})=-i\theta(t_1-t_2)\sqrt{z(\mathbf{k}-e\mathbf{F}t_1)z(\mathbf{k}-e\mathbf{F}t_2)}\times\exp\left[-i\int_{t_2}^{t_1}dt(\varepsilon(\mathbf{k}-e\mathbf{F}t)+\sigma(\mathbf{k}-e\mathbf{F}t))\right] \qquad (3.5)$$

applies even if the nonlinear terms of the gradient expansion of the self-energy are taken into account. In such a case the effective self-energy σ and the wave-function renormalization have to be treated as field dependent. Since we want to remove the linear field dependence of the effective self-energy, we define the field-dependent generalization of σ as

$$\sigma(\mathbf{k})=\tfrac{1}{2}[\sigma_{\mathrm{right}}(\mathbf{k})+\sigma_{\mathrm{left}}(\mathbf{k})] . \qquad (3.6)$$

The field-dependent z is defined by (2.16). Note that in (3.6) the instant of the scattering event is placed in the center of time-integration regions, in contrast to the definitions (2.3) and (2.12). By this property the definition (3.6) is close in spirit to the pole approximation used within Wigner mixed energy-time representation.

To prove a validity of (3.5) beyond the linear term we need to show that the relation (1.12) is more general than the way in which it was derived in the Introduction suggests. From (2.19) [using (2.4) and (2.11)] one obtains

$$\frac{\partial z^{-1}(t)}{\partial t}=i\{z^{-1}(t)[H_0(t)+\sigma_{\mathrm{right}}(t)]-[H_0(t)+\sigma_{\mathrm{left}}(t)]z^{-1}(t)\} . \qquad (3.7)$$

In the homogeneous field described by the vector potential all operators commute so that Eq. (3.7) provides an equivalent to Eq. (1.12),

$$z(t)\frac{\partial z^{-1}(t)}{\partial t}=i[\sigma_{\mathrm{right}}(t)-\sigma_{\mathrm{left}}(t)] . \qquad (3.8)$$

Note that the validity of (3.8) is restricted to a single-band model and homogeneous fields. However, stationarity is not used in this step.

From (3.8) and the definition (3.6) one can see that the difference between the right (or left) self-energy and the σ is equal to the half of the logarithmic derivative of the wave-function renormalization. Now one can follow the steps used in (1.13) and prove the validity of (3.5) for any field strength.

D. Symmetry in the orientation of the electric field

The simplicity of (3.5) within the linear approximation results from the symmetry of the effective self-energies and the wave-function renormalization with respect to change of the sign of the electrical field. Now we will show that the wave-function renormalization and the effective self-energy σ [defined by (3.6)] do not change if the sign of the electric field is flipped. In other words, z and σ are even function of the electrical field. This property follows from our model: Einstein (dispersionless) phonons and momentum-independent electron-phonon interaction.

The symmetry of the wave-function renormalization and the effective self-energy follows from the symmetry

$$\sigma_{\mathrm{left}}(\mathbf{k};[-\mathbf{F}])=\sigma_{\mathrm{right}}(\mathbf{k};[\mathbf{F}]) , \qquad (3.9)$$

where the square brackets denote the functional dependence. We will prove this symmetry from the definitions (2.3) and (2.12). We start from (2.3).

The retarded self-energy $\Sigma(t_3,t_4)$ depends only on the time difference t_3-t_4 independently from the field strength. At the same time Σ does not depend on the sign of the field. This property is shown in Appendix A; see (A9). One can understand the absence of the dependence on t_3+t_4 from the fact that the self-energy depends only on a local density of states that is a momentum-independent quantity. Since the dependence on t_3+t_4 can be moved into the momentum dependence, see (3.1), the momentum independence implies the independence from t_3+t_4. The independence of $\Sigma(t_3,t_4)$ from t_3+t_4 allows us to rearrange (2.3) as

$$\sigma_{\mathrm{right}}(\mathbf{k})=\int_{-3\tau}^{0}dt_4\Sigma(0,t_4)U_{\mathrm{right}}(t_4,0;\mathbf{k})=\int_{0}^{3\tau}dt_4U_{\mathrm{right}}(-t_4,0;\mathbf{k})\Sigma(t_4,0) , \qquad (3.10)$$

where we set $t_3=0$ without losing generality, see (3.1), and take the substitution $t_4\to -t_4$. We have used the diagonality to interchange the order of the product. Clearly we need to turn U_{right} into U_{left}.

First we will show that the symmetry (3.8) is satisfied if the propagator is approximated by its free particle value. Then we will use the iterative argument to show that the symmetry applies to any order. Flipping of the direction of the electrical field in the free-particle propagator is identical to the inversion of the time flow,

$$U_0(-t_4,0;\mathbf{k};[\mathbf{F}])=\exp\left[-i\int_{-t_4}^{0}d\bar{t}\,\varepsilon(\mathbf{k}-e\mathbf{F}\bar{t})\right]=\exp\left[-i\int_{0}^{t_4}d\bar{t}\,\varepsilon(\mathbf{k}+e\mathbf{F}\bar{t})\right]=U_0(0,t_4;\mathbf{k};[-\mathbf{F}]) . \qquad (3.11)$$

After a substitution of (3.11) into (3.10) one finds that the symmetry (3.9) is satisfied within the free-particle approximation of the propagator.

Now we assume an iteration of Eqs. (2.3) and (2.4) that starts from the free-particle propagator in the first step. Let us assume that the symmetry (3.9) is satisfied in some order; we will show that it is satisfied in the next one as well. The propagator $U_{\text{right}}(-t_4,0;\mathbf{k};[\mathbf{F}])$ can be recognized as

$$\begin{aligned} U_{\text{right}}(-t_4,0;\mathbf{k};[\mathbf{F}]) &= \exp\left[-i\int_{-t_4}^{0} d\tilde{t}\,(\varepsilon(\mathbf{k}-e\mathbf{F}\tilde{t})+\sigma_{\text{right}}(\mathbf{k}-e\mathbf{F}\tilde{t};[\mathbf{F}]))\right] \\ &= \exp\left[-i\int_{0}^{t_4} d\tilde{t}\,(\varepsilon(\mathbf{k}+e\mathbf{F}\tilde{t})+\sigma_{\text{right}}(\mathbf{k}+e\mathbf{F}\tilde{t};[\mathbf{F}]))\right]. \end{aligned} \tag{3.12}$$

According to (3.9) the right effective self-energy can be replaced by the left effective self-energy

$$\sigma_{\text{right}}(\mathbf{k}+e\mathbf{F}\tilde{t};[\mathbf{F}])=\sigma_{\text{left}}\{\mathbf{k}-e(-\mathbf{F})\tilde{t};[-\mathbf{F}]\}\ . \tag{3.13}$$

After the substitution of (3.13) into (3.12) one recovers the relation

$$U_{\text{right}}(-t_4,0;\mathbf{k};[\mathbf{F}])=U_{\text{left}}(0,t_4;\mathbf{k};[-\mathbf{F}])\ , \tag{3.14}$$

which substituted into (3.10) proves the symmetry (3.9) in a higher order of the iteration. Thus the symmetry (3.9) holds for any order.

From the symmetry (3.9) the even dependence of $\sigma(0;\mathbf{k})$ on the field strength follows immediately,

$$\sigma(\mathbf{k})=\tfrac{1}{2}[\sigma_{\text{right}}(\mathbf{k})+\sigma_{\text{left}}(\mathbf{k})]=\tfrac{1}{2}[\sigma_{\text{right}}(\mathbf{k};[\mathbf{F}])+\sigma_{\text{right}}(\mathbf{k};[-\mathbf{F}])]\ . \tag{3.15}$$

We prove the even field dependence of $z(\mathbf{k})$ from the definition (2.16). Let us write the wave-function renormalization with the flipped sign of the field

$$z^{-1}(\mathbf{k};[-\mathbf{F}])=1-i\int_{0}^{3\tau}dt_3\int_{-3\tau}^{0}dt_4\,U_{\text{left}}(0,t_3;\mathbf{k};[-\mathbf{F}])\Sigma(t_3,t_4)U_{\text{right}}(t_4,0;\mathbf{k};[-\mathbf{F}])\ . \tag{3.16}$$

From the symmetry (3.14) one obtains

$$z^{-1}(\mathbf{k};[-\mathbf{F}])=1-i\int_{0}^{3\tau}dt_3\int_{-3\tau}^{0}dt_4\,U_{\text{right}}(-t_3,0;\mathbf{k};[\mathbf{F}])\Sigma(t_3,t_4)U_{\text{left}}(0,-t_4;\mathbf{k};[\mathbf{F}])\ . \tag{3.17}$$

After substitutions $t_3 \rightarrow -t_3$ and $t_4 \rightarrow -t_4$ Eq. (3.17) reads

$$z^{-1}(\mathbf{k};[-\mathbf{F}])=1-i\int_{-3\tau}^{0}dt_3\int_{0}^{3\tau}dt_4\,U_{\text{left}}(0,t_4;\mathbf{k};[\mathbf{F}])\Sigma(-t_3,-t_4)U_{\text{right}}(t_3,0;\mathbf{k};[\mathbf{F}])\ . \tag{3.18}$$

Since the self-energy depends only on the difference in time, $\Sigma(-t_3,-t_4)=\Sigma(t_4,t_3)$. Equation (3.18) is identical to (2.16) evaluated for the field with an unflipped sign, therefore

$$z^{-1}(\mathbf{k};[-\mathbf{F}])=z^{-1}(\mathbf{k};[\mathbf{F}])\ . \tag{3.19}$$

The even symmetry of $\sigma(\mathbf{k})$ and $z(\mathbf{k})$ shows that the main field effect (linear order) is given by the time dependence of the momentum argument in the formula (1.14). Within the linear approximation of the field effect on the self-energy the WKB formula is correct. The intrinsic field effect requires a study of the field dependence of $\sigma(\mathbf{k})$. The electrical fields for which the majority of electrons is in the central valley of GaAs is below 3×10^5 V/m. For these fields the field effect on the scattering is rather small and the nonlinear corrections are negligible. More details and results of the numerical evaluation of the right and left effective self-energies are in Appendix A; see Fig. 1.

IV. CONCLUSIONS

In the paper we have discussed the field effect on the total scattering rate in semiconductors. We have shown that the definition of the instantaneous approximation of the self-energy, a necessary step to introduce scattering rates, is not unique. The particular choice of the definition of the instantaneous approximation is very important because the linear order of the field dependence of the instantaneous self-energy is determined by this choice. In particular, there is a definition within which the field effect on the instantaneous approximation of the self-energy is even in powers of the field strength so that the linear order contribution is zero. Since the superlinear terms are small for electric field typically met in GaAs devices, one can use equilibrium values of the self-energy and the wave-function renormalization.

ACKNOWLEDGMENTS

This work was supported in part by the U.S. Office of Naval Research.

APPENDIX A: QUASIPARTICLE FORMATION IN THE ELECTRICAL FIELD

In this appendix we discuss the characteristic time scales that appear in the Dyson equation if the stationary homogeneous electrical field is taken into account. There are two main questions of the discussion: (i) Is there any characteristic time scale that will allow us to use the instantaneous approximation of the self-energy instead of the true double-time function? (ii) How should the instantaneous approximation of the self-energy be defined beyond the gradient expansion, i.e., if the pole approximation (1.3) is unacceptable? The first question addresses the possibility of introducing the effective self-energies (2.3) and (2.12) at all. The second question addresses forms of (2.3) and (2.12), in particular, why *two* definitions are necessary, while the instantaneous scattering rates are uniquely defined by the Fermi golden rule.

1. Free-electron Hamiltonian within the vector gauge

For the sake of simplicity we will assume in our discussion one parabolic band of electrons with the dispersion relation $\varepsilon(\mathbf{k})=k^2/2m$. In our estimates we take $m=5\times10^{-32}$ kg, that is, the mass in the Γ valley of the GaAs. An extension of the model including intervalley scattering leads to a modification of the regions of validity of the instant approximation; however, the overall picture does not change. This extension is presented in Appendix C.

For the electrical field $\mathbf{F}$ we take 3×10^5 V/m for the field effect on the scattering rate and 3×10^4 V/m for the case when the electron is decelerated so that it crosses the optic-phonon-emission threshold. Both fields are below the onset of the Gunn effect. The smaller magnitude of the field in the case of the threshold crossing is chosen to amplify the effect [see the discussion following Eq. (A23)].

The electrical field is described within the vector gauge $\mathbf{A}(t)=-\mathbf{F}t$. The vector gauge has the advantage of keeping the Hamiltonian transparently translation invariant; therefore the Hamiltonian is diagonal in the momentum representation $\mathbf{k}$. The price one has to pay is the time dependence of the Hamiltonian that appears in spite of the stationarity of the field.[1,6] The free-electron Hamiltonian within the vector gauge reads

$$H_0(t;\mathbf{k})=\varepsilon[\mathbf{k}+e\,\mathbf{A}(t)]=\frac{|\mathbf{k}-e\mathbf{F}t|^2}{2m}\ . \qquad \text{(A1)}$$

2. Dyson equation

The retarded Green function is the solution of the Dyson equation

$$G(t_1,t_2;\mathbf{k})=G_0(t_1,t_2;\mathbf{k})+\int_{t_2}^{t_1}dt_3\int_{t_2}^{t_3}dt_4G_0(t_1,t_3;\mathbf{k})\times\Sigma(t_3,t_4)G(t_4,t_2;\mathbf{k})\ . \qquad \text{(A2)}$$

In (A2) we have eliminated the momentum dependence of the self-energy restricting our attention to the interaction local in space such as the optic-phonon scattering that will be the main concern of this paper.

Beside Eq. (A2) there is an alternative form of the Dyson equation that reads

$$G(t_1,t_2;\mathbf{k})=G_0(t_1,t_2;\mathbf{k})+\int_{t_2}^{t_1}dt_4\int_{t_4}^{t_1}dt_3G(t_1,t_3;\mathbf{k})\times\Sigma(t_3,t_4)G_0(t_4,t_2;\mathbf{k})\ . \qquad \text{(A3)}$$

The two forms (A2) and (A3) are strictly equivalent, but *ad hoc* estimates of the field effect depend on which form was used as a starting point. We will show that the consistency of both approaches can be maintained only if the wave-function renormalization is taken into account.

3. Free-electron Green function

The free-electron Green function $G_0(t_1,t_3)$ has a solution of the free-particle Dyson equation

$$\left[i\frac{\partial}{\partial t_1}-H_0(t_1;\mathbf{k})\right]G_0(t_1,t_3;\mathbf{k})=\delta(t_1-t_3)\ , \qquad \text{(A4)}$$

that is[11]

$$\begin{aligned}G_0(t_1,t_3;\mathbf{k})&=-i\theta(t_1-t_3)\exp\left[-i\int_{t_3}^{t_1}dt_5H_0(t_5;\mathbf{k})\right]\\&=-i\theta(t_1-t_3)e^{-i(k_x^2/2m)(t_1-t_3)}e^{-i(k_y^2/2m)(t_1-t_3)}e^{-i(k_z^2/2m)(t_1-t_3)}e^{i(k_zeF/2m)(t_1^2-t_3^2)}e^{-i(1/3)[(eF)^2/2m](t_1^3-t_3^3)}\ .\end{aligned} \qquad \text{(A5)}$$

where we have associated the direction of the field with the z direction.

4. Self-energy in the external field

The typical lifetime of electrons in the central Γ valley above the optic-phonon-emission threshold is 2×10^{-13} s.[12] Our parametrization and estimates will be based on this value.

In our model we adopt two simplifications to the self-energy: (i) We assume only single optic-phonon interactions. Optic phonons are described by a single flat band, i.e., by the Einstein model, and a constant interaction vertex γ. (ii) The density of electrons is infinitely small and therefore the phonon distribution can be assumed to be in equilibrium. Under these simplifications the self-energy reads

$$\Sigma(t_3,t_4)=\gamma^2D(t_3-t_4)\int\frac{d\mathbf{k}}{(2\pi)^3}G(t_3,t_4;\mathbf{k})\ , \qquad \text{(A6)}$$

where the optic-phonon correlation function D is

$$D(t)=\frac{N}{2\omega_0}e^{-i\omega_0 t}+\frac{N+1}{2\omega_0}e^{i\omega_0 t}\,, \tag{A7}$$

$N=1/[\exp(\omega_0/k_B T)-1]$ is the Bose-Einstein distribution, and ω_0 is the optic-phonon frequency.

As the simplest approximation of the self-energy one can neglect the self-consistency and use the free-electron Green function for the internal line of the self-energy. Below we will take self-consistency into account in an approximative way.

The local element of the free-electron Green function $\int dk\, G_0$ can be evaluated from (A5) with the help of the Fresnel integrals

$$\int \frac{d\mathbf{k}}{(2\pi)^3}G_0(t_3,t_4;\mathbf{k})=-i\theta(t_3-t_4)n_b[i(t_3-t_4)]^{-3/2} \times e^{-i(1/12)[(eF)^2/2m](t_3-t_4)^3}\,, \tag{A8}$$

where the prefactor is $n_b=(m/2\pi)^{3/2}$.

The density of states of the infinite parabolic band is growing in the high-energy region. This limiting behavior causes a divergence in the real part of the self-energy. To remove this divergence we will use an exponential cutoff of the density of states; therefore the actual density of states

$$-2\,\mathrm{Im}\int d\mathbf{k}(2\pi)^{-3}G_0(\omega;\mathbf{k})=4\sqrt{\pi}n_b\sqrt{\omega}$$

will be replaced by

$$-2\,\mathrm{Im}\int d\mathbf{k}(2\pi)^{-3}G_0(\omega;\mathbf{k})=4\sqrt{\pi}n_b\sqrt{\omega}\exp(-\omega/\omega_{\mathrm{cut}})\,,$$

where $\omega_{\mathrm{cut}}\sim 2$ eV is of the order of the bandwidth. The exponential cutoff in the time representation results in the removal of the short-time singularity of the density of states by replacing $[i(t_3-t_4)]^{-3/2}$ by the regular function $[1/\omega_{\mathrm{cut}}+i(t_3-t_4)]^{-3/2}$. For simplicity of the notation the cutoff is eliminated from the formulas in the paper, but it is used in plots. Accordingly, we will ignore the singular behavior in the limit $t_3-t_4\rightarrow 0$. More details about the cutoff and the self-energy used can be found in Ref. 7.

According to (A6) and (A8), the self-energy is a function only of the time difference t_3-t_4,

$$\Sigma_0(t_3-t_4)=-i\theta(t_3-t_4)\frac{\alpha}{2\sqrt{\pi}}[Ne^{-i\omega_0(t_3-t_4)}+(N+1)e^{i\omega_0(t_3-t_4)}][i(t_3-t_4)]^{-3/2}e^{-i(1/12)[(eF)^2/2m](t_3-t_4)^3}\,. \tag{A9}$$

In (A9) we have used the prefactor $\alpha=2\sqrt{\pi}\gamma^2 n_b/\omega_0$, which corresponds to the parametrization of the self-energy usually used within the energy representation where $-2\,\mathrm{Im}\Sigma(\omega)=\alpha[N\theta(\omega+\omega_0)\sqrt{\omega+\omega_0}+(N+1)\theta(\omega-\omega_0)\sqrt{\omega-\omega_0}]$.

Note that the self-energy does not depend on $\mathbf{F}$ in the linear order. This confirms the assumption made in Secs. II and III that self-energy has its equilibrium value within the lowest-order approximation.

5. Self-consistent self-energy

In the formula (A6) we have used the free-electron Green function to evaluate the self-energy. As we have shown in (A9), the nonanalytical square-root behavior of the density of states near the band edge results in the power-law decay of the self-energy. Since the long-time decay of the self-energy is crucial in a discussion of the instantaneous approximation, one can improve the formula (A6) by taking into account the final lifetime of the electron in the vicinity of the band edge. In the simplest manner one can approximate the scattering rate by a momentum-independent constant that is fitted to the crucial value at the band edge. Then the free-electron Green function $G_0(t_3,t_4;\mathbf{k})$ in (A6) is replaced by

$$G(t_3,t_4;\mathbf{k})\approx G_0(t_3,t_4;\mathbf{k})\exp[-(t_3-t_4)/2\tau_0]\,,$$

where the lifetime near the band edge τ_0 approximately reads $1/\tau_0=-2\,\mathrm{Im}\Sigma(\omega=0)$. Within this approximation the self-energy (A9) has an additional multiplicative time dependence

$$\Sigma(t_3-t_4)=\Sigma_0(t_3-t_4)\exp[-(t_3-t_4)/2\tau_0]\,. \tag{A10}$$

The lifetime near the band edge τ_0 is entirely due to absorption processes. Accordingly, τ_0 depends on the temperature via the Bose-Einstein distribution. Since at the room temperature $N=0.27$, the lifetime on the band edge can be neglected for electrons above the optic-phonon-emission threshold. Below the threshold and in its vicinity the self-consistency provides an important correction that supports the applicability of the instantaneous approximation of the self-energy.

Let us check that the lifetime on the band edge is the relevant one for the long-time behavior of the self-energy. To this end we need to evaluate the local element of the Green function with the lifetime included. Within the pole approximation the Green function reads

$$\begin{aligned}\int \frac{d\mathbf{k}}{(2\pi)^3}G(t_3,t_4;\mathbf{k})&=-i\theta(t_3-t_4)\int \frac{d\mathbf{k}}{(2\pi)^3}\exp\left[-i\int_{t_4}^{t_3}dt\,(\mathbf{k}-e\mathbf{F}t)^2/2m\right]\exp\left[-\int_{t_4}^{t_3}dt/2\tau(\mathbf{k}-e\mathbf{F}t)\right]\\&=-\theta(t_3-t_4)e^{-i(1/12)[(eF)^2/2m](t_3-t_4)^3}\\&\quad\times\int\frac{dk_x}{2\pi}\int\frac{dk_y}{2\pi}e^{-i(k_x^2+k_y^2)(t_3-t_4)/2m}\\&\quad\times\int\frac{dk_z}{2\pi}e^{-i[k_z-eF(t_3+t_4)/2]^2(t_1-t_2)/2m}\exp\left[-\int_{t_4}^{t_3}dt/2\tau(\mathbf{k}-e\mathbf{F}t)\right]\,.\end{aligned} \tag{A11}$$

For large t_3-t_4 the integration over the perpendicular part of the momentum has an appreciable value only for $k_{x,y}^2<2m/(t_3-t_4)$, therefore only the vicinity of the band edge in the perpendicular momentum counts. In the parallel direction the momentum in the center of the time interval has to be close to the band edge, thus the zero-momentum value of the lifetime is the best approximation for the decay term, which confirms the approximation (A10).

6. Instantaneous approximation of the self-energy

Now we are ready to discuss the two questions stated in the introduction to this appendix. We will start with the second one: Assuming that the instantaneous approximation exists, what is its form? To answer this question we will follow the method developed in Ref. 7. By the following discussion we aim to motivate the definitions (2.3) and (2.12).

First, we introduce the effective self-energy $\sigma_{\text{right}}(t;\mathbf{k})$ that is instantaneous (local in the time variables or energy independent) and approximate the true double-time (i.e., energy-dependent) self-energy Σ. After the substitution of this effective self-energy into the Dyson equation (A2),

$$G(t_1,t_2;\mathbf{k})=G_0(t_1,t_2;\mathbf{k})+\int_{t_2}^{t_1}dt_3G_0(t_1,t_3;\mathbf{k})\times\sigma_{\text{right}}(t_3;\mathbf{k})G(t_3,t_2;\mathbf{k}) \ . \tag{A12}$$

One finds by comparing (A12) with (A2) that σ_{right} should satisfy

$$\sigma_{\text{right}}(t_3;\mathbf{k})G(t_3,t_2;\mathbf{k})=\int_{t_2}^{t_3}dt_4\Sigma(t_3,t_4)G(t_4,t_2;\mathbf{k}) \ . \tag{A13}$$

The condition (A13) cannot be satisfied for arbitrary t_3 and t_2, but there is a region of times for which (A13) holds. To find the region of t_3 and t_2 where the approximation (A13) applies we divide the right-hand side by $G(t_3,t_2;\mathbf{k})$ and plot as a test function

$$s_{\text{right}}(t_3;\mathbf{k};t_2)=\int_{t_2}^{t_3}dt_4\Sigma(t_3,t_4)G(t_4,t_2;\mathbf{k})/G(t_3,t_2;\mathbf{k}) \ . \tag{A14}$$

The plot of s_{right} as a function of t_2 for fixed t_3 and $\mathbf{k}$ is in Fig. 1. Apparently there is a plateau in which the approximation (A14) holds and the plateau value of s_{right} is identified as σ_{right}.

Before we discuss test function s_{right} in the analytical way, let us check whether the condition (A13) results if one starts from (A3). To distinguish these two forms we introduce a new effective self-energy $\sigma_{\text{left}}(t;\mathbf{k})$. After the substitution of $\sigma_{\text{left}}(t;\mathbf{k})$ into (A3) and the similar comparison with the nonapproximative form (A3) one finds that $\sigma_{\text{left}}(t;\mathbf{k})$ should satisfy

$$G(t_1,t_3;\mathbf{k})\sigma_{\text{left}}(t_4;\mathbf{k})=\int_{t_4}^{t_1}dt_3G(t_1,t_3;\mathbf{k})\Sigma(t_3,t_4) \ . \tag{A15}$$

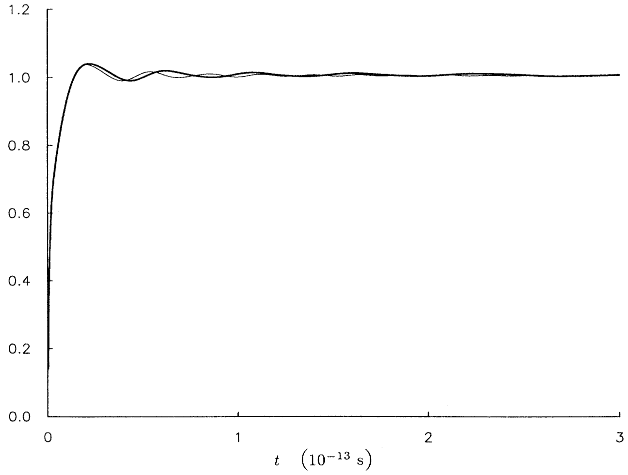

FIG. 1. Test functions $s_{\text{right}}(0;\mathbf{k};t)$ (thick line) and $s_{\text{left}}(0;\mathbf{k};t)$ (thin line) vs t for the electric field $F=3\times10^5$ V/m. For comparison with the equilibrium value we plot $\text{Im}s/\text{Im}\sigma(\mathbf{k})$ so that the equilibrium value is reproduced if the plateau reaches 1. The characteristic times are quasiparticle formation $\tau_{\text{QF}}=\hbar/[\varepsilon(\mathbf{k})-\omega_0]=10^{-14}$ s, quasiparticle lifetime $\tau=-\hbar/2\,\text{Im}\sigma(\mathbf{k})=1.5\times10^{-13}$ s. The momentum $\mathbf{k}=(0,0,k_z)$ corresponds to the energy $\varepsilon(\mathbf{k})=4\omega_0=0.16$ eV and is chosen so that the field decelerates the electron. Both curves can be approximated by a plateau with the value 1 for $t>2\tau_{\text{QF}}$. The averaged value of s_{left} is a little smaller than the averaged value of s_{right} due to the field effect on the scattering; however, in both cases the field effect is very small.

That is different from (A13). Introducing the similar test function

$$s_{\text{left}}(t_4;\mathbf{k};t_2)=\int_{t_4}^{t_1}dt_3G(t_1,t_3;\mathbf{k})\Sigma(t_3,t_4)/G(t_1,t_4;\mathbf{k}) \ , \tag{A16}$$

one finds that there is a region of the validity of (A15), but the plateau value of s_{left} is different from s_{right}. Thus the effective self-energies σ_{right} and σ_{left} are *different*. The test function s_{left} is plotted in Fig. 1 for $t_4=t_3$ [t_4 is from formula (A16), while t_3 is from formula (A14)] and the same value of $\mathbf{k}$ as s_{right}. From Fig. 1 one can see that the difference between the plateau value of s_{right} and s_{left} is rather small. This is due to a small magnitude of the field effect on the scattering. We did not use a stronger electric field to make the difference between s_{right} and s_{left} more visible since then the electron is driven across the intervalley-scattering threshold and the asymptotic behavior becomes more complex.

The inconsistency of the conditions (A13) and (A15) does not mean that the instantaneous approximation is not possible. The only restriction that follows from this disagreement is that one has only to specify which form of the Dyson equation is being used.

7. Estimate of the applicability of the instantaneous approximation

In Fig. 1 one can see that the instantaneous approximation fails for very short times. Similarly the instantaneous approximation fails in the very long-time region that is absent in Fig. 1 to keep the appropriate scale for the more important short-time region. The long-time failure can be seen from the formula (A20) and the reader can find more details in Ref. 7. The region of applicability can be estimated analytically from the test function s_{right}.

For the purpose of the estimate we can neglect the field effect on the scattering rate and use the Green function (1.6) in the formula (A14). Within the approximation (1.6) the ratio of propagators simplifies as

$$G(t_4,t_2;\mathbf{k})/G(t_3,t_2;\mathbf{k}) = \exp\left[i\int_{t_4}^{t_3} dt(\varepsilon(\mathbf{k}-e\mathbf{F}t)+\sigma(\mathbf{k}-e\mathbf{F}t)) \right] . \tag{A17}$$

Within (A17) the ratio of the Green functions is independent from t_2 and the entire t_2 dependence of s_{right} comes from the integration limit. Note that the ratio of the Green function is an exponentially increasing function of the difference time t_3-t_4.

Now we are ready to use the explicit form of the self-energy (A9). By the substitution of (A9) and (A17) into (A14) one finds

$$s_{\mathrm{right}}(t_3;\mathbf{k};t_2) = -i\frac{\alpha}{2\sqrt{\pi}}\int_{t_2}^{t_3} dt_4[Ne^{-i\omega_0(t_3-t_4)}+(N+1)e^{i\omega_0(t_3-t_4)}][i(t_3-t_4)]^{-3/2} \times e^{-i(1/12)[(eF)^2/2m](t_3-t_4)^3} e^{-(t_3-t_4)/2\tau_0} \exp\left[i\int_{t_4}^{t_3} dt(\varepsilon(\mathbf{k}-e\mathbf{F}t)+\sigma(\mathbf{k}-e\mathbf{F}t)) \right] . \tag{A18}$$

The self-energy, and thus the test function, involves two analytical pieces corresponding to the emission and the absorption of the phonon. These two pieces have to be discussed separately. We will concentrate on the emission part (proportional to $N+1$), which is larger and, moreover, has the threshold inside the electron band. Equation (A18) can be simplified by the substitution $t_4=t_3-t$,

$$s_{\mathrm{right}}(t_3;\mathbf{k};t_2) = -\frac{\alpha(N+1)}{2\sqrt{i\pi}}\int_0^{t_3-t_2} dt\, t^{-3/2}\exp\left[i\int_{t_3-t}^{t_3} dt'(\varepsilon(\mathbf{k}-e\mathbf{F}t')-\omega_0+\sigma(\mathbf{k}-e\mathbf{F}t')) \right] e^{-i(1/12)[(eF^2)/2m]t^3} e^{-t/2\tau_0} , \tag{A19}$$

where we have already omitted the absorption part.

Let us assume first that the momentum $\mathbf{k}$ is such that the condition $\varepsilon(\mathbf{k}-e\mathbf{F}t)-\omega_0-(eFt)^2/8m \gg -\mathrm{Im}\sigma(\mathbf{k}-e\mathbf{F}t)$ is satisfied for all t in the interval $t_2<t<t_3$. Within the linear term in the field F this condition means that the energy of the electron is above the optic-phonon-emission threshold during the time interval (t_2,t_3). The integrand in (A19) is a product of the oscillating exponential function

$$\exp\left[i\int_{t_3-t}^{t_3} dt'(\varepsilon(\mathbf{k}-e\mathbf{F}t')-\omega_0)-i(eF)^2t^3/(24m) \right]$$

and the real envelope function

$$t^{-3/2}\exp\left[\int_{t_3-t}^{t_3} dt'/2\tau(\mathbf{k}-e\mathbf{F}t')-t/2\tau_0 \right] .$$

In the crude estimate the amplitude of the deviation of the test function $s_{\mathrm{right}}(t_3;\mathbf{k};t_2)$ from the average plateau value is proportional to the product of the envelope function for $t=t_3-t_2$ and $2/\pi$ times the half-period of the oscillation $\pi/(\varepsilon(\mathbf{k}-e\mathbf{F}t_2)-\omega_0)$. For details see Ref. 7. The amplitude of the deviation $\delta s_{\mathrm{right}}(t_3;\mathbf{k};t_2)=s_{\mathrm{right}}(t_3;\mathbf{k};t_2)-\sigma(\mathbf{k}-e\mathbf{F}t_3)$ of the self-energy from the plateau value then can be estimated as

$$|\delta s_{\mathrm{right}}(t_3;\mathbf{k};t_2)| \approx \frac{\alpha(N+1)}{2\pi^{3/2}}(t_3-t_2)^{-3/2}\exp\left[-\int_{t_2}^{t_3} dt(1/2\tau(\mathbf{k}-e\mathbf{F}t)-1/2\tau_0) \right] \times \frac{2\pi}{\varepsilon(\mathbf{k}-e\mathbf{F}t_2)-\omega_0-[eF(t_2-t_3)]^2/8m} . \tag{A20}$$

The instantaneous approximation applies in the time region where the plateau value is a good approximation of the test function s_{right}, i.e., in the region where the amplitude of the oscillation is small comparing to the self-energy. Since the imaginary part of the self-energy has a meaning of the total scattering rate while the real part provides only an unimportant correction of the dispersion relation, we relate the condition of applicability to the imaginary part only:

$$|\delta s_{\mathrm{right}}(t_3;\mathbf{k};t_2)| \ll -2\,\mathrm{Im}\sigma_{\mathrm{emission}}(\mathbf{k}-e\mathbf{F}t_3) .$$

This condition restricts the time t_2 from both sides.

From below the increase of the power term $(t_3 - t_2)^{-3/2}$ restricts the applicability to the region of the order of one period of the oscillation, from above the exponential term restricts applicability to few (typically more than five) quaisparticle lifetimes. As it is shown in Ref. 7, the period of oscillation therefore can be interpreted as the quasiparticle formation time τ_{QF}, and τ_{div} is a few times the quasiparticle lifetime.

8. Crossing of the threshold

In the above discussion we have eliminated the cases in which the electron passes through the optic-phonon-emission threshold. Here we discuss the case when the energy of the electron at the time t_3 is above the threshold while at t_2 the electron energy was below. The test function s_{right} for this case is plotted in Fig. 2.

In Fig. 2 there is one new feature: the test function jumps from one plateau value to another one. The jump is in the vicinity of the time when the energy of the electron crosses the threshold.

In the vicinity of the threshold the period of the oscillating function gets large and we cannot use the estimate (A19) of the deviation from the plateau value. However, for sufficiently strong fields it is possible to estimate the size of the step from the saddle-point approximation. The time t_C at which the electron energy effectively passes through the threshold is found from the condition

$$\varepsilon(\mathbf{k} - e\mathbf{F}t_C) - \omega_0 - [eF(t_C - t_3)]^2/8m = 0 \ . \qquad \text{(A21)}$$

Note that the quadratic term modifies the condition for t_C comparing to the zero-field limit. The dependence of t_C on t_3 is not intuitively clear to us. In the vicinity of t_C it is advantageous to express the argument of the oscillating function in (A19) in powers of $t_3 - t - t_C$,

$$\int_{t_3-t}^{t_3} dt'(\varepsilon(\mathbf{k} - e\mathbf{F}t') - \omega_0) - \frac{(eF)^2 t^3}{24m} = \int_{t_C}^{t_3} dt'(\varepsilon(\mathbf{k} - e\mathbf{F}t') - \omega_0) - \frac{(eF)^2(t_3 - t_C)^3}{24m}$$
$$+ \left[\frac{e\mathbf{F}(\mathbf{k} - e\mathbf{F}t_C)}{2m} + \frac{(eF)^2}{8m}(t_3 - t_C) \right] (t_3 - t - t_C)^2 - \frac{5(eF)^2}{24m}(t_3 - t - t_C)^3 \ . \qquad \text{(A22)}$$

To have a crude estimate of the magnitudes of individual terms let us use t_C in the zero-field limit. From the condition (A21) one finds that $k_z^C = k_z - eFt_C \sim (2m\omega_0 - k_x^2 - k_y^2)^{1/2}$ and thus the value of k_z^C ranges from zero to $\sqrt{2m\omega_0}$. The characteristic time associated with the linear term in the field is $\tau_F = [m/(eFk_x^C)]^{1/2}$, which ranges from 2×10^{-13} s to infinity for the field $F = 3 \times 10^4$ V/m. For small parallel momenta there is a region in which $\tau_F < 1/[-2\,\mathrm{Im}\Sigma(\omega_0)] = 8.6 \times 10^{-13}$ s, thus the saddle-point approximation applies. Note that the validity of the saddle-point approximation becomes better for higher fields.

Beside the quadratic term there is a cubic term $(eF)^2/6m\,(t_3 - t - t_C)^3$. This term has the characteristic time $\tau_{F^2} = [2m/(eF)^2]^{1/3} = 10^{-12}$ s. We will restrict attention to the case $\tau_F \ll \tau_{F^2}$ and neglect the cubic term.

For the estimate we take the simplest saddle-point approximation of the integral in (A19) at the vicinity of the threshold. Thus the estimate of the magnitude $\Delta s_{\mathrm{right}}$ of the step of the test function is

$$|\Delta s_{\mathrm{right}}(\mathbf{k})| = \tau_F \tfrac{1}{2} \alpha (N+1)(t_3 - t_C)^{-3/2}$$
$$\times \exp\left[-\int_{t_C}^{t_3} dt\, \mathrm{Im}\sigma(\mathbf{k} - e\mathbf{F}t) \right] . \qquad \text{(A23)}$$

The step $\Delta s_{\mathrm{right}}(\mathbf{k})$ picks up the complex prefactor from the first term in (A22), therefore the actual step observed in the imaginary or real part of the test function can be both positive or negative and (A23) is the upper limit of the magnitude. The most important field dependence is the prefactor $\tau_F \sim 1/\sqrt{F}$. Hence the step increases for decreasing fields.

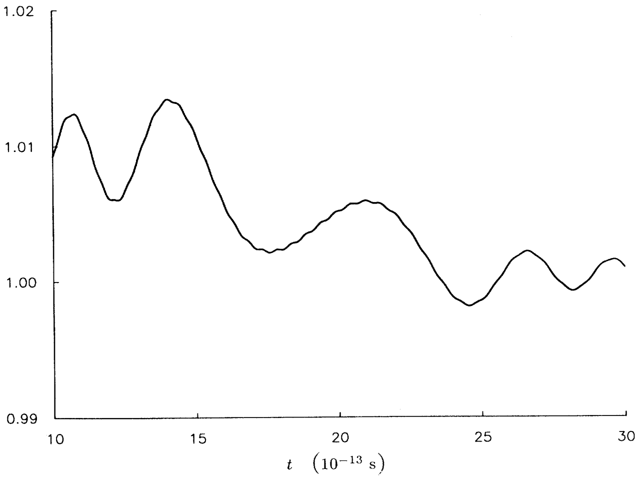

FIG. 2. A detail of the test functions $s_{\mathrm{right}}(0;\mathbf{k};t)$ vs t for the electric field $F = 3 \times 10^4$ V/m. The characteristic times are quasiparticle formation $\tau_{\mathrm{QF}} = \hbar/[\varepsilon(\mathbf{k}) - \omega_0] = 3.6 \times 10^{-14}$ s, quasiparticle lifetime $\tau = -\hbar/2\,\mathrm{Im}\sigma(\mathbf{k}) = 2.2 \times 10^{-13}$ s. The momentum $\mathbf{k} = (0,0,k_z)$ corresponds to the energy $\varepsilon(\mathbf{k}) = 1.9\omega_0$ and is chosen so that the field decelerates the electron. The time t_C at which the electron effectively crosses the threshold of the optical phonon emission is 1.9×10^{-12} s. The maximal value of the step $|\Delta s_{\mathrm{right}}(\mathbf{k})|$ predicted by the formula (3.24) is 3×10^{-2}; the actual value is smaller because of the phase factor that reduces the projection into the imaginary part. The small high-frequency oscillations are caused by the phonon-absorption contribution.

From (A23) one can see that the magnitude of the step is appreciable in the long-time region where the instantaneous approximation of the self-energy already fails and if the time t_3 is very close to the passing through the threshold. In the short-time region one finds that the step is negligible [i.e., $\Delta s_{\text{right}}(\mathbf{k}) \ll \sigma(t_3;\mathbf{k})$] if $t_3 - t_C \gg (\tau_F^2/\omega_0)^{1/3} = 4\times 10^{-14}$ s. According to this restriction of t_3 the instantaneous approximation does not hold if the momentum $\mathbf{k} - eFt_3$ is in the vicinity of the threshold that spreads over $\omega_0/5$ [if $\varepsilon(\mathbf{k}-eFt_3) - \omega_0 < \omega_0/5$]. We note that the applicability of the instantaneous approximation is problematic in this energy strip already in the zero-field case because the imaginary part of the self-energy is crudely $\omega_0/5$ at the threshold.

9. Conclusions

In this appendix we have shown that there is a time region within which the instantaneous approximation of the self-energy holds. This region covers the most important time scale for the transport properties since it starts at a fraction of the quasiparticle lifetime and exceeds a few quasiparticle lifetimes.

The instantaneous approximation can be used even if the electron passed through the phonon-emission threshold, assuming that the final electron state is already sufficiently far from the threshold.

APPENDIX B: QUASIPARTICLE EIGENSTATES IN EQUILIBRIUM

In the equilibrium the effective propagators U_{right} and U_{left} can be conveniently expressed with help of the set of wave functions which are eigenstates of the operators $H_0 + \sigma_{\text{right}}$ and $H_0 + \sigma_{\text{left}}$, respectively. This set of wave functions is not orthonormal because the effective self-energy is not a Hermitian operator.

The right and left states are solutions of the equations

$$[\varepsilon(\alpha) - H_0 - \sigma_{\text{right}}]\phi_{\text{right}}(\alpha) = 0 \ , \tag{B1}$$

$$\phi_{\text{left}}(\alpha)[\varepsilon(\alpha) - H_0 - \sigma_{\text{left}}] = 0 \ . \tag{B2}$$

In addition to the right and left states we need their conjugates. Conjugated functions are solutions of the equations

$$\bar{\phi}_{\text{right}}(\alpha)[\varepsilon(\alpha) - H_0 - \sigma_{\text{right}}] = 0 \ , \tag{B3}$$

$$[\varepsilon(\alpha) - H_0 - \sigma_{\text{left}}]\bar{\phi}_{\text{left}}(\alpha) = 0 \ . \tag{B4}$$

The right and left states are connected to the conjugated left and right states, respectively, by the wave-function renormalization z. This relation is recovered by taking the derivative with respect to t_1 of both sides of (2.19) from which one finds that

$$(H_0 + \sigma_{\text{right}})z = z(H_0 + \sigma_{\text{left}}) \ . \tag{B5}$$

From (B5) one can check that

$$\phi_{\text{right}}(\alpha) = z\bar{\phi}_{\text{left}}(\alpha) \ , \tag{B6}$$

$$\phi_{\text{left}}(\alpha) = \bar{\phi}_{\text{right}}(\alpha) z \ , \tag{B7}$$

are permissible connection relations.

The normalization of right and left states differs from the standard normalization of eigenstates of the Schrödinger equation because the operators $H_0 + \sigma_{\text{right}}$ and $H_0 + \sigma_{\text{left}}$ are non-Hermitian. The choice of the normalization is restricted by the conjugacy and the connection given by (B6) and (B7). The conjugacy of ϕ and $\bar{\phi}$ follows from Eqs. (B1) and (B2). Multiplying Eq. (B1) by $\bar{\phi}_{\text{right}}(\beta)$ with $\beta \neq \alpha$ one finds

$$\begin{aligned}\bar{\phi}_{\text{right}}(\beta)[\varepsilon(\alpha) - H_0 - \sigma_{\text{right}})]\phi_{\text{right}}(\alpha) \\ = \bar{\phi}_{\text{right}}(\beta)[\varepsilon(\alpha) - \varepsilon(\beta)]\phi_{\text{right}}(\alpha) = 0 \ .\end{aligned} \tag{B8}$$

The check of the orthogonality of the left states is similar. To satisfy the orthogonality we take the normalization conditions as

$$\bar{\phi}_{\text{right}}(\beta)\phi_{\text{right}}(\alpha) = \delta_{\beta,\alpha} \ , \tag{B9}$$

$$\bar{\phi}_{\text{left}}(\beta)\phi_{\text{left}}(\alpha) = \delta_{\beta,\alpha} \ . \tag{B10}$$

The set (B6), (B7), (B9), and (B10) provides the required normalization condition.

1. Effective self-energy in equilibrium

With help of the right states one can express the effective propagator $U_{\text{left}}(-t_3)$ as

$$U_{\text{right}}(t_4, t_3)\phi_{\text{right}}(\alpha) = e^{-i\varepsilon(\alpha)(t_4 - t_3)}\phi_{\text{right}}(\alpha) \ . \tag{B11}$$

One can check (B11) from (2.4). The representation (B11) of the effective propagator allows us to express the effective self-energy in terms of poles. By the substitution of (B11) in the definition (2.3) one finds

$$\sigma_{\text{right}}\phi_{\text{right}}(\alpha) = \int_{t_3 - 3\tau}^{t_3} dt_4 \Sigma(t_3, t_4)\phi_{\text{right}}(\alpha) e^{-i\varepsilon(\alpha)(t_4 - t_3)} \ . \tag{B12}$$

For the states $\phi_{\text{right}}(\alpha)$ for which the integration over t_4 reaches the plateau value (B12) gives the pole value of the self-energy. Projecting (B12) to a single state one finds

$$\sigma_{\text{right}}\phi_{\text{right}}(\alpha) = \Sigma\,(\varepsilon(\alpha))\phi_{\text{right}}(\alpha) \ . \tag{B13}$$

From (B12) one finds that states $\phi_{\text{right}}(\alpha)$ solve the equation

$$[\varepsilon(\alpha) - H_0 - \Sigma(\varepsilon(\alpha))]\phi_{\text{right}}(\alpha) = 0 \ , \tag{B14}$$

so that the states $\phi_{\text{right}}(\alpha)$ are quasiparticle eigenstates. In contrast, the conjugated states cannot in general be associated with the quasiparticle eigenstates.

2. Wave-function renormalization in equilibrium

The expression (2.16) for the wave-function renormalization is in a nonstandard form. We will evaluate (2.16) in terms of the quasiparticle states to demonstrate its relation to the standard approach. Let us take the element of (2.16) as

$$\phi_{\mathrm{left}}(\beta)z^{-1}\phi_{\mathrm{right}}(\alpha)=\phi_{\mathrm{left}}(\beta)\phi_{\mathrm{right}}(\alpha)-i\int_0^{3\tau}dt_3\int_{-3\tau}^{0}dt_4\phi_{\mathrm{left}}(\beta)U_{\mathrm{left}}(0,t_3)\Sigma(t_3,t_4)U_{\mathrm{right}}(t_4,0)\phi_{\mathrm{right}}(\alpha)\,, \tag{B15}$$

where we have set $t_2=0$ for the simplicity of the notation. This choice is possible since we are dealing with the equilibrium. According to (B11) and the similar expression for U_{left}, the projections of the effective propagators reduce to the simple exponentials; thus one finds

$$\phi_{\mathrm{left}}(\beta)z^{-1}\phi_{\mathrm{right}}(\alpha)=\phi_{\mathrm{left}}(\beta)\phi_{\mathrm{right}}(\alpha)-i\int_0^{3\tau}dt_3\int_{-3\tau}^{0}dt_4 e^{i\varepsilon(\beta)t_3}e^{-i\varepsilon(\alpha)t_4}\phi_{\mathrm{left}}(\beta)\Sigma(t_3,t_4)\phi_{\mathrm{right}}(\alpha)\,. \tag{B16}$$

For the retarded self-energy one can use the spectral representation that gives

$$\phi_{\mathrm{left}}(\beta)z^{-1}\phi_{\mathrm{right}}(\alpha)=\phi_{\mathrm{left}}(\beta)\phi_{\mathrm{right}}(\alpha)-\int\frac{d\omega}{2\pi}\phi_{\mathrm{left}}(\beta)\Gamma(\omega)\phi_{\mathrm{right}}(\alpha)\int_0^{3\tau}dt_3 e^{i[\varepsilon(\beta)-\omega]t_3}\int_{-3\tau}^{0}dt_4 e^{-i[\varepsilon(\alpha)-\omega]t_4}\,. \tag{B17}$$

Here the spectral function of the self-energy is $\Gamma(\omega)=i[\Sigma(\omega)-\Sigma^A(\omega)]$.

The integrations over t_3 and t_4 can be evaluated easily. There are two kinds of the contributions, the terms appearing on the integration limits $t_{3,4}\rightarrow 0$ and the terms following from the integration limits at $t_3\rightarrow 3\tau$ and $t_4\rightarrow -3\tau$. The oscillating terms from the artificial integration limits do not survive the integration over the energy ω at least for states $\phi_{\mathrm{left}}(\beta)$ and $\phi_{\mathrm{right}}(\alpha)$ that are not close to the nonanalytical points of the self-energy. This independence on the artificial integration limit is our main assumption. Neglecting the terms from the artificial limits one finds

$$\begin{aligned}\phi_{\mathrm{left}}(\beta)z^{-1}\phi_{\mathrm{right}}(\alpha)&=\phi_{\mathrm{left}}(\beta)\phi_{\mathrm{right}}(\alpha)\\&\quad-\int\frac{d\omega}{2\pi}\phi_{\mathrm{left}}(\beta)\Gamma(\omega)\phi_{\mathrm{right}}(\alpha)\frac{1}{\varepsilon(\alpha)-\varepsilon(\beta)}\\&\quad\times\left[\frac{1}{\omega-\varepsilon(\alpha)}-\frac{1}{\omega-\varepsilon(\beta)}\right]\,.\end{aligned} \tag{B18}$$

The integral over the energy ω defines the analytical continuation of the spectral function of the self-energy into the retarded complex energy half-plane. Thus the formula (2.16) finally leads to the expression

$$\phi_{\mathrm{left}}(\beta)z^{-1}\phi_{\mathrm{right}}(\alpha)=\phi_{\mathrm{left}}(\beta)\left[1-\frac{\Sigma(\varepsilon(\alpha))-\Sigma(\varepsilon(\beta))}{\varepsilon(\alpha)-\varepsilon(\beta)}\right]\phi_{\mathrm{right}}(\alpha)\,. \tag{B19}$$

The formula (B19) is close in the form to the generalized Ward identity.[13] In the diagonal elements the ratio in large parentheses has to be evaluated by the l'Hôpital rule, which simplifies the formula (B19) to the known form

$$\phi_{\mathrm{left}}(\alpha)z^{-1}\phi_{\mathrm{right}}(\alpha)=\phi_{\mathrm{left}}(\alpha)\left[1-\left.\frac{\partial\Sigma(\omega)}{\partial\omega}\right|_{\omega=\varepsilon(\alpha)}\right]\phi_{\mathrm{right}}(\alpha)\,. \tag{B20}$$

APPENDIX C: EFFECT OF THE INTERVALLEY SCATTERING

In this appendix we extend the discussion presented in Appendix A by including the scattering into L valley via the emission or absorption of optic phonons. We will not repeat the entire theory but only list model features that are necessary to include and compare numerical results.

1. Free-electron Hamiltonian in L minima

In GaAs there are three minima around L points in the Brillouin zone. The dispersion relation in these minima is strongly anisotropic; we will use two different masses m_r and m_t for its description. The effective mass in the radial direction from the Γ point is larger, we use $m_r=2.5m$, the effective mass in the directions parallel to the zone boundary are close to the effective mass of the Γ valley, thus we use $m_t=m$. Both values were extracted from Ref. 12 together with the L-minima bottom energy $\varepsilon_L=0.3$ eV (the value is related to the bottom of the Γ valley). We denote k_L the distance of the L point from the Γ point; however, the numerical value of k_L does not matter because of the dispersionless phonons.

In the absence of the electrical field all minima are degenerate. Within the special choice of field direction along the Γ-L line, two minima remain degenerate while the minimum in the direction of the field has to be distinguished. We associate the direction of the field with the coordinate z. [In Ref. 12 the line Γ-L has a direction vector (1,1,1) and its rotations, while we use (0,0,1) and its rotations.] For the minimum in the direction of the field the free-particle dispersion reads

$$H_r(t;\mathbf{k})=\varepsilon_L+\frac{k_x^2+k_y^2}{2m_t}+\frac{|k_z-k_L-e\mathbf{F}t|^2}{2m_r}\,. \tag{C1}$$

Two other minima experience the field in the direction of the Brillouin-zone boundary. We express the dispersion for the minimum in the x direction

$$H_t(t;\mathbf{k})=\varepsilon_L+\frac{(k_x-k_L)^2}{2m_r}+\frac{k_y^2}{2m_t}+\frac{(k_z-e\mathbf{F}t)^2}{2m_t}\,. \tag{C2}$$

2. Contribution of L minima to the local Green function

The self-energy of the intervalley scattering has the form (A6) with the only difference being that the integration over momenta is performed in the vicinity of L points. Similarly to Appendix A we use the simplification of infinite bands in the vicinity of L points and include the cutoff by the artificial exponential factors. The cutoff is eliminated from formulas but not from the numerical calculation.

The contribution of L minima to the local element of the free-electron Green function is

$$\int \frac{d\mathbf{k}}{(2\pi)^3} G_{L0}^R(t_3,t_4;\mathbf{k}) = -i\theta(t_3-t_4)e^{-i\varepsilon_L(t_3-t_4)} n_L[i(t_3-t_4)]^{-3/2} \times \left(e^{-i(1/12)[(eF)^2/2m_r](t_3-t_4)^3} + 2e^{-i(1/12)[(eF)^2/2m_t](t_3-t_4)^3}\right), \tag{C3}$$

where the prefactor is $n_L = (2\pi)^{-3/2} m_t \sqrt{m_r}$. In the same manner as in Appendix A, we can introduce the final lifetime correction to the local element of the Green function,

$$\int \frac{d\mathbf{k}}{(2\pi)^3} G_L^R(t_3,t_4;\mathbf{k}) = \int \frac{d\mathbf{k}}{(2\pi)^3} G_{L0}^R(t_3,t_4;\mathbf{k}) e^{-(t_3-t_4)/2\tau_L}, \tag{C4}$$

where τ_L is the lifetime at the bottom of the L minima. This correction is even more important here since τ_L is shorter than lifetime τ in the central valley below the intervalley-scattering threshold.

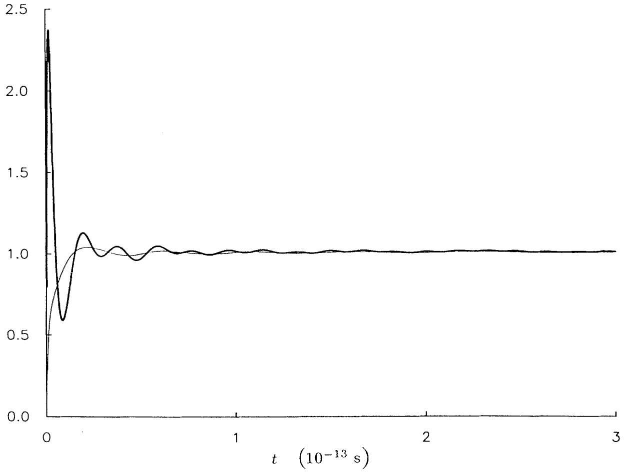

FIG. 3. Comparison of the test functions $s_{\text{right}}(0;\mathbf{k};t)$ with the intervalley scattering turned on (thin line) and off (thick line). All other parameters are identical to those used in Fig. 1; therefore the thin line is exactly the same as the thick line in Fig. 1. The remarkably different behavior of $s_{\text{right}}(0;\mathbf{k};t)$ in the short-time region is caused by a large density of states in the L minima. The magnitude of the oscillations due to the intervalley terms drops more rapidly with time than the magnitude of the oscillations due to the intravalley contribution as a result of the shorter lifetime at the bottom of the L minima. In the short-time region the importance of the intervalley scattering is apparent.

3. Test function

In Appendix A we discussed the applicability of the instantaneous approximation with the help of the test functions s_{right}^R and s_{left}^R. Here we discuss the effect of the intervalley scattering included on these functions.

The modification of the self-energy is obtained by adding the local Green function (C3) to the local Green function due to the central minimum in the formula (A6). The resulting self-energy has six terms corresponding to emission-absorption, intravalley-intervalley, and intervalley terms split into radial and tangential terms. All of these terms are of the same form except for values of parameters, therefore the analytical discussion would be a trivial modification of the discussion in Appendix A. Individual contributions enter the test functions with different magnitudes and periods of oscillation. In Fig. 3 one can see how the test function is changed if the intervalley scattering is on or off. The large deviation in the short-time region is due to the fact that the intervalley

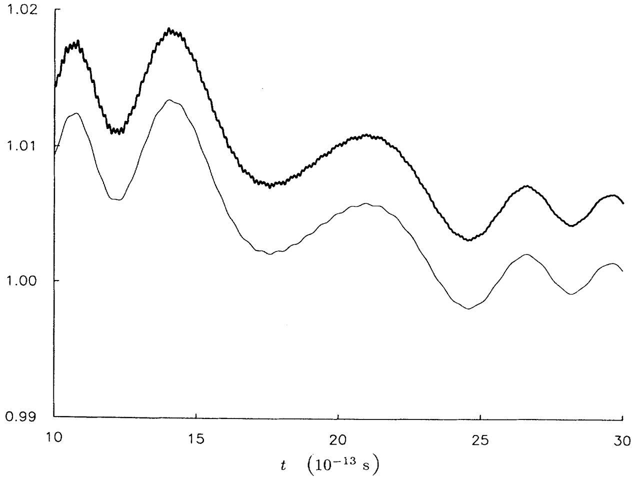

FIG. 4. Comparison of the test functions $s_{\text{right}}(0;\mathbf{k};t)$ for the intervalley scattering on (thin line) and off (thick line). All other parameters are identical to those used in Fig. 2. The constant shift of the value is due to the tails of the intervalley-scattering self-energy. There is no influence of the intervalley-scattering terms on the value of the step; therefore the presence of the intervalley scattering is not important for the crossing of the phonon-emission threshold.

self-energy has $3(m_t^2 m_r/m^3)^{1/2}$ larger prefactor. In the long-time region the exponential decay of the intervalley scattering takes over and the oscillating contributions disappear, leaving only a constant shift due to tails of the self-energy below the thresholds.

In Fig. 4 the crossing of the optic-phonon-emission threshold is shown. There is no influence of the intervalley scattering on the value of the step that appears when the threshold is crossed. One can see that with intervalley scattering on there is small oscillations of the test function that disappear with increasing time. These oscillations have a small amplitude compared to the main contribution since amplitudes of individual oscillations depend on the distance from the thresholds.

---

[1] I. B. Levinson, Zh. Eksp. Teor. Fiz. **57**, 660 (1969) [Sov. Phys.—JETP **30**, 362 (1970)].

[2] J. R. Barker, J. Phys. C **6**, 2663 (1973).

[3] A.P. Jauho and J. W. Wilkins, Phys. Rev. B **29**, 1919 (1984).

[4] F. S. Khan, J. H. Davies, and J. W. Wilkins, Phys. Rev. B **36**, 2578 (1987).

[5] L. P. Kadanof and G. Baym, *Quantum Statistical Mechanics* (Benjamin, Reading, MA, 1962).

[6] D. Calecki and N. Pottier, J. Phys. (Paris) Colloq. **42**, C7-271 (1981).

[7] P. Lipavský, F. S. Khan, A. Kalvová, and J. W. Wilkins, Phys. Rev. B (to be published).

[8] J. H. Davies and J. W. Wilkins, Phys. Rev. B **38**, 1667 (1988).

[9] R. Bertoncini, A. M. Kriman, and D. K. Ferry, Phys. Rev. B **40**, 3371 (1989).

[10] A. T. Dorsey, K. W. Jacobsen, Z. H. Levine, and J. W. Wilkins, Phys. Rev. B **40**, 3417 (1989).

[11] J. H. Davies and J. W. Wilkins, Phys. Rev. B **38**, 1667 (1988).

[12] K. Hess, Adv. Electron. Electron Phys. **59**, 239 (1982).

[13] J. R. Schrieffer, *Theory of Superconductivity* (Benjamin, New York, 1964), see material in the vicinity of p. 229.