High-field transport in semiconductors. II. Collision duration time

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The collision duration time τ_{CD} (a short time scale $<10^{-14}$ s) is estimated for an equilibrium nondegenerate semiconductor. For interaction with nonpolar optical phonons, it is shown that there is one common scale τ_{CD} for both the scattering-in and scattering-out integrals. This time is defined so that if $\tau_{CD} \ll \tau$ (τ is the quasiparticle lifetime) subsequent scattering events do not interfere. We find that the best estimator for this time, which is independent of the temperature, is the Landau criterion proposed for metals.

I. INTRODUCTION

Transport theory of nondegenerate semiconductors is largely based on the Boltzmann equation. To derive the Boltzmann equation from the quantum description of electrons interacting with phonons or impurities one has to assume that individual interaction events are independent of each other, i.e., that the collision is completed before another interaction starts. Formally, we say that the Boltzmann equation is derived under the assumption that the collision duration time $\tau_{CD} \ll \tau$, where τ is the quasiparticle lifetime or the mean free time.[1]

In spite of its importance a precise definition of the collision duration time is not in print. There are few intuitive estimators of the collision duration time τ_{CD} that show a variety of trends and values. We will briefly discuss them for a local interaction, such as the interaction with polar optical phonons or with neutral impurities. All our estimates are appropriate for the Γ valley in GaAs and the electron energy ~ 0.2 eV. This high value of the energy was chosen for the discussion since it is appropriate for the high-energy tail of the distribution for which it is difficult to justify the Boltzmann equation.

1. Classical estimate

Within classical mechanics an estimator of the collision duration time is the time the electron spends in the region of the interaction potential, assuming that the velocity of the electron is equal to the velocity of the initial or final state.[1] Since the interaction with optical phonons is localized to one elementary cell ($a \sim 10^{-9}$ m) and electron velocity v is 10^6 m/s, one finds

$$\tau_{CD}^{(1)} \approx \frac{a}{v} \sim 10^{-15} \text{ s} . \tag{1.1}$$

In the parabolic band the classical estimator for an electron of energy ω is proportional to $1/\sqrt{\omega}$.

2. Energy derivative of phase shift

The theory of elastic scattering associates the "collision delay time" $\tau_{CD}^{(2)}$ with the energy derivative of the phase shift.[2] In the lowest-order approximation such an interpretation of the collision duration gives[3]

$$\tau_{CD}^{(2)} \approx \pi\hbar \left| V \frac{dN(\omega)}{d\omega} \right| \sim 10^{-16} \text{ s} , \tag{1.2}$$

where N is the density of states per atomic unit ($\sim\sqrt{\omega}$), and $V \sim 1$ eV was used for the impurity potential. This value of the collision delay time differs from the classical estimator $\tau_{CD}^{(1)}$ in that it depends on the strength of the impurity potential. On the other hand, both have the same energy dependence $1/\sqrt{\omega}$.

3. Applicability of the pole approximation to the spectral function

Alternative quantum-mechanical estimators are based on the applicability of the pole approximation to the spectral function A of the single-particle Green function[3]

$$A(\omega;\mathbf{k}) = -2 \operatorname{Im} \frac{1}{\omega - \varepsilon(\mathbf{k}) - \Sigma^R(\omega)} \approx -2 \operatorname{Im} \frac{1}{\omega - \varepsilon(\mathbf{k}) - \Sigma^R(\varepsilon(\mathbf{k}))} , \tag{1.3}$$

where the energy ω and the kinetic energy $\varepsilon(\mathbf{k})$ are independent.

Since the real part of the self-energy is usually model

dependent, it is better to base the estimator entirely on the spectral function of the self-energy $\Gamma(\omega) = -2\,\mathrm{Im}\Sigma^R(\omega)$. The spectral function $A(\omega;\mathbf{k})$ has a Lorentzian peak of the width $\Gamma(\varepsilon(\mathbf{k}))$ in the vicinity of $\omega=\varepsilon(\mathbf{k})$. The pole approximation is applicable if the change of the self-energy within the width $\Gamma(\omega)$ is small relatively to its value

$$\Gamma(\omega)\left|\frac{d\Gamma(\omega)}{d\omega}\right| \ll \Gamma(\omega) , \tag{1.4}$$

which can be rearranged as

$$\frac{\hbar}{\Gamma(\omega)}\left|\frac{d\Gamma(\omega)}{d\omega}\right| \ll \frac{\hbar}{\Gamma(\omega)} . \tag{1.5}$$

The right-hand side of (1.5) is the quasiparticle lifetime $\tau=1/\Gamma$, so that the expression on the left-hand side can be associated with τ_{CD},

$$\tau_{\mathrm{CD}}^{(3)} \approx \hbar\left|\frac{d\,\ln\Gamma(\omega)}{d\omega}\right| \sim 10^{-15}\ \mathrm{s} . \tag{1.6}$$

Since the spectral function of the self-energy is typically proportional to the density of states, one can then replace Γ by N in (1.3), which gives $\tau_{\mathrm{CD}}^{(3)} \sim \hbar/(2\omega)$. This definition is not only independent of the interaction potential, but the energy dependence is different from (1.2). This characteristic time directly relates to the scattering-out integral of the Boltzmann equation. See the discussion in Sec. III.

4. *Applicability of the pole approximation to the correlation function of the self-energy*

The pole approximation of the correlation function of the self-energy $A(\omega;\mathbf{k})\Sigma^<(\omega)=A(\omega;\mathbf{k})\Sigma^<(\varepsilon(\mathbf{k}))$ was used by Kadanoff and Baym[4] to derive the scattering-in integral of the Boltzmann equation from the nonequilibrium Green functions. The applicability of the pole approximation to the correlation function of the self-energy $\Sigma^<$,

$$\Sigma^<(\omega)=f_{\mathrm{FD}}(\omega)\Gamma(\omega) , \tag{1.7}$$

where f_{FD} is the Fermi-Dirac distribution and $\Gamma(\omega)=-2\,\mathrm{Im}\Sigma^R(\omega)$, requires that the energy dependence of the correlation function of the self-energy is small relative to the pole value in the vicinity of the pole.[3,5] This relative energy dependence can be characterized by a time

$$\tau_{\mathrm{CD}}^{(4)} \approx \hbar\left|\frac{d\,\ln f_{\mathrm{FD}}(\omega)\Gamma(\omega)}{d\omega}\right| \sim 10^{-14}\ \mathrm{s} . \tag{1.8}$$

In (1.8) the dominant part of the logarithmic derivative arises from the Fermi-Dirac distribution and one can simplify the criterion to the inverse temperature

$$\tau_{\mathrm{CD}}^{(4)} \approx \hbar/k_B T \sim 10^{-14}\ \mathrm{s} , \tag{1.9}$$

where we have used room temperature for T.

This definition relates to the scattering-in integral of the Boltzmann equation. The value and trend with energy are essentially different from the scattering-out integral (1.6). Incidentally, in the early 1950s the criterion $\hbar/k_B T < \tau$ was supposed to limit the validity of the Landau Fermi-liquid theory for metals.[5] We should note that this criterion is extremely restrictive at low temperatures and also at high energies when τ becomes very small. Later, it was shown by Landau that this criterion is irrelevant and he proposed the correct criterion ($\tau_{\mathrm{CD}} \sim 1/E_F$), which is much less restrictive in metals.

5. *Landau criterion*

Finally there is an intuitive estimator proposed by Landau[5] within the theory of Fermi liquid that the relevant characteristic time is given by the reciprocal energy distance from the nearest nonanalytical point in the density of states. The strongest nonanalytical points are typically close to the band edge and thus one finds, if ω measures the nearest separation from the nonanalytical point in the density of states,

$$\tau_{\mathrm{CD}}^{(5)} \approx \frac{2\pi\hbar}{\omega} \sim 10^{14}\ \mathrm{s} . \tag{1.10}$$

For simple metals this characteristic time simplifies to $2\pi\hbar/E_F$, where E_F is the Fermi energy.

In this paper we will ask a simple question: what characteristic time scale allows scattering events to be treated as independent? Since the scattering events enter the transport equation (for instant, the Boltzmann equation) via the scattering-in and scattering-out integrals, one has to discuss the characteristic time scale in two different frameworks.

The scattering-out term is the simpler one because it involves only the single-particle Green function. For the discussion of the scattering-in term one has to use the entire transport equation, which includes statistical information such as the temperature. We show in this paper that (at least at the equilibrium) the noninterference of subsequent processes, either involved in the scattering out or scattering in, is given by a single characteristic time scale. The best estimator of this time scale is the Landau suggestion (1.10).

The paper is organized as follows. The model, approximation, and formalism introduced in Sec. II are used to evaluate, in Sec. III, the quasiparticle formation of the single-particle Green function and, in Sec. IV, the characteristic time scale that corresponds to the interference of individual scattering events. In Sec. V we assume that both new times are short compared to the quasiparticle lifetime and prove a validity of the Boltzmann equation in this limit. In Sec. VI we discuss the completed collision approximation in the formulation used for high-field transport theory. Section VII contains conclusions.

II. ELECTRON-PHONON COUPLED SYSTEM

As an archetypical problem, we study a nondegenerate system of noninteracting electrons coupled to phonons. In this section we introduce the full set of nonequilibrium Green functions and specify the approximations we use. The set of equations for nonequilibrium Green functions is an input of our paper, and our aim is to discuss the

characteristic time scales that are involved in this set at equilibrium.

A. Definitions of Green functions

Our conventions are as follows.[6] The correlation functions are defined for electrons and phonons, respectively, as

$$G^{<}(t;\mathbf{k})=\mathrm{Tr}[\hat{\rho}\psi^{\dagger}(0;\mathbf{k})\psi(t;\mathbf{k})] , \tag{2.1}$$

$$G^{>}(t;\mathbf{k})=\mathrm{Tr}[\hat{\rho}\psi(t;\mathbf{k})\psi^{\dagger}(0;\mathbf{k})] , \tag{2.2}$$

and

$$D^{<}(t)=\mathrm{Tr}[\hat{\rho}u(0)u(t)] , \tag{2.3}$$

$$D^{>}(t)=\mathrm{Tr}[\hat{\rho}u(t)u(0)] , \tag{2.4}$$

where $\psi^{\dagger}$ (ψ) is a creation (annihilation) operator of an electron, u is an atomic position deviation, and $\hat{\rho}$ is a grand-canonical ensemble-averaging operator. In the definition of the phonon correlation function we have already taken into account that we will be concerned only with the flat optical-phonon mode and thus we exclude the momentum dependence of $D^{>}$ and $D^{<}$.

The retarded and advanced operators are composed of the correlation parts as[6]

$$Z^{R}(t)=-i\Theta(t)[Z^{>}(t)\pm Z^{<}(t)] \tag{2.5}$$

and

$$Z^{A}(t)=i\Theta(-t)[Z^{>}(t)\pm Z^{<}(t)] , \tag{2.6}$$

where the upper (lower) sign is for fermionlike (bosonlike) operators. While, as in (2.5) and (2.6), we put $\hbar=1$ throughout the paper, for the reader's convenience we write $\hbar$ explicitly whenever we recall material parameters that are fitted to the central valley of conductivity band in GaAs.

B. Approximations of Green functions

To express approximations we take advantage of the diagrammatical language and use the rules[6] of the generalized Kadanoff-Baym formalism to write down equations determined by the diagrams. The set of equations that will be the focus of our interest follows from the dressing of the electron Green function.[7] It is diagrammatically expressed by the equation on Fig. 1.

The Dyson equation for the single-electron Green function G^{R} thus reads

$$G^{R}(t;\mathbf{k})=G_0^{R}(t;\mathbf{k})+\int_0^{t}dt_1\int_0^{t_1}dt_2 G_0^{R}(t-t_1;\mathbf{k}) \times\Sigma^{R}(t_1-t_2)G^{R}(t_2;\mathbf{k}) . \tag{2.7}$$

The self-energy Σ^{R} consists of the analytical parts

$$\Sigma^{>,<}(t)=\gamma^2 D^{>,<}(t)g^{>,<}(t) , \tag{2.8}$$

where γ is the coupling constant, and

$$g^{>,<}(t)=\int\frac{d\mathbf{k}}{(2\pi)^3}G^{>,<}(t;\mathbf{k}) \tag{2.9}$$

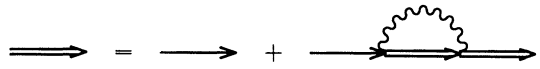

FIG. 1. Dressing of single-electron Green function; the Dyson equation.

is the local (in space) correlation function.

According to the definition (2.5), Σ^{R} can be reorganized as

$$\Sigma^{R}(t)=-i\theta(t)[\gamma^2 D^{>}(t)g^{>}(t)+\gamma^2 D^{<}(t)g^{<}(t)] =\gamma^2 D^{>}(t)g^{R}(t)-\gamma^2 D^{R}(t)g^{<}(t) . \tag{2.10}$$

The last form in (2.10) is the most suitable for the low electron-density limit, where the second term of (2.10) vanishes.

The transport equation for the correlation function, called the generalized Kadanoff-Baym equation,

$$G^{<}(t;\mathbf{k})=\int_{-\infty}^{t}dt_1\int_{-\infty}^{0}dt_2 G^{R}(t-t_1;\mathbf{k}) \times\Sigma^{<}(t_1-t_2)G^{A}(t_2;\mathbf{k}) , \tag{2.11}$$

will be the focus of our interest.

C. Spectral representation

For the sake of convenience we describe the model property in the energy representation. Our convention for the energy representation is

$$Z(\omega)=\int_{-\infty}^{\infty}dt\, e^{i\omega t}Z(t) . \tag{2.12}$$

The equilibrium correlation functions are related to the single-particle Green functions via the Schwinger boundary condition[4]

$$G^{<}(\omega;\mathbf{k})=f_{\mathrm{FD}}(\omega)i[G^{R}(\omega;\mathbf{k})-G^{A}(\omega;\mathbf{k})] , \tag{2.13}$$

$$\Sigma^{<}(\omega)=f_{\mathrm{FD}}(\omega)i[\Sigma^{R}(\omega)-\Sigma^{A}(\omega)] . \tag{2.14}$$

Our main interest will be nondegenerate systems where the Fermi-Dirac distribution f_{FD} reduces to the Boltzmann distribution

$$f_{\mathrm{FD}}(\omega)\approx n_d e^{-\omega/k_B T} , \tag{2.15}$$

where $n_d=n(2\pi/mk_B T)^{3/2}$ includes the electron density n and the normalization of the Boltzmann distribution with respect to a density of electron states to satisfy $\int(d\omega/2\pi)g^{<}(\omega)=n$.

D. Spectral function of the self-energy

In the self-energy, given by the last line of (2.10), we assume that the semiconductor is nondegenerate and the concentration of electrons in the conduction band vanishes, i.e., $n\to 0$. The second term of (2.10) thus vanishes and the energy representation of the self-energy reads

$$\Sigma^{R}(\omega)=\gamma^2\int\frac{dz}{2\pi}D^{>}(z)g^{R}(\omega-z) . \tag{2.16}$$

The correlation function of Einstein phonons is purely real,[7]

$$D^{>}(z)=N\frac{1}{2\omega_0}\delta(z+\omega_0)+(N+1)\frac{1}{2\omega_0}\delta(z-\omega_0)\ , \quad (2.17)$$

where ω_0 is the phonon frequency (or energy since $\hbar=1$) and N is the Bose-Einstein distribution $N=[\exp(\omega_0/k_BT)-1]^{-1}$.

For simplicity we neglect the dressing of the line inside the self-energy. In other words, we approximate the local Green function g^R in (2.16) by the free-particle Green function g_0^R. For the parabolic band the integration over momentum [see Eq. (2.9)] thus results in the free-electron density of states

$$-2\,\mathrm{Im}g_0^R(\omega)=n_b\theta(\omega)\sqrt{\omega}\ , \quad (2.18)$$

where $n_b=(\sqrt{2}/\pi)m^{3/2}$; note the norm n_b depends on the effective mass of electrons.

Since in a real material the integration over momentum does not go to infinity but is limited by the Brillouin zone, we introduce an artificial exponential cutoff

$$-2\,\mathrm{Im}g_0^R(\omega)=n_b\theta(\omega)\sqrt{\omega}e^{-\omega/\omega_{\mathrm{cut}}}\ , \quad (2.19)$$

where ω_{cut} is of the order of the energy bandwidth ($\sim$2 eV). This form of cutoff is not motivated by any physical property of the system. In fact, in GaAs the density of states increases more rapidly than the square root in the upper part of the conductivity band and moreover involves the additional nonanalytical points due to X and L minima. We use the exponential cutoff because it removes fast oscillating parts in the time dependence of the self-energy and leads to convenient analytic expressions.

Upon the substitution of (2.17) and (2.19) into the imaginary part of (2.16) one finds

$$\begin{aligned}-2\,\mathrm{Im}\Sigma^R(\omega)=&N\alpha\theta(\omega+\omega_0)\sqrt{\omega+\omega_0}e^{-(\omega+\omega_0)/\omega_{\mathrm{cut}}}\\&+(N+1)\alpha\theta(\omega-\omega_0)\\&\times\sqrt{\omega-\omega_0}e^{-(\omega-\omega_0)/\omega_{\mathrm{cut}}}\ , \quad (2.20)\end{aligned}$$

where $\alpha=\gamma^2n_b/2\omega_0$. Figure 2 shows that two nonanalytical points at $-\omega_0$ and at ω_0 are common for both self-energies, the spectral function of the self-energy $\mathrm{Im}\Sigma^R(\omega)$, and the correlation function of the self-energy $\Sigma^{<}(\omega)$. The nonanalytical point at $-\omega_0$, due to the phonon absorption part of the self-energy, lies below the band edge and thus it is always far from the electron energy. The nonanalytical point at ω_0 is due to the phonon emission, and it will be in the center of our interest.

III. PROPERTIES OF THE RETARDED GREEN FUNCTION $G^R(t_1,t_2)$

Here we discuss properties of the retarded Green function $G^R(t_1,t_2)$ in equilibrium. Although the energy representation is advantageous in equilibrium, we will discuss the Green function in a time representation to naturally demonstrate the characteristic times.

A. Retarded self-energy in the time representation

The retarded self-energy in the time representation is obtained from (2.20) by inverse Fourier transformation

$$\Sigma^R(t)=-i\theta(t)\int\frac{d\omega}{2\pi}e^{-i\omega t}(-2)\mathrm{Im}\Sigma^R(\omega)\ . \quad (3.1)$$

In (3.1) the entire retarded self-energy in the time representation is evaluated only from the imaginary part of its

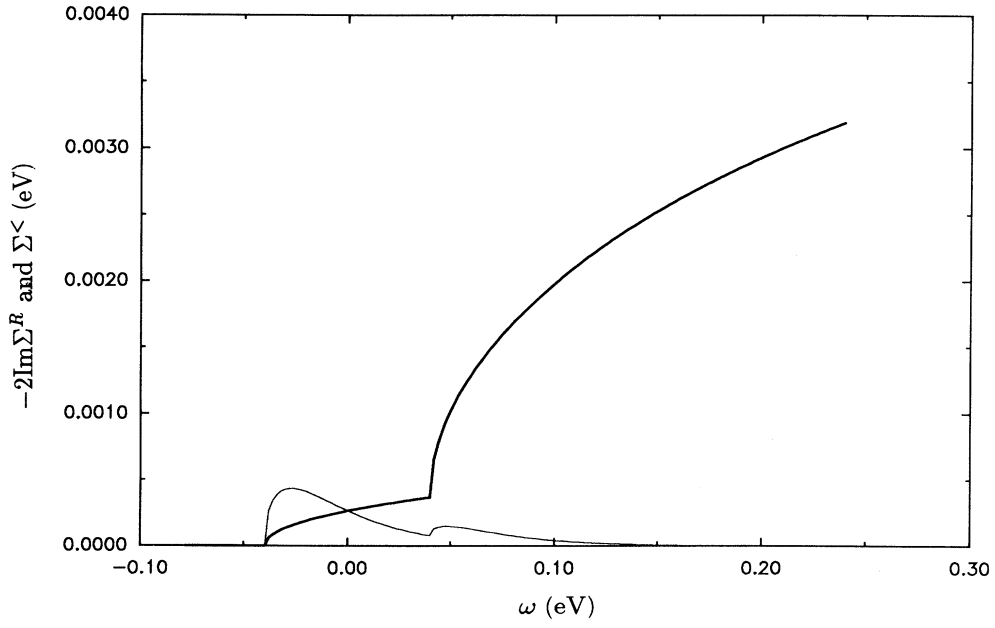

FIG. 2. Spectral function of the self-energy $-2\,\mathrm{Im}\Sigma^R(\omega)$, thick line; and the correlation function of the self-energy $\Sigma^{<}(\omega)$, thin line. To make the two scales comparable we set $n_d=1$ in the plot. The parameters we use are $\omega_0=0.04$ eV, $\alpha=2.5\times10^{-3}$ eV$^{1/2}$, and $T=300$ K.

energy representation. This simplification is possible because of the following identities known from spectral representation. First, the retarded self-energy can be expressed as $\Sigma^R(t)=\theta(t)[\Sigma^R(t)-\Sigma^A(t)]$ because the advanced self-energy is zero for $t>0$. In the energy representation the difference of the retarded and advanced self-energies is the spectral function of the self-energy $\Sigma^R(\omega)-\Sigma^A(\omega)=-i(-2)\mathrm{Im}\Sigma^R(\omega)$ because the retarded and advanced functions are conjugated to one another, $\Sigma^R(\omega)=\overline{\Sigma}^A(\omega)$, where the overbar denotes the complex conjugate.

From (3.1) and (2.20) one finds

$$\Sigma^R(t)=-i\theta(t)\frac{\alpha}{4\sqrt{\pi}}[Ne^{i\omega_0 t}+(N+1)e^{-i\omega_0 t}]\times\left[\frac{1}{\omega_{\mathrm{cut}}}+it\right]^{-3/2}. \tag{3.2}$$

The cutoff energy ω_{cut} does not affect the self-energy except for the times $t\lesssim\hbar/\omega_{\mathrm{cut}}$. For $\omega_{\mathrm{cut}}=2$ eV, $\hbar/\omega_{\mathrm{cut}}\sim10^{-16}$ s.

We note that the self-energy (3.2) has the form of the original expression (2.10) $\Sigma^R(t)=\gamma^2D^>(t)g_0^R(t)$. Since $D^>(t)$ is a sum of two exponentials, the power-law decay follows entirely from the time dependence of the Green function $g_0^R(t)$. This power-law behavior of $g_0^R(t)$ follows from the square-root behavior of the density of states (2.19).

B. Pole (quasiparticle) approximation in equilibrium

In semiclassical transport theory the scattering-out term is treated as an instantaneous event. Within the double-time Green-function approach the scattering-out events are described by $\Sigma^R(t)$, which is certainly not an instantaneous function of time; see (3.2). Our aim therefore is to approximate the self-energy $\Sigma^R(t)$ by some effective self-energy $\sigma^R(t;\mathbf{k})$ that is instantaneous but may be momentum dependent,

$$\sigma^R(t;\mathbf{k})=\delta(t)\sigma^R(\mathbf{k}) . \tag{3.3}$$

In the energy representation this instantaneous approximation means that σ^R is energy independent. The standard way to obtain such an approximation is the pole approximation[4,8]

$$\sigma^R(\mathbf{k})=\Sigma^R(\omega)|_{\omega=\varepsilon_0(\mathbf{k})+\sigma^R(\mathbf{k})} , \tag{3.4}$$

where $\varepsilon_0(\mathbf{k})=k^2/2m$ is the free-electron dispersion relation.

The real part of the effective self-energy $\sigma^R(\mathbf{k})$ is an effective potential and the correction of the dispersion relation (correction to the mass of electron). The quasiparticle energy dispersion is then

$$\varepsilon(\mathbf{k})=\varepsilon_0(\mathbf{k})+\mathrm{Re}\sigma^R(\mathbf{k}) . \tag{3.5}$$

The imaginary part relates to the quasiparticle lifetime[8]

$$\frac{1}{\tau(\mathbf{k})}=-2\,\mathrm{Im}\sigma^R(\mathbf{k}) . \tag{3.6}$$

C. Green function in the pole approximation

We introduce the pole approximation using the time representation, which will permit a discussion of the applicability of the pole approximation in terms of characteristic times. Within the pole (quasiparticle) approximation

$$\Sigma^R(\omega)\approx\sigma^R(\mathbf{k}) , \tag{3.7}$$

the Dyson equation for the retarded Green function (2.7) reads

$$G_q^R(t;\mathbf{k})=G_0^R(t;\mathbf{k})+\int_0^t dt_1 G_0^R(t-t_1;\mathbf{k})\sigma^R(\mathbf{k})G_q^R(t_1) . \tag{3.8}$$

The exact solution of (3.8) for the quasiparticle approximation for G^R is

$$G_q^R(t;\mathbf{k})=-i\theta(t)e^{-i\varepsilon(\mathbf{k})t}e^{-t/2\tau(\mathbf{k})} . \tag{3.9}$$

D. Limits of validity of pole approximation: Test function

The form (3.8) of the pole-approximated Dyson equation permits direct comparison with the nonapproximated Dyson equation (2.7). If the effective Green function G_q^R should approximate the asymptotic behavior of the Green function G^R, the terms containing self-energies in (2.7) and (3.8) must lead to the same result,

$$\int_0^{t_1} dt_2\Sigma^R(t_1-t_2)G^R(t_2;\mathbf{k})=\sigma^R(\mathbf{k})G_q^R(t_1;\mathbf{k}) . \tag{3.10}$$

The condition (3.10) cannot be satisfied for a general time argument t_1, but there is a region in which approximation (3.10) is valid. To gain insight into regions of applicability of (3.10) we approximate the Green function $G^R(t_2;\mathbf{k})$ in the left-hand side of (3.10) by the effective Green function $G_q^R(t_2;\mathbf{k})$. Dividing both sides of (3.10) by $G_q^R(t_1;\mathbf{k})$, one then finds that the approximation (3.10) is possible in the region where the test function

$$s^R(t_1;\mathbf{k})=\int_0^{t_1} dt_2\Sigma^R(t_1-t_2)G_q^R(t_2;\mathbf{k})[G_q^R(t_1;\mathbf{k})]^{-1} \tag{3.11}$$

is independent of t_1. In particular, if $s^R(t_1;\mathbf{k})$ is independent of t_1 over a dominant part of the integral over t_1 in (2.7) or (3.8), then the pole approximation holds and $\sigma^R(\mathbf{k})$ is equal to $s^R(t_1;\mathbf{k})$ in the region of the latter's constant value.

E. Limits of validity of pole approximation: Numerical results

Consider the test function (3.11) for our model. Since $t_{1,2}>0$ in (3.11), the product of the retarded Green function with its inverse, according to (3.9), is

$$G_q^R(t_2;\mathbf{k})[G_q^R(t_1;\mathbf{k})]^{-1}=e^{-i\varepsilon(\mathbf{k})(t_2-t_1)}e^{-(t_2-t_1)/2\tau(\mathbf{k})} . \tag{3.12}$$

But the substitution of (3.12) and (3.2) into (3.11) one finds

$$s^R(t_1;\mathbf{k})=-i\int_0^{t_1}dt_2\frac{\alpha}{4\sqrt{\pi}}[Ne^{i\omega_0(t_1-t_2)}+(N+1)e^{-i\omega_0(t_1-t_2)}]\left[\frac{1}{\omega_{\rm cut}}+i(t_1-t_2)\right]^{-3/2}e^{i\varepsilon(\mathbf{k})(t_1-t_2)}e^{(t_1-t_2)/2\tau(\mathbf{k})}$$
$$=-i\int_0^{t_1}dt\frac{\alpha}{4\sqrt{\pi}}[Ne^{i\omega_0 t}+(N+1)e^{-i\omega_0 t}]\left[\frac{1}{\omega_{\rm cut}}+it\right]^{-3/2}e^{i\varepsilon(\mathbf{k})t}e^{t/2\tau(\mathbf{k})}. \quad (3.13)$$

The function $s^R(t_1;\mathbf{k})$ can be separated into the absorption part, the term with $Ne^{i\omega_0 t}$, and the emission part, the term with $(N+1)e^{-i\omega_0 t}$. The emission part of the function $s^R(t_1;\mathbf{k})$ is plotted in Fig. 3. The behavior of the absorption part is very similar.

The main features of the function $s^R(t_1;\mathbf{k})$ are as follows. The function oscillates around the pole value of the self-energy with a constant period $\tau_{\rm QF}$

$$\tau_{\rm QF}(\mathbf{k})=\frac{2\pi}{\varepsilon(\mathbf{k})-\omega_0}. \quad (3.14)$$

[Below we show that the period of oscillations is a good estimator of the quasiparticle formation time. Indeed, it is identical with Landau's proposal (1.10).] The amplitude of the oscillations is minimal at the time $t_1=3\tau$, increases as $t_1^{-3/2}$ for $t_1\to 0$, and exponentially diverges for $t_1\to\infty$. There is a deviation from this behavior in a very short-time region $t_1\lesssim 2\pi\hbar/\omega_{\rm cut}$ because of the complex shift of the time $(t-i/\omega_{\rm cut})$ in (3.13).

With respect to validity of the pole approximation there are three important regions apparent in Fig. 3. In the region $t_1<\tau_{\rm QF}=2.8\times10^{-14}$ s the function $s^R(t_1;\mathbf{k})$ differs from the pole approximation of the self-energy.

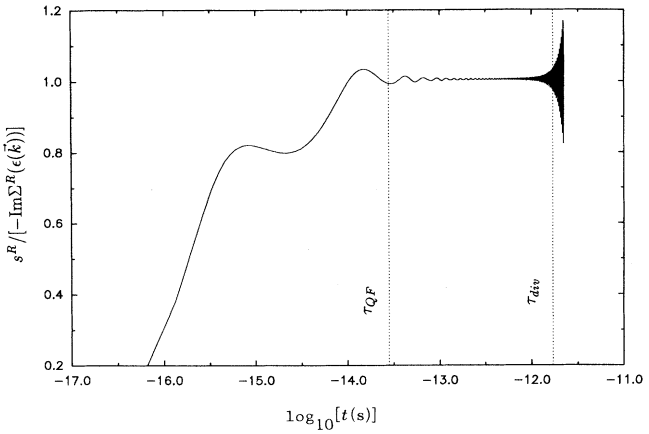

FIG. 3. Test of the pole approximation for scattering out. The figure contains a plot of the emission part of the test function s^R vs $\log_{10}[t\ (\mathrm{s})]$ for the energy $\varepsilon(\mathbf{k})=0.2$ eV. There are three regions of interest. For $t<\tau_{\rm QF}$, $\log_{10}(\tau_{\rm QF})=-13.55$ from (3.14), the quasiparticle is forming, and s^R varies strongly. In the interval $\tau_{\rm QF}<t<\tau_{\rm div}$, $\log_{10}(\tau_{\rm div})=-11.77$ from (3.20), there is a plateau on which the pole approximation is acceptable. For $t>\tau_{\rm div}$ the power-law tails of the propagator take over the exponentially decaying quasiparticle part, which results in the breakdown of the pole approximation. The quasiparticle lifetime is $\tau=1.9\times10^{-13}$ s, therefore the smallest amplitude of the oscillations is around $\log_{10}(3\tau)=-12.24$.

The middle region $\tau_{\rm QF}(\mathbf{k})<t_1<\tau_{\rm div}=1.7\times10^{-12}$ s provides a plateau in which the pole approximation of the self-energy holds. For $t_1>\tau_{\rm div}$ the integral starts to oscillate with an exponentially increasing amplitude. An estimate for $\tau_{\rm div}$ will later be provided by (3.20).

F. Limits of validity of pole approximation: Analytical estimate

The characteristic features of Fig. 3 can be derived analytically. The emission part of the integrand in (3.13) is a product of the oscillating term $\exp\{i[\varepsilon(\mathbf{k})-\omega_0]t\}$ and the envelope function which is

$$(\alpha/4\sqrt{\pi})(N+1)t^{-3/2}\exp[t/2\tau(\mathbf{k})]$$

for $t\gg 1/\omega_{\rm cut}$.

The minimum of the envelope function is at $t=3\tau(\mathbf{k})$. As a result, around the point $3\tau(\mathbf{k})$ the oscillation of $s^R(t_1;\mathbf{k})$ has the smallest amplitude. Since the envelope function monotonically increases with $|t-3\tau(\mathbf{k})|$, the amplitude of fluctuations of $s^R(t_1;\mathbf{k})$ increase as one goes from the "center" of plateau at $3\tau(\mathbf{k})$ toward the limiting regions either of very short times (before the quasiparticle is formed) or to very long times (when quasiparticles are decaying and power-law tails are dominant).

An estimator of the amplitude $\Delta s^R(t_1;\mathbf{k})$ of oscillations of $s^R(t_1;\mathbf{k})$ can be obtained by integrating t over a half-period $\Delta t=\pi/[\varepsilon(\mathbf{k})-\omega_0]\equiv\tau_{\rm QF}(\mathbf{k})/2$ of the oscillating term in the vicinity of t_1,

$$\Delta s^R(t_1;\mathbf{k})=|s^R(t_1+\Delta t/2;\mathbf{k})-s^R(t_1-\Delta t/2;\mathbf{k})|$$
$$=\int_{t_1-\Delta t/2}^{t_1+\Delta t/2}dt\frac{\alpha(N+1)}{4\sqrt{\pi}}t^{-3/2}e^{t/2\tau(\mathbf{k})}\times\cos[\varepsilon(\mathbf{k})(t-t_1)]. \quad (3.15)$$

We approximate the time dependence of the envelope function by its value at $t=t_1$,

$$\Delta s^R(t_1;\mathbf{k})\approx\frac{\alpha(N+1)}{4\sqrt{\pi}}t_1^{-3/2}e^{t_1/2\tau(\mathbf{k})}\frac{2}{\varepsilon(\mathbf{k})-\omega_0}. \quad (3.16)$$

This approximation is valid if $\tau_{\rm QF}$, which is the period of oscillation of the oscillating function, is smaller than t_1 and τ, the scales on which the envelope function varies.

The pole approximation of self-energy is correct for times t_1 for which the test function is constant, i.e., the oscillation is negligible. The amplitude of the oscillation has to be compared with both the real and imaginary parts of the pole approximation of the self-energy; however, obtaining the correct value of the imaginary part of the pole approximation is more important since it has the direct meaning of the scattering rate while the real part produces only a small correction to the band structure.

We do not discuss the validity of the pole approximation of the real part since the artificial energy cutoff (2.19) does not guarantee the right behavior of the real part of the self-energy anyway. Thus our criterion of the validity of the pole approximation is to require that the amplitude of oscillation $\Delta s^R(t_1;\mathbf{k})$ be small compared to the pole approximation of the emission part of the self-energy $-2\,\mathrm{Im}\Sigma^R_{\mathrm{em}}(\varepsilon(\mathbf{k}))$ in the plateau

$$\frac{\Delta s^R_{\mathrm{em}}(t_1;\mathbf{k})}{-2\,\mathrm{Im}\Sigma^R_{\mathrm{em}}(\varepsilon(\mathbf{k}))} \ll 1 \ . \tag{3.17}$$

From (2.20) one finds the pole approximation of the emission part of the self-energy

$$-2\,\mathrm{Im}\Sigma^R_{\mathrm{em}}(\varepsilon(\mathbf{k}))=\alpha(N+1)\sqrt{\varepsilon(\mathbf{k})-\omega_0} \ , \tag{3.18}$$

where we have neglected the bandwidth cutoff ω_{cut}, since we assume $\varepsilon(\mathbf{k}) \ll \omega_{\mathrm{cut}}$, and we have left out the function $\theta(\varepsilon(\mathbf{k})-\omega_0)$, since we are interested in the high-energy part of the spectra.

By substituting (3.16) and (3.18) into (3.17) one finds that for t_1 to lie in the plateau the following condition has to be satisfied:

$$\frac{(2\pi)^2}{\sqrt{2}}\left[\frac{t_1}{\tau_{\mathrm{QF}}(\mathbf{k})}\right]^{3/2} e^{-t_1/2\tau(\mathbf{k})} \gg 1 \ . \tag{3.19}$$

The criterion works for both short- and long-time limits of the pole approximation. In the short-time region one can neglect the exponential term and thus one recovers the condition $t_1 > \tau_{\mathrm{QF}}(\mathbf{k})$. That is why the period of the oscillation of the test function plays the role of the quasiparticle formation time. In the long-time region the exponential term prevails. Then one can approximate t_1 by $3\tau(\mathbf{k})$ in the power-law term and one finds that the pole approximation applies for t_1 smaller than $3\tau(\mathbf{k})\ln[3\tau(\mathbf{k})/\tau_{\mathrm{QF}}(\mathbf{k})]$. The region of the validity of the pole approximation thus reads

$$\frac{2\pi}{\varepsilon(\mathbf{k})-\omega_0} \equiv \tau_{\mathrm{QF}}(\mathbf{k}) < t_1 < \tau_{\mathrm{div}}(\mathbf{k})$$

$$\equiv 3\tau(\mathbf{k})\ln\left[\frac{3\tau(\mathbf{k})}{\tau_{\mathrm{QF}}(\mathbf{k})}\right] \ . \tag{3.20}$$

Let us summarize the results of this section. We have formulated the criterion for the pole approximation of the retarded self-energy in terms of the semiasymptotic behavior of the test function $s^R(t_1;\mathbf{k})$. In the plateau region of s^R, given by (3.20), the pole approximation applies and the instantaneous approximation of the self-energy holds. The criterion (3.20) provides both the short-time limit and the long-time limit. In the short-time limit the criterion defines the quasiparticle formation time and our result confirms Landau's criterion (1.10).

The time scale τ_{QF} that allows us to use the instantaneous approximation of the retarded self-energy appears in the transport equation in the scattering-out term.[3] In this sense we have confirmed that the estimator (1.10) applies to the scattering-out integral and one can identify τ_{QF} with τ_{CD} with respect to scattering out. However, the discussion of the propagator does not provide a criterion for the scattering-in term.

IV. CHARACTERISTIC TIMES OF THE TRANSPORT EQUATION FOR THE CORRELATION FUNCTION IN EQUILIBRIUM

In this section we discuss characteristic times that appear in the transport equation for the correlation function $g^<(t)$ in equilibrium. Our main aim is to show under what conditions the double-time integration in (2.11) can be reduced to a single-time integration, i.e., under what criterion the quantum scattering in can be described by scattering rates of instantaneous events as it is done using the Fermi golden rule. We shall show that the relevant time scale for scattering in is identical to the time scale for scattering out, i.e., the quasiparticle formation time τ_{QF} defined by (3.14). In particular we shall show that the semiclassical Boltzmann equation for semiconductors is valid under the condition $\tau_{\mathrm{QF}} < \tau$. This condition is identical to the one proposed by Landau[5] for metals.

A. Transport equation for the correlation function local in space

An important property of the correlation function of the self-energy is that it depends only on those elements of the correlation function $G^<(t,\mathbf{k})$ that are local in space, $g^<(t)$. Indeed, from (2.10) and (2.11) one finds that $g^<(t)$ satisfies the closed equation

$$g^<(t)=\int_{-\infty}^{t} dt_1 \int_{-\infty}^{0} dt_2 \gamma^2 D^<(t_1-t_2) \times \int \frac{d\mathbf{k}}{(2\pi)^3} G^R(t-t_1;\mathbf{k}) \times G^A(t_2;\mathbf{k}) g^<(t_1-t_2) \ . \tag{4.1}$$

We note that the local element $g^<(t)$ provides only very limited information about the system. On the other hand, once $g^<(t)$ is evaluated, the generalized Kadanoff-Baym equation (2.11) is no longer a difficult self-consistent equation since $\Sigma^<$ is known. Accordingly, $g^<$ involves exactly that information necessary to evaluate the self-consistent transport equation and thus the characteristic time scale of (4.1) is the relevant one that determines the noninterference of individual scattering events within the generalized Kadanoff-Baym equation.

There are two time integrations in (4.1). We shall show that for a given t there is a region in the plane of t_1 and t_2, where the integration provides the dominant contribution; see Fig. 4. In the rest of this section we show the following three points: (i) the dominant region is a strip centered around $t_1-t_2=t$; (ii) the width of this strip is $|t_1-t_2-t| < \tau_{\mathrm{QF}}$; and (iii) in the limit of an infinitesimally narrow strip, i.e., for $\tau_{\mathrm{QF}} \ll \tau$, two subsequent scattering events do not interfere. From these three points it follows that the τ_{QF} can be interpreted as the collision duration time that confirms Landau's criterion (1.10).

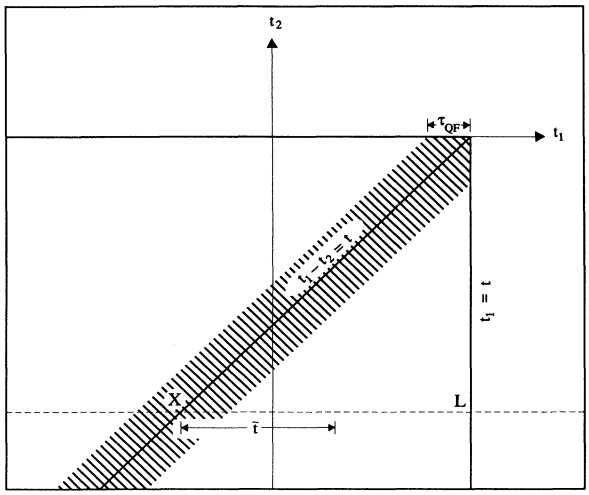

FIG. 4. Dominant region of the double-time integration in the GKB equation (4.1). The cross-hatched area represents the strip $|t_1-t_2-t|<\tau_{QF}$, which gives the dominant contribution.

B. Spectral representation of initial states of scattering in

The correlation function of the self-energy, $\Sigma^<(t_1-t_2)$, is the quantum-mechanical analog of the scattering-in integral of the Boltzmann equation. The correlation function $g^<(t_1-t_2)$ in (4.1) represents a sum over all the initial (incoming) states of the scattering-in event. The validity of the approximation of the quantum-mechanical scattering by instantaneous scattering rates, known from the Fermi golden rule, depends on the energy of the incoming electron. Within the language of characteristic times we can say that the collision duration time depends on the energy of the incoming electron. To separate incoming states with different energies we express the correlation function $g^<(t_1-t_2)$ in the energy representation

$$g^<(t_1-t_2)=\int \frac{d\omega}{2\pi} e^{-i\omega(t_1-t_2)} g^<(\omega) . \tag{4.2}$$

Using the free-particle approximation (2.19), we find the equilibrium correlation function from (2.13) $[i(G^R-G^A)=-2\,\mathrm{Im}G^R]$ and (2.19)

$$g^<(\omega)=f_{FD}(\omega)n_b\theta(\omega)\sqrt{\omega}e^{-\omega/\omega_{cut}} . \tag{4.3}$$

The approximation (4.3) will be used only for the internal line of the correlation function of the self-energy in the right-hand side of (4.1).

To discuss the collision duration time as a function of the energy of the incoming electron we define the response function $P(t;\omega)$ as

$$P(t;\omega)=\int_{-\infty}^{t} dt_1 \int_{-\infty}^{0} dt_2 \gamma^2 D^<(t_1-t_2) \times \int \frac{d\mathbf{k}}{(2\pi)^3} G^R(t-t_1;\mathbf{k}) G^A(t_2;\mathbf{k}) \times e^{-i\omega(t_1-t_2)} . \tag{4.4}$$

The reduced transport equation (4.1) in terms of the response function reads

$$g^<(t)=\int_0^\infty \frac{d\omega}{2\pi} P(t;\omega) g^<(\omega) . \tag{4.5}$$

The collision duration time we want to discuss is a property $P(t;\omega)$.

C. Response function

The response function $P(t;\omega)$ involves the retarded and advanced Green functions. For simplicity we take the quasiparticle approximation (3.9) of the propagators

$$G^R(t-t_1;\mathbf{k})G^A(t_2;\mathbf{k}) = -i\theta(t-t_1)e^{-i\varepsilon(\mathbf{k})(t-t_1)}e^{-(t-t_1)/2\tau(\mathbf{k})} \times i\theta(-t_2)e^{-i\varepsilon(\mathbf{k})t_2}e^{t_2/2\tau(\mathbf{k})} ; \tag{4.6}$$

here we have used $G^A(t_2)=\bar{G}^R(-t_2)$, where the overbar indicates the complex conjugate. Within the quasiparticle approximation (4.6) the response function reads

$$P(t;\omega)=\int_{-\infty}^{t} dt_1 \int_{-\infty}^{0} dt_2 \gamma^2 D^<(t_1-t_2) \int \frac{d\mathbf{k}}{(2\pi)^3} e^{-i[\omega-\varepsilon(\mathbf{k})](t_1-t_2)} e^{-i\varepsilon(\mathbf{k})t} e^{-(t-t_1-t_2)[1/2\tau(\mathbf{k})]} . \tag{4.7}$$

The characteristic times of the response function P are mostly determined by the integration over the final momentum $\mathbf{k}$. Since the momentum dependence of the integrand in (4.7) appears only via the quasiparticle energy $\varepsilon(\mathbf{k})$, it is advantageous to express the integration over momentum in terms of the integration over final-state energy. Within the free-particle approximation of the quasiparticle energy, $\varepsilon(\mathbf{k})=\varepsilon_0(\mathbf{k})$, the response function reads

$$P(t;\omega)=n_b\int_0^\infty \frac{dz}{2\pi}\sqrt{z}e^{-z/\omega_{cut}} \int_{-\infty}^{t} dt_1 \int_{-\infty}^{0} dt_2 \gamma^2 D^<(t_1-t_2) e^{-i(\omega-z)(t_1-t_2)} e^{-izt} e^{-(t-t_1-t_2)[1/2\tau(z)]} , \tag{4.8}$$

where we have introduced the exponential cutoff $\exp(-z/\omega_{cut})$ to simulate the finite size of the Brillouin zone.

The phonon correlation function

$$D^<(t_1-t_2)=N\frac{1}{2\omega_0}e^{-i\omega_0(t_1-t_2)}+(N+1)\frac{1}{2\omega_0}e^{i\omega_0(t_1-t_2)} \tag{4.9}$$

has the absorption part (first term) and the emission part (second term). Again we discuss below only the emission part; the absorption part is similar. Finally, the part of the response function we will study is

$$P_e(t;\omega)=(N+1)\alpha\int_0^\infty \frac{dz}{2\pi}\sqrt{z}\,e^{-z/\omega_{\rm cut}}\int_{-\infty}^{t}dt_1\int_{-\infty}^{0}dt_2 e^{-i(\omega-z-\omega_0)(t_1-t_2)}e^{-izt}e^{-(t-t_1-t_2)[1/2\tau(z)]}\,, \tag{4.10}$$

where $\alpha=\gamma^2 n_b/2\omega_0$.

D. Characteristic time scale of scattering in

As already mentioned there are two time integrations in the response function P and we want to find the region of t_1 and t_2 producing the dominant contribution. Towards this end we assume $t>0$ and keep t_2 fixed while integrating over t_1. The path of this integration corresponds to the dashed line in Fig. 4. Our aim will be to show that the main contribution to the integral comes from the region around the point of intersection of the dashed line and the line $t_1-t_2=t$ indicated by the point X in the figure. First we interchange the order of t_1 and t_2 integrations in (4.10) and then change the t_1 variable via the substitution $t'=t_1-t_2-t$ to get

$$P_e(t;\omega)=(N+1)\alpha e^{-i(\omega-\omega_0)t}\int_0^\infty \frac{dz}{2\pi}\sqrt{z}\,e^{-z/\omega_{\rm cut}}\int_{-\infty}^{0}dt_2 e^{t_2/\tau(z)}\int_{-\infty}^{-t_2}dt' e^{-i\{\omega-z-\omega_0+i/[2\tau(z)]\}t'}\,. \tag{4.11}$$

Now the dashed line in Fig. 4 is the path of the t' integration and the point X corresponds to $t'=0$.

Our approach will roughly be the following. First we replace the upper limit of the t' integration by an independent variable $\tilde{t}$ and study the integration over the path shown by the dashed line. We then show that this integration, as a function of $\tilde{t}$, exhibits a step-function-like jump at $\tilde{t}=0$. This indicates that the main contribution to the t' integration is coming from the region $t'\sim 0$, which is indicated by the point X in the figure. To accomplish this we introduce the test function

$$p_{\rm test}(\tilde{t},t_2;\omega)=\int_0^\infty \frac{dz}{2\pi}\sqrt{z}\,e^{-z/\omega_{\rm cut}}e^{t_2/\tau(z)}\times\int_{-\infty}^{\tilde{t}}dt' e^{-i\{\omega-z-\omega_0+i/[2\tau(z)]\}t'}\,. \tag{4.12}$$

In terms of this test function the response function (4.11) reads

$$P_e(t;\omega)=(N+1)\alpha e^{-i(\omega-\omega_0)t}\int_{-\infty}^{0}dt_2 p_{\rm test}(-t_2,t_2;\omega)\,. \tag{4.13}$$

The behavior of $p_{\rm test}(\tilde{t},t_2;\omega)$ as a function of $\tilde{t}$, obtained numerically, is demonstrated in Fig. 5 for two different choices of the time t_2. In Fig. 5(a) we have 10^{-13} s$=|t_2|<\tau=2.6\times10^{-13}$ s and one can see that the real part of the test function $p(\tilde{t},t_2;\omega)$ has a jump at $\tilde{t}=0$ which is nearly a step function. The deviation of the real part from the step function is in a region of the order of the quasiparticle formation time $\tau_{\rm QF}(\omega)=4\times10^{-14}$ s. The imaginary part of $p(\tilde{t},t_2;\omega)$ does not have the step contribution and oscillates around the value zero. As discussed above, the presence of this step shows that the main contribution to the integration is coming from the region around point X of width $\tau_{\rm QF}(\omega)$. In Sec. IV E these results will be confirmed analytically. In Fig. 5(b) the condition 10^{-12} s$=|t_2|>\tau=2.6\times10^{-13}$ s is satisfied and the step contribution is not so obvious. However, one can see that the test function oscillates around a nonzero value on the right-hand side of the figure. This value provides a step contribution. The magnitude of the step is relatively small compared to the oscillations; note that the scale in Fig. 5(b) is two orders of magnitude smaller than in Fig. 5(a). This reflects the fact that large values of $t_2\gg\tau$ contribute little to the generalized Kadanoff-Baym equation (4.1). Note that the position of the jump corresponds to the point X in Fig. 4 while the value of the test function entering the response function (4.13) is for the value $\tilde{t}=-t_2$ which corresponds to the point L in Fig. 4. The important fact is that at point L the value of the oscillations in the test function have a small magnitude and are smaller than the magnitude of the step. Below we discuss the approximation of neglecting the oscillations completely and keeping only the step. It will be shown in Sec. V that this approximation leads to the Boltzmann equation.

E. Pole (quasiparticle) approximation of scattering in

In this section we shall try to understand the features of the test function $p_{\rm test}(\tilde{t},t_2;\omega)$ (4.12) analytically. Integrating over t' gives

$$p_{\rm test}(\tilde{t},t_2;\omega)=\int_0^\infty \frac{dz}{2\pi}\sqrt{z}\,e^{-z/\omega_{\rm cut}}e^{t_2/\tau(z)}\times\frac{i}{\omega-z-\omega_0+i/[2\tau(z)]}\times e^{-i\{\omega-z-\omega_0+i/[2\tau(z)]\}\tilde{t}}\,. \tag{4.14}$$

Now we discuss the properties of the integral over energy z via contour integration in the complex plane; the contours are shown in Fig. 6. The choice of the contours depends on the sign of $\tilde{t}$. For $\tilde{t}<0$ the integrand of (4.14) goes to zero exponentially in the right lower half-plane, i.e., for $\mathrm{Re}z>0$ and $\mathrm{Im}z<0$. This region is contoured by the curve $c_<$ with clockwise orientation. For $\tilde{t}>0$ the integrand of (4.14) goes to zero exponentially in the right upper half-plane, i.e., for $\mathrm{Re}z>0$ and $\mathrm{Im}z>0$. This region is contoured by the curve $c_>$, which has a counterclockwise orientation.

We express the integration along the real axis from 0 to ∞ as the integral over the contour minus the integral along the imaginary axis,

$$p_{\text{test}}(\tilde{t},t_2;\omega)=\theta(-\tilde{t})\left[-2\pi i\sum \text{Res}_{c_<}-\int_{-i\infty}^{0}\frac{dz}{2\pi}\sqrt{z}\,e^{-z/\omega_{\text{cut}}}e^{t_2/\tau(z)}\frac{i}{\omega-z-\omega_0+i/[2\tau(z)]}e^{-i(\omega-z-\omega_0+i\{1/[2\tau(z)]\})\tilde{t}}\right]$$
$$+\theta(\tilde{t})\left[2\pi i\sum \text{Res}_{c_>}+\int_{0}^{i\infty}\frac{dz}{2\pi}\sqrt{z}\,e^{-z/\omega_{\text{cut}}}e^{t_2/\tau(z)}\frac{i}{\omega-z-\omega_0+i/[2\tau(z)]}e^{-i(\omega-z-\omega_0+i\{1/[2\tau(z)]\})\tilde{t}}\right]. \tag{4.15}$$

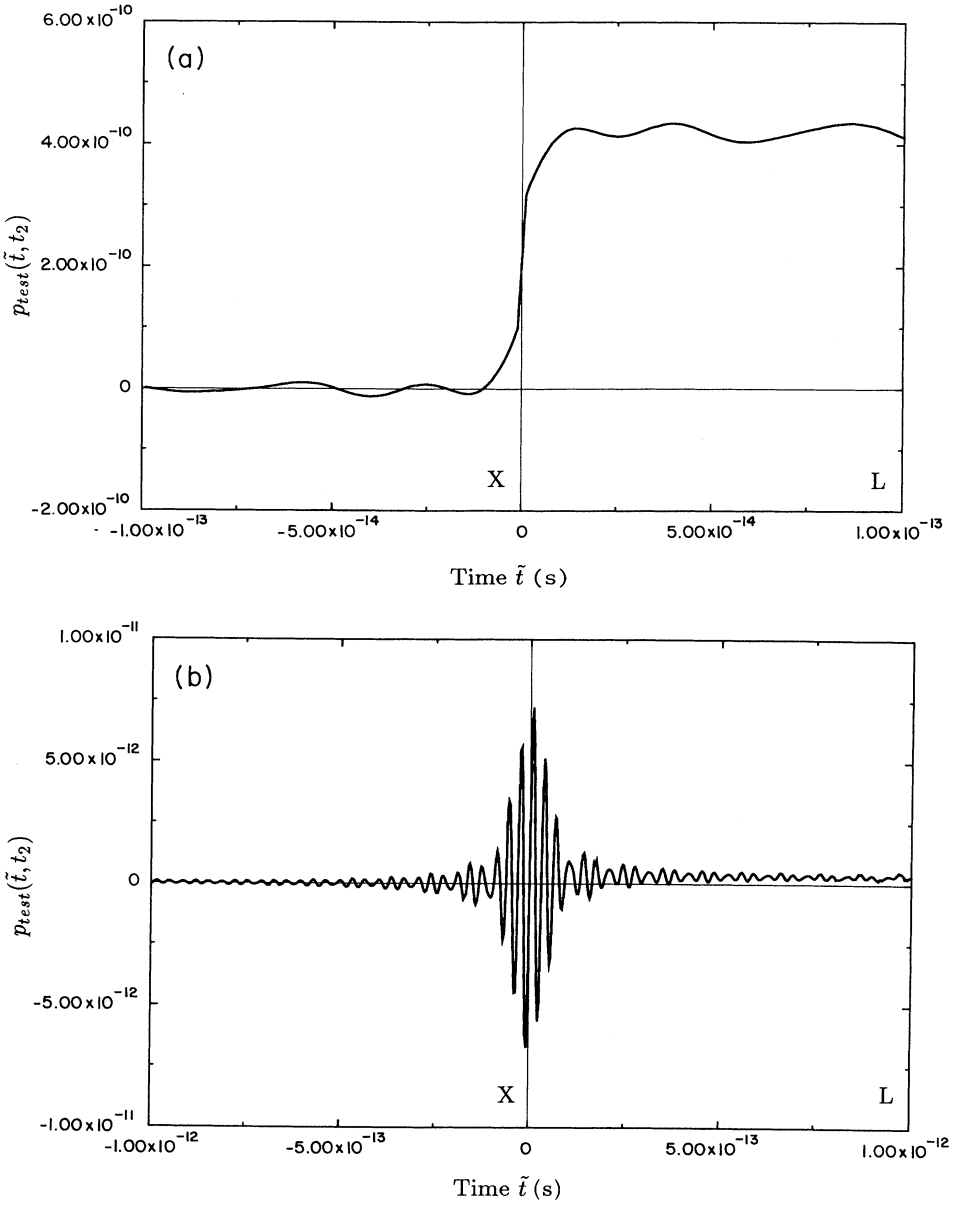

FIG. 5. Test of the pole approximation for scattering in. The real part of the test function $p_{\text{test}}(\tilde{t},t_2)$ is plotted for the energy $\omega=0.2$ eV and for two different values of t_2 (time separation between consecutive scatterings). In (a) we have $t_2=-10^{-13}$ s, which corresponds to scattering events close to one another, since the quasiparticle lifetime is $\tau=2.6\times10^{-13}$ s. In (b) the value $t_2=-10^{-12}$ s is used, which corresponds to a long free flight between two events. For both values of t_2, $\bar{p}$ behaves like $\theta(\tilde{t})$. The sharp jump at $\tilde{t}=0$ (represented by point X in Fig. 4) develops within the time scale $\hbar/\omega_{\text{cut}}\sim10^{-16}$ s (where ω_{cut} is the band cutoff), which is too short to be readable from the figure. The oscillations have the period $\tau_{\text{QF}}\sim4\times10^{-14}$ s and the *asymptotic* value is achieved for $\tilde{t}>\tau_{\text{QF}}$ in (a) and after few periods in (b). Comparing (a) and (b) one sees that the oscillations are relatively larger for bigger $|t_2|$. Note that the scales are different in (a) and (b).

The only pole occurs in the contour $c_>$ for $z=\omega-\omega_0+i/2\tau(\omega-\omega_0)$ with contribution

$$p_{\rm pole}(\tilde{t},t_2;\omega)=\theta(\tilde{t})\theta(\omega-\omega_0)\sqrt{\omega-\omega_0}e^{-(\omega-\omega_0)/\omega_{\rm cut}}e^{t_2/\tau(\omega-\omega_0)}, \tag{4.16}$$

where we have omitted the complex shift of the energy in the square root. We observe from (4.16) that the pole contributes to the reduced transport equation (4.1) in the form of a sharp θ function in time $\tilde{t}$.

Both integrations along the imaginary axis (we call them the off-pole contributions $p_{\rm off}$) are of the same order and have similar behavior. The region $\tilde{t}>0$ is more important because $\tilde{t}=-t_2$ in (4.13) and $t_2<0$, thus we focus on the case $\tilde{t}>0$. We can estimate the integral along the imaginary axis in (4.15) for values of $\tilde{t}$ satisfying $|\tilde{t}|>2\pi/|\omega-\omega_0|$, i.e., for $|\tilde{t}|>\tau_{\rm QF}(\omega)$. For these values of $\tilde{t}$ the integration is important only in the region $\mathrm{Im}z\ll\omega-\omega_0$ and one can approximate z by zero except for the term $\sqrt{z}\exp(iz\tilde{t})$,

$$\begin{aligned}p_{\rm off}(\tilde{t},t_2;\omega)&=\int_0^{i\infty}\frac{dz}{2\pi}\sqrt{z}e^{-z/\omega_{\rm cut}}e^{t_2/\tau(z)}\frac{-i}{z-\omega+\omega_0-i/2\tau(z)}e^{-i\{\omega-z-\omega_0+i[1/2\tau(z)]\}\tilde{t}}\\&\approx e^{t_2/\tau(0)}\frac{\sqrt{i}}{-\omega+\omega_0-i/2\tau(0)}e^{-i\{\omega-\omega_0+i[1/2\tau(0)]\}\tilde{t}}\int_0^{\infty}\frac{dz}{2\pi}\sqrt{z}e^{-z\tilde{t}}.\end{aligned} \tag{4.17}$$

Since

$$\int_0^{\infty}\frac{dz}{2\pi}\sqrt{z}e^{-z\tilde{t}}=\frac{1}{4\sqrt{\pi}}\tilde{t}^{-3/2} \tag{4.18}$$

one obtains the oscillating tail of the test function $p_{\rm test}$, which has the same form as the deviations from the pole value of the retarded self-energy (see Sec. III). The prefactor of (4.18) falls exponentially slower than (4.16) since $1/\tau(0)<1/\tau(\omega-\omega_0)$. In the estimates that follow we will neglect $1/\tau(0)$ compared to $1/\tau(\omega-\omega_0)$.

Now we are in a position to estimate the width of the shaded region of Fig. 4 for fixed t_2. The shaded area indicates the region where the off-pole contribution cannot be neglected compared with the pole contribution. According to (4.17) and (4.16) one finds

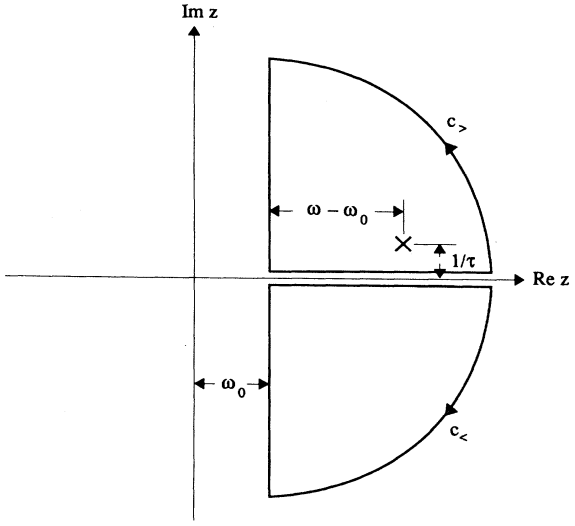

FIG. 6. Contours $c_<$ and $c_>$ in the complex plane used to solve (4.14). The cross represents the pole enclosed by the contour $c_>$.

$$\left|\frac{p_{\rm off}(\tilde{t},t_2;\omega)}{p_{\rm pole}(\tilde{t},t_2;\omega)}\right|\approx\frac{1}{8\pi^2\sqrt{2}}e^{-t_2/\tau(\omega-\omega_0)}\times\left[\frac{\tilde{t}}{2\pi/(\omega_2-\omega_0)}\right]^{-3/2}. \tag{4.19}$$

The energy distance from the nonanalytical point $\omega-\omega_0$ can be expressed in terms of the quasiparticle formation time $\tau_{\rm QF}$. Then, from the requirement that the off-pole contribution has to be negligible compared to the pole contribution, one gets

$$\tilde{t}>\tau_{\rm QF}(\omega)e^{2|t_2|/3\tau(\omega-\omega_0)}. \tag{4.20}$$

Thus we see that for t_2 of order τ or less, the dominant region of integration represented by the shaded area in Fig. 4 has a width of order $\tau_{\rm QF}$.

In the reduced transport equation $\tilde{t}$ is replaced by $-t_2$, see (4.13); we can also ask the question that for which values of t_2 the pole approximation is valid? By replacing $\tilde{t}$ by $|t_2|$ in (4.20) we get

$$\tau_{\rm QF}(\omega)<|t_2|<\tfrac{3}{2}\tau(\omega-\omega_0)\ln\left[\frac{\tau(\omega-\omega_0)}{\tau_{\rm QF}(\omega)}\right]. \tag{4.21}$$

One can see that the criterion (4.21) gives less than half the value of the upper limit compared to the criterion (3.20) for the retarded self-energy, the order of magnitude, however, is the same. The upper limit on $|t_2|$ shows that for large enough values of $|t_2|$ the off-pole contribution cannot be neglected compared the pole part. This is due to the fact that the pole contribution decays faster than the off-pole contribution; however, the magnitude of both these contributions is small in this region.

V. RELATION OF THE POLE APPROXIMATION TO THE BOLTZMANN EQUATION

The pole approximation (4.16) brings about a strong simplification in the physical processes involved in the correlation function $g^<$. In fact, in this section we show that within the pole approximation the quantum generalized Kadanoff-Baym equation (4.1) reduces to the semi-

classical Boltzmann equation and establishes the criterion for validity to be $\tau_{QF} \ll \tau$.

A. The generalized Kadanoff-Baym equation within the pole approximation

Now we write $g^<$ within the pole approximation (4.16) derived in Sec. IV. Equation (4.16) includes only the emission part; the absorption part is obtained by replacing $-\omega_0 \leftrightarrow \omega_0$ and $(N+1) \leftrightarrow N$. The substitution of (4.16) into (4.13) gives

$$P(t;\omega)=\int_{-\infty}^{0} dt_2 [e^{-i(\omega-\omega_0)t}(N+1)\alpha\theta(\omega-\omega_0) \times\sqrt{\omega-\omega_0}e^{t_2/\tau(\omega-\omega_0)} +e^{-i(\omega+\omega_0)t}N\alpha\theta(\omega+\omega_0)\sqrt{\omega+\omega_0} \times e^{t_2/\tau(\omega+\omega_0)}] . \tag{5.1}$$

Note that (4.13) involves an integration over all values of negative t_2, whereas the pole approximation is only valid under the condition (4.21). However, since the main region of integration in (4.13) extends over a quasiparticle lifetime τ, the error introduced by the pole approximation is only of the order τ_{QF}/τ. Using (5.1) in (4.5) gives

$$g^<(t)=\int_0^\infty \frac{d\omega}{2\pi}\alpha\sqrt{\omega}e^{-i\omega t} \times\int_{-\infty}^{0} dt_2 e^{t_2/\tau(\omega)}[(N+1)g^<(\omega+\omega_0) +Ng^<(\omega-\omega_0)] , \tag{5.2}$$

where we have used the substitutions $\omega\rightarrow\omega\pm\omega_0$ in the emission and absorption parts. Finally, to get this equation closer to the semiclassical Boltzmann equation, we express the ω integration as an integration over the momentum $\mathbf{k}$ [note that $\alpha=(\gamma^2/2\omega_0)(\sqrt{2}/\pi)m^{3/2}$],

$$g^<(t)=\int \frac{d\mathbf{k}}{(2\pi)^3}e^{-i\varepsilon_0(\mathbf{k})t} \times\int_{-\infty}^{0} dt_2 e^{t_2/\tau(\mathbf{k})}\frac{\gamma^2}{2\omega_0}[(N+1)g^<(\varepsilon_0(\mathbf{k})+\omega_0) +Ng^<(\varepsilon_0(\mathbf{k})-\omega_0)] . \tag{5.3}$$

B. Connection with the Boltzmann equation

Now we want to show that (5.3) is equivalent to the semiclassical Boltzmann equation. First we define a function $f(\mathbf{k})=f(\varepsilon_0(\mathbf{k}))$ by [see (2.9)]

$$g^<(t)=\int \frac{d\mathbf{k}}{(2\pi)^3}e^{-i\varepsilon_0(\mathbf{k})t}f(\mathbf{k}) . \tag{5.4}$$

Then we substitute this into the left-hand and right-hand sides of (5.3), after using inverse of (4.2), and find that $f(\mathbf{k})$ must satisfy the following equation:

$$f(\mathbf{k})=\int_{-\infty}^{0} dt_2 e^{t_2/\tau(\mathbf{k})}\frac{\gamma^2}{2\omega_0}\int\frac{d\mathbf{k}'}{(2\pi)^3}f(\mathbf{k}')[N\delta(\varepsilon_0(\mathbf{k})-\varepsilon_0(\mathbf{k}')-\omega_0)+(N+1)\delta(\varepsilon_0(\mathbf{k})-\varepsilon_0(\mathbf{k}')+\omega_0)] . \tag{5.5}$$

This equation is exactly the Boltzmann equation in its integral form.[9] We want to remind the reader that the arguments given in this section do not depend on the temperature. Indeed we have derived the Boltzmann equation *subject to the Landau criterion* $\tau_{QF} \ll \tau$ *and not* $\hbar/k_B T \ll \tau$.

VI. THE COMPLETED COLLISION APPROXIMATION

Up to now we have discussed the integrated transport equation (4.1). This is not a standard approach to the derivation of the Boltzmann equation or its more general derivation from quantum-field theory (in our paper represented by the generalized Kadanoff-Baym equation). In derivations of the Boltzmann equation one usually assumes that the strip which gives the dominant contribution to the double-time integral in (2.11) is defined by $|t_1-t_2|<\tau_{CD}$. This assumption is used to derive the completed collision approximation;[3,10] the same assumption leads to a simplification which is called the generalized Kadanoff-Baym ansatz.[11] We will see that using this assumption leads to an overly restrictive, temperature-dependent criterion for the validity of the Boltzmann equation. In this section we will discuss this assumption and demonstrate its failure in the high-energy tails of the distribution.

A. Correlation function of the self-energy in the time representation

The time dependence of the correlation function of the self-energy $\Sigma^<(t)$ can be evaluated analytically from the spectral representation

$$\Sigma^<(t)=\int\frac{d\omega}{2\pi}e^{i\omega t}\Sigma^<(\omega) . \tag{6.1}$$

According to the Schwinger boundary condition (2.14) and the formula for the spectral function of the self-energy (2.21), one finds $\{i[\Sigma^R(\omega)-\Sigma^A(\omega)] = -2\,\mathrm{Im}\Sigma^R(\omega)\}$

$$\Sigma^{<}(\omega)=n_d e^{-\omega/k_B T}\alpha[\theta(\omega+\omega_0)N\sqrt{\omega+\omega_0}\,e^{-(\omega+\omega_0)/\omega_{\rm cut}} +\theta(\omega-\omega_0)(N+1) \times\sqrt{\omega-\omega_0}\,e^{-(\omega-\omega_0)/\omega_{\rm cut}}] , \tag{6.2}$$

which results in

$$\Sigma^{<}(t)=\frac{\alpha n_d}{4\sqrt{\pi}}[Ne^{-i\omega_0 t}+(N+1)e^{i\omega_0 t}]\left[\frac{1}{k_B T}+it\right]^{-3/2} , \tag{6.3}$$

where we have neglected $\hbar/\omega_{\rm cut}\sim 10^{-16}$ s compared to the inverse temperature $\hbar/k_B T\sim 3\times 10^{-14}$ s. One can see, that up to times $t\sim\hbar/k_B T\sim 3\times 10^{-14}$ s, the time dependence of $\Sigma^{<}(t)$ is dominated by the temperature.

B. Completed collision approximation

It is interesting to see how the integrations over times t_1 and t_2 in the entire generalized Kadanoff-Baym equation (2.11) proceed for different final momenta $\mathbf{k}$, i.e., for different energies of the final state. We study this to determine when the completed collision approximation[3,10] is satisfied and thus when the Boltzmann equation can be derived with the help of this approximation.

To derive the Boltzmann equation in the way used in high-field transport theory,[3,10] one keeps the momentum dependence while the energy is integrated out, i.e., one sets $t=0$. Thus the generalized Kadanoff-Baym equation (2.11) reads, using quasiparticle approximation for $G^{R,A}$ (4.6),

$$G^{<}(0;\mathbf{k})=\int_{-\infty}^{0}dt_2\int_{t_2}^{0}dt_1 e^{i\varepsilon(\mathbf{k})(t_1-t_2)}e^{(t_1+t_2)/2\tau(\mathbf{k})}\Sigma^{<}(t_1-t_2)+\int_{-\infty}^{0}dt_1\int_{t_1}^{0}dt_2 e^{i\varepsilon(\mathbf{k})(t_1-t_2)}e^{(t_1+t_2)/2\tau(\mathbf{k})}\Sigma^{<}(t_1-t_2) , \tag{6.4}$$

where we have separated the regions $t_1>t_2$ and $t_1<t_2$ as is necessary if one uses the generalized Kadanoff-Baym ansatz.[3,11]

The correlation function of the self-energy satisfies $\Sigma^{<}(t_1-t_2)=\overline{\Sigma}^{<}(t_2-t_1)$, where the overbar denotes the complex conjugate [this can be checked from (6.3)]. Interchanging t_1 and t_2 in the second term of (6.4) one finds that the second term is a complex conjugate of the first term. Here the correlation function $G^{<}(0;\mathbf{k})$ is a real function [this can be checked independently from (2.13) by integration over energy ω]. Thus we express (6.4) as

$$\begin{aligned}G^{<}(0;\mathbf{k})&=2\,\mathrm{Re}\int_{-\infty}^{0}dt_2\int_{t_2}^{0}dt_1 e^{i\varepsilon(\mathbf{k})(t_1-t_2)}e^{(t_1+t_2)/2\tau(\mathbf{k})}\Sigma^{<}(t_1-t_2)\\&=\int_{-\infty}^{0}dt_2 e^{t_2/\tau(\mathbf{k})}2\,\mathrm{Re}\int_{t_2}^{0}dt_1 e^{i\varepsilon(\mathbf{k})(t_1-t_2)}e^{(t_1-t_2)/2\tau(\mathbf{k})}\Sigma^{<}(t_1-t_2) .\end{aligned} \tag{6.5}$$

This equation has a structure similar to the Boltzmann equation in the integral form.[3] To get the Boltzmann equation one needs two simplifications. (i) The time diagonal element of the correlation function $G^{<}(0;\mathbf{k})$ has to be interpreted as the Boltzmann distribution $f(\mathbf{k})$. This is true in the weak-coupling limit we assume here. (ii) The correlation function of the self-energy $\Sigma^{<}(\omega)$ has to be approximated by some energy-independent function, which is its pole value $\Sigma^{<}(\varepsilon(\mathbf{k}))$. Since energy independence in the double-time representation corresponds to a function proportional to the δ function in the difference time, the Boltzmann equation for equilibrium in our notation reads [see (5.5) and substitute (5.4) in (2.8) for comparison]

$$f(\mathbf{k})=\int_{-\infty}^{0}dt_2 e^{t_2/\tau(\mathbf{k})}\Sigma^{<}(\omega)|_{\omega=\varepsilon(\mathbf{k})} . \tag{6.6}$$

From a straightforward comparison of (6.6) and the second line of (6.5) one finds that the Boltzmann equation holds if

$$2\,\mathrm{Re}\int_{t_2}^{0}dt_1 e^{i\varepsilon(\mathbf{k})(t_1-t_2)}e^{(t_1-t_2)/2\tau(\mathbf{k})}\Sigma^{<}(t_1-t_2) =\Sigma^{<}(\omega)|_{\omega=\varepsilon(\mathbf{k})} . \tag{6.7}$$

This equation cannot be satisfied for a general time argument t_2 and a general momentum $\mathbf{k}$. Equation (6.7) is called the *completed collision approximation.*

C. Numerical test of the completed collision approximation

We keep the time t_2 as a parameter and test the possibility to integrate out t_1 in (6.5), i.e., the applicability of the completed collision approximation (6.7). To simplify the notation we introduce the test functions

$$\begin{aligned}c_{\rm em}(t_2;\mathbf{k})&=\frac{\alpha(N+1)n_d}{4\sqrt{\pi}}\int_{t_2}^{0}dt_1 e^{i\varepsilon(\mathbf{k})(t_1-t_2)}e^{-(t_1-t_2)/2\tau(\mathbf{k})}e^{i\omega_0(t_1-t_2)}\left[\frac{1}{k_B T}+i(t_1-t_2)\right]^{-3/2}\\&=\frac{\alpha(N+1)n_d}{4\sqrt{\pi}}\int_{0}^{-t_2}dt\, e^{i\varepsilon(\mathbf{k})t}e^{t/2\tau(\mathbf{k})}e^{i\omega_0 t}\left[\frac{1}{k_B T}+it\right]^{-3/2}\end{aligned} \tag{6.8}$$

and

$$c_{\rm ab}(t_2;\mathbf{k})=\frac{\alpha N n_d}{4\sqrt{\pi}}\int_{t_2}^{0}dt_1 e^{i\varepsilon(\mathbf{k})(t_1-t_2)}e^{-(t_1-t_2)/2\tau(\mathbf{k})}e^{i\omega_0(t_1-t_2)}\left[\frac{1}{k_BT}+i(t_1-t_2)\right]^{-3/2}$$
$$=\frac{\alpha N n_d}{4\sqrt{\pi}}\int_{0}^{-t_2}dt\, e^{i\varepsilon(\mathbf{k})t}e^{t/2\tau(\mathbf{k})}e^{i\omega_0 t}\left[\frac{1}{k_BT}+it\right]^{-3/2}, \tag{6.9}$$

where we have used the explicit form of the correlation function of the self-energy (6.3) and separated the emission and the absorption parts.

Equation (6.5) in terms of the test function reads

$$G^{<}(0;\mathbf{k})=2\,\mathrm{Re}\int_{-\infty}^{0}dt_2 e^{-t_2/\tau(\mathbf{k})}\times[c_{\rm em}(t_2;\mathbf{k})+c_{\rm ab}(t_2;\mathbf{k})]\ . \tag{6.10}$$

Again we are looking for a plateau in the test functions $c_{\rm em}(t_2;\mathbf{k})$ and $c_{\rm ab}(t_2;\mathbf{k})$ which will provide us with the time region within which scattering in can be approximated by instantaneous scattering events. The plot of the test function $2\,\mathrm{Re}c_{\rm em}(t_2;\mathbf{k})$ is shown in Fig. 7. We note that in the short-time region $t<10^{-14}$ s the test function for the completed collision approximation $c_{\rm em}$ increases more slowly than the test function for the pole approximation s^R introduced in Sec. III (compare Figs. 3 and 7). This follows from the presence of the inverse temperature $1/k_BT\sim 3\times10^{-14}$ s in (3.13). Aside from the slow increase in the short-time region, the amplitude of oscillation around the pole value is larger than the amplitude of oscillations around the pole value of the retarded self-energy. This second feature is discussed analytically in Sec. VI D.

D. Analytical criterion for the validity of the completed collision approximation

For the analytic discussion of the completed collision approximation we can advantageously use the results of Sec. III. One can again define the amplitude of the oscillation of the rest function $c_{\rm em}(t_2;\mathbf{k})$ as

$$\Delta c_{\rm em}(t_2;\mathbf{k})=|c_{\rm em}(t_2+\Delta t/2;\mathbf{k})-c_{\rm em}(t_2-\Delta t/2;\mathbf{k})|$$
$$=\int_{t_2-\Delta t/2}^{t_2+\Delta t/2}dt\frac{\alpha(N+1)n_d}{4\sqrt{\pi}}t^{-3/2}e^{t/2\tau(\mathbf{k})}\times\cos[\varepsilon(\mathbf{k})(t-t_2)]\ , \tag{6.11}$$

where $\Delta t=\tau_{\rm QF}/2$ and we have used that $1/k_BT\ll\tau(\mathbf{k})$ to simplify the power-law term of the integrand; therefore we have excluded the short-time region from this discussion. The "center" of the plateau is again at $3\tau(\mathbf{k})$. The amplitude of the oscillation of the test function $c_{\rm em}$ is equal to the amplitude of the oscillation of the test function s^R except for the prefactor

$$\Delta c_{\rm em}(t_2;\mathbf{k})=n_d\Delta s^R(t_2;\mathbf{k})\ , \tag{6.12}$$

therefore we can use (3.16).

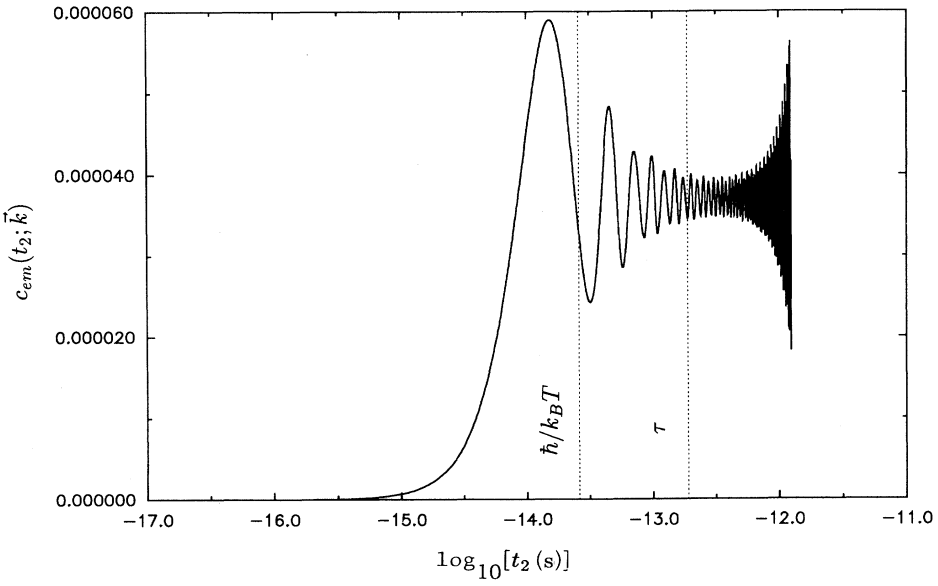

FIG. 7. Test of the completed oscillation approximation. Test function $c_{\rm em}(t_2;\mathbf{k})$ is evaluated at the energy $\varepsilon_0(\mathbf{k})=0.2$ eV. This energy corresponds to $7.7k_BT$ at room temperature, and thus is in the high-energy tail of the electronic distribution. For the most important time $t_2<\tau$, the figure does not contain a region that can be identified by a reasonable plateau, even though the naive criterion $\hbar/k_BT=2.6\times10^{-14}\ll\tau=2\times10^{-13}$ s is well satisfied.

The criterion of the validity of the completed collision approximation is derived from the requirement that the amplitude of the oscillation $2\Delta c_{\rm em}(t_2;\mathbf{k})$ (the factor of 2 comes from $2\,{\rm Re}c_{\rm em}$) be small compared to the pole approximation $\Sigma^<_{\rm em}(\varepsilon(\mathbf{k}))$,

$$\frac{2\Delta c_{\rm em}(t_2;\mathbf{k})}{\Sigma^<_{\rm em}(\varepsilon(\mathbf{k}))} \ll 1 \ . \tag{6.13}$$

From (2.14) one finds

$$\Sigma^<_{\rm em}(\varepsilon(\mathbf{k}))=n_d e^{-\varepsilon(\mathbf{k})/k_BT}(-2){\rm Im}\Sigma^R_{\rm em}(\varepsilon(\mathbf{k})) \ , \tag{6.14}$$

and thus the criterion (6.13) becomes

$$\frac{2\Delta c_{\rm em}(t_2;\mathbf{k})}{\Sigma^<_{\rm em}(\varepsilon(\mathbf{k}))}=\frac{1}{e^{-\varepsilon(\mathbf{k})/k_BT}}\frac{\Delta s^R_{\rm em}(t_2;\mathbf{k})}{-{\rm Im}s^R_{\rm em}[3\tau(\mathbf{k});\mathbf{k}]} \ll 1 \ . \tag{6.15}$$

Using the estimate of $\Delta s^R_{\rm em}/{\rm Im}s^R_{\rm em}$ derived in Sec. III one finds that the time region within which the completed oscillation approximation is accurate is set by the condition

$$\frac{(2\pi)^2}{\sqrt{2}}\left[\frac{t_2}{\tau_{\rm QF}(\mathbf{k})}\right]^{3/2} e^{-t_2/2\tau(\mathbf{k})} \gg e^{\varepsilon(\mathbf{k})/k_BT} \ . \tag{6.16}$$

This criterion has the same left-hand side as (3.19), but the right-hand side depends strongly on the energy $\varepsilon(\mathbf{k})$. For energies that are of the order of averaged thermal energy k_BT the right-hand side of (6.16) is of the order of unity and one finds that in the long-time region the completed collision approximation is satisfied with the same accuracy as the pole approximation of the retarded self-energy. In the high-energy region $\varepsilon(\mathbf{k})>k_BT$ the right-hand side of (6.16) is large and the plateau region disappears entirely and the completed collision approximation does not apply at all.

Note that it is not the large amplitude of the oscillation $\Delta c_{\rm em}(t_2;\mathbf{k})$ which leads to the failure of the completed collision approximation at high energies, but instead, it is the small value of the total scattering-in rate which makes these oscillations relatively large (see Fig. 2). Although, after integrating over momenta, the discrepancy introduced by the completed collision approximation at high energies becomes relatively small again, one has to be careful about this approximation if one studies phenomena that depend exclusively on the high-energy tails of the distribution. In contrast, we have shown in Sec. V B that within the pole approximation of the scattering-in integral, the Boltzmann equation holds even in the high-energy tails of the distribution where the completed collision approximation fails.

VII. CONCLUSIONS

In the paper we have discussed the characteristic time scales that appear in the generalized Kadanoff-Baym (GKB) equation (transport equation for the single-electron correlation function) in equilibrium. We used parameters appropriate for a nondegenerate electron gas in GaAs. By analyzing the GKB transport equation we have identified a short-time $\tau_{\rm CD}$ collision duration time, which has the property that if the condition $\tau_{\rm CD}\ll\tau$ is satisfied, then the Boltzmann equation is valid, where τ is the quasiparticle life time. This time scale is identical with the quasiparticle formation time $\tau_{\rm QF}$, associated with the single-particle propagator. The best estimate of this time scale is given by the inverse energy separation from the nearest nonanalytical point in the local density of states; this is in agreement with the argument proposed by Landau for the elastic scattering in metals. In particular, the inverse temperature does not enter into the criterion for the validity of the Boltzmann equation.

We have also discussed the validity of the completed collison approximation, which is an alternative way to derive the Boltzmann equation. We show that this approximation is overly restrictive and fails in the high-energy tails of the distribution function where $\varepsilon_0(\mathbf{k})\ll k_BT$, and where the Boltzmann equation is still valid.

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