PHYSICAL REVIEW B VOLUME 43, NUMBER 1 1 JANUARY 1991

Scale-invariant disorder in fracture and related breakdown phenomena

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(Received 26 July 1990)

We introduce and discuss the concept of scale-invariant disorder in connection with breakdown and fracture models of disordered brittle materials. We show that in the case of quenched-disorder models where the local breaking thresholds are randomly sampled, only two numbers determine the scaling properties of the models. These numbers characterize the behavior of the distribution of thresholds close to zero and to infinity. We review briefly some results obtained in the literature and show how they fit into this framework. Finally, we address the case of an annealed disorder, and show via a mapping onto a quenched-disorder model, that our analysis is also valid there.

I. INTRODUCTION

It has been long known that material properties may be strongly influenced by the presence of disorder. However, the sensitivity to the disorder is widely different, according to which property one is interested in. Usually, transport properties, for example, conductance and elastic constants, are much less sensitive than breakdown properties such as material strength. In breakdown processes, such as the seemingly simple case of brittle fracture—i.e., fracture that do not involve local plastic deformations—extreme sensitivity to rare events makes the problem very difficult to handle theoretically. This has led to transport properties having been much more studied. As a result, they are therefore today much better understood than the breakdown processes. However, recently the breakdown problems have been approached within the same statistical-physics framework as the transport problems, and a number of interesting results have been found.[1]

In particular, several recent numerical studies of the breakdown of networks of either elastic or electrical elements have been done with the aim of investigating the relation between disorder and the global properties of the entire network, such as evolution of breaking stress or strain, the damage at peak stress, or the total damage at the breakdown point. At the outset, it was expected that these global properties of the networks should depend strongly on the type of disorder that was put into the models. However, this turned out not to be the case, but a rather puzzling picture emerged: For wide classes of disorders—here in the maximum loads each bond in the network is able to sustain before breaking down—the global properties are very little sensitive,[2-4] in fact so little that one might suspect *universality*, in much the same way as has been found in connection with transport properties in disordered materials.[5] For other distributions, on the other hand, the dependence of the global properties on the disorder was very strong.

The universality manifests itself through scaling laws between the global properties of the networks and the size of the networks. These scaling laws are governed by nontrivial scaling exponents, and universality means that the exponents are independent of the details of the particular breakdown model that is used. The suggestion of universality was based on numerical measurements of the exponents involved in three very different models, one based on an electrical network of random fuses,[3] the second one based on a network of elastic beams, each having random breaking thresholds,[2] and the third model consisting of a network of central-force springs—i.e., springs that rotate frictionless about their end points. Also in this case the springs were assigned random maximum loads they could sustain before breaking.[2] In each of these three models several different distributions for the randomness of the breaking thresholds were investigated. The exponents describing the breakdown processes seen in these models as a function of external load were found to be rather insensitive to the disorder (we will come back to this statement more precisely later) and even to which model was used.

In Ref. 4, a study was made of the transition from an ohmic to a superconducting state of a network of superconductors, as a current through it is lowered. In this case the disorder is introduced through the current thresholds of each bond in the network, below which the bond becomes superconducting. Also in this case, scaling behavior was found, with the corresponding exponents being insensitive to the distribution of thresholds.

The appearance of such a universality in breakdown processes such as fracture would be quite important, as these are not scaling exponents describing a physical system near a critical point—i.e., in a limited region of some parameter space—but rather describing the *typical* situation, much in the same way that the Kolmogorov scaling theory is able to describe universal features in fully developed turbulence.[6]

It is the aim of this paper to investigate under what circumstances the scaling laws appear, and what the source is for the observed insensitivity to the disorder.

In Sec. II, we discuss the random-fuse model[7] in more detail and summarize the scaling results of Refs. 2 and 3 as well as other works on this model that did *not* show such scaling behavior.[8,9]

In Sec. III we introduce the notion of *scale-invariant* disorder and relate the appearance of the scaling laws in the breakdown process to this scale invariance. Two initial distributions having the same scale invariant components are expected to behave in a similar way.

The meaning of "scale-invariant distributions" may be understood as follows: The current distribution in a network of fuses broadens as the breakdown process proceeds. If we pick one well-defined point in the breakdown process, for example right before the network breaks apart, the width of the current distribution depends on the system size in a *logarithmic* way. The threshold distribution must be as broad as the current distribution, and thus its width must also increase logarithmically with the system size. This is so, since when the current distribution becomes broader than the threshold distribution a single crack will occur, breaking the entire network apart without much further widening of the current distribution. Those components of the disorder that depend on the system size, at least logarithmically, form the scale-invariant part of the distribution. Those distributions that were used in works where no scaling laws were observed[8,9] had a scale-invariant part that was equivalent to *no disorder*.

When no size dependence is present in the disorder distribution, such as is the case in *correlated* distributions, the scale-invariant distributions can simply be characterized by power laws close to 0 and ∞. The exponents reported in Refs. 2 and 3 depend weakly on the exponent of these power laws, and therein lies the reason for the universality: The scaling behavior of the breakdown process is universal in that it only depends on the exponent of the scale-invariant component of the threshold distribution. We are thus dealing with a weaker form of universality than that found near critical points.

In Sec. IV, we show how results previously obtained in the literature fit into our analysis. The distribution of *weak* bonds is important in the beginning of the breakdown process. However, the distribution of the *strongest* bonds also play a role in the development of the breakdown process, in that they arrest or redirect the development of large cracks. This is also discussed in some detail in Sec. IV.

Algorithms where the disorder is introduced *before* the breakdown process starts, we will refer to as *quenched-disorder* models. The model we discuss in Sec. V belongs to the class of models we may refer to as *annealed-disorder* models, where there is no disorder at the beginning of the breakdown process, but is introduced by stochasticity in the bond-breaking algorithm. The particular model we describe in this section corresponds to a random-fuse model with annealed disorder.[10] This model cannot be compared directly to the fuse model discussed so far. However, we use an algorithm to *reconstruct*[11] the quenched disorder that would have produced the *same* breaking pattern—spatially and temporally—with the random-fuse model. The disorder we find has a nontrivial scale-invariant component, which results from the current distribution that has appeared in the network throughout the breakdown process and thus fit the picture we are proposing well.

The annealed-disorder model with an added connectedness requirement for the cracks that develop is equivalent to the *dual*[12] of the dielectric-breakdown model of Niemeyer *et al.*,[13] i.e., equivalent to the scalar version of the fracture-growth model of Louis and Guinea[14] and of Hinrichsen, Hansen, and Roux.[15] Thus, we are able to investigate the connection between quenched disorder and annealed disorder in Sec. VI and show that they can all be fitted into the same mold. Using the above-mentioned correspondence between quenched and annealed disorders, we studied quenched and annealed single-crack models. There is no apparent difference in the scaling properties between these two models.[11] Thus, also in this case, which touches closely upon the dielectric-breakdown model and thus diffusion-limited aggregation (DLA), we find an underlying scale-invariant disorder.

In Sec. VII, we draw some general conclusions from these results.

II. THE FUSE MODEL AND OTHER QUENCHED-DISORDER MODELS

The scaling properties we discuss in this section have been seen in network models with *quenched* disorder, such as the fuse model:[7] We imagine a network where each bond is an electrical fuse; i.e., it has a constant conductance equal to, say, unity up to a threshold current t. If the current flowing through the fuse, i, exceeds this threshold current, the fuse "burns" out; i.e., it turns irreversibly into an insulator with zero conductance. This is illustrated in Fig. 1(a). The threshold t is drawn from some probability distribution $p(t)$. We now imagine sending a current through the network, I, which is slowly being increased, and record several parameters describing the subsequent breakdown process. "Slow" here means that the currents have time to relax into the pattern dictated by the Kirchhoff equations between each time a new fuse burns out—this is in some sense in a quasistatic limit.

In practice, however, since the Kirchhoff equations are linear, we solve them with $I=1$, and then search for the bond for which the ratio i/t is maximum. This is the

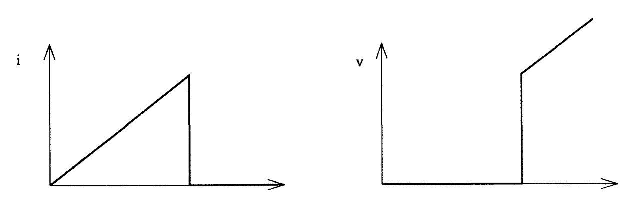

FIG. 1. The characteristics of a single bond in (a) the fuse model and (b) the conductor-superconductor model.

next bond to break, since this is the one that will first reach its maximum load as the external current I is increased. This ratio gives the inverse of the external current through the network, which is necessary to break this bond. Let us call it I_c,

$$I_c = \min\left[\frac{t}{i}\right] . \tag{1}$$

The corresponding voltage across the network is V_c. If we now record V_c versus I_c, the characteristic of the network, and, say, I_c versus n, the number of blown fuses, we find a functional dependence between them:

$$\frac{I_c}{L^\alpha} = F\left[\frac{V_c}{L^\beta}\right] = G\left[\frac{n}{L^\gamma}\right] , \tag{2}$$

where L is the linear size of the network. Thus, all L dependence is absorbed into the three reduced variables I_c/L^α, V_c/L^β, and n/L^γ. The exponents α, β, and γ turn out to be rather insensitive to the probability distribution $p(i_t)$ the thresholds have been picked from, *as long as it is broad enough*. It is one of the major subjects of this paper to investigate what "broadness" means in this connection. It was found that $\alpha \approx \beta \approx 0.75$ and $\gamma \approx 1.75$ within more or less 10% in the elastic models;[2] the exponents found in the electrical-fuse model[3] are listed in Table I.

Earlier work on the random-fuse model did not show such a scaling behavior as Eq. (2) indicates.[8,9] In Ref. 8, a uniform distribution between $1-w/2$ and $1+w/2$ was investigated. Kahng *et al.*[8] found that the properties of the breakdown process were strongly dependent on w. For $w<2$ they found that the number of bonds necessary to break the network apart grew as L, whereas at maximum current, this number was finite (independent of the system size). In the case of $w \to 2$, the number of bonds necessary to break the network apart grew faster than L (assuming a two-dimensional network).

That the number necessary to break the network apart scales as L indicates that the breakdown is caused by one unstable crack growing through the network, as was described in Sec. I. This is very different from the scaling found in Refs. 2 and 3 where the number of bonds necessary to break the network apart scales as L^γ, where $\gamma \approx 1.7-1.8$. However, in a network *without* disorder, we expect the number of broken bonds at breakdown to scale as L. This is so, since the bonds that will be closest to breakdown will be the bonds carrying the maximum current—and these are throughout the entire breakdown process located at the crack tips.

Duxbury and co-workers[9] have studied a threshold distribution of the form $p(i_t) = p\delta(1-i_t) + (1-p)\delta(i_t)$, where $\delta(i)$ is the Dirac δ function. Also in this case it was found that the number of bonds that had to burn out in order to break the network apart was a vanishing fraction of the total number of bonds.

In a recent study,[16] the following question was asked: Suppose the first bond has broken. At what distance is it most likely that the next one will break? With a threshold distribution of the same type as the one used by Kahng *et al.*[8] it was found that for $w<2$ the most likely distance between the first and the second bond that burns out, $\Delta(w)$, is finite. As $w \to 2$, this distance goes to infinity. The third bond that breaks is, for $w<2$, likely to break at a distance from the first two of the same order as $\Delta(w)$, since the breaking of the first two increases the current distribution in this area. This argument may be repeated over and over, and we see that the network breaks apart in a way reminiscent of the way a network without disorder breaks apart, but with an *effective* lattice constant equal to $\Delta(w)$. Only in the limit $w \to 2$ is it impossible to describe the rupture process in this way.

Using the same type of approach, different distributions were considered. Since the distribution $p(t)$ close to 0 seemed to be an important feature, cases where $p(t)$ followed a power-law close to the origin $p(t) \propto t^{-1-\beta}$ were also analyzed. It was found that, depending on the value of β, two regimes could appear. If $\beta > 2$, then an uncontrolled breaking of the structure is expected after a finite number of bonds have broken, whereas for $\beta < 2$, the early stage of fracture is a random dilution of the network (diffuse damage).

TABLE I. The values of the exponents α, β, and γ as a function of the disorder ϕ_0 as measured by de Arcangelis and Herrmann (Ref. 3). The last row in this table refers to the corresponding figure in Ref. 3 from which the data were taken. In Ref. 3, Fig. 1 showed I/L^α as a function of V/L^β, Fig. 2 showed I/L^α as a function of n/L^γ, Fig. 3 the average number of bonds cut to reach the maximum of the I-V characteristics, and Fig. 4 the total number of bonds broken before the networks fall apart. In these two last figures, γ was determined from data collapse with different sizes L.

ϕ_0	α		β	γ		
0.5		0.80		1.85		1.82
1	0.89	0.89	0.83	1.80	1.65	1.73
2	0.90	0.90	0.85	1.70	1.65	1.58
5	0.92		0.88		1.36	1.32
10	0.90		0.85		0.95	1.11
Fig. in Ref. 3	1	2	1	2	3	4

III. BROADNESS IN THE LANGUAGE OF MULTIFRACTALS

In Ref. 3 it was shown numerically that the current distribution right before the network breaks apart is multifractal when the threshold distribution was chosen to be broad. The same was found for the stress distribution in the elastic networks at rupture.[2] This means that the logarithmically binned histogram of the currents show the following scaling form:

$$N(i,L) \sim L^{f(\alpha)} , \tag{3}$$

where

$$i \sim L^{-\alpha} . \tag{4}$$

If $f(\alpha)$ is a more complicated function of α than of the form $a+b\alpha$, then the current distribution is multifractal.[17] Such a distribution indicates that the network at the final breaking stage is "critical," i.e., that there are no length scales in the problem apart from L. The broadness of the current distribution indicated by Eq. (3) does not develop suddenly in the breakdown process but rather gradually. There is some correlation length ξ that marks the length scale at which the network crosses over from showing the scaling properties of a system with a narrow current distribution, and the scaling behavior of a system with a multifractal current distribution. Then, at a "time" in the breakdown process when the correlation length is less than L, the current distribution is[18]

$$N(i,L,\xi) \sim L^{2+[f(\alpha)-2]\ln\xi/\ln L} , \tag{5}$$

where

$$i \sim L^{-1+(1-\alpha)\ln\xi/\ln L} . \tag{6}$$

This is a result that follows directly from finite-size scaling.

What we have accomplished through Eqs. (5) and (6) is to identify an intensive—in a thermodynamical sense—formulation of the development of the current distribution through the breakdown process. We may use the intensive variable

$$\tau = \frac{\ln\xi}{\ln L} \tag{7}$$

rather than n—the number of broken bonds—as a "time" parameter in the process. Likewise, an intensive current is

$$\alpha(\tau) = 1-(\alpha-1)\tau , \tag{8}$$

and an intensive histogram is

$$f(\alpha(\tau),\tau) = 2-[f(\alpha)-2]\tau . \tag{9}$$

These variables are those that describe the current distribution in the limit $L\to\infty$. [There is no consensus in the literature as to how these quantities are to be defined. For example, Mandelbrot[19] uses a definition corresponding to $f=\ln(tp)/\ln L$ rather than $f=\ln(L^2tp)/\ln L$. The convention we are using corresponds to that of the *random-resistor-network* literature.]

The breakdown process is governed by Eq. (1). Let us now write the threshold distribution in intensive variables rather than the extensive ones, t and $p(t)$. This we do in a way similar to that of the current distribution, Eqs. (3) and (4):

$$L^2 tp(t) = L^{f_t(\alpha_t)} , \tag{10}$$

where

$$t \sim L^{-\alpha_t} . \tag{11}$$

The extra factor t in Eq. (10) is a result of binning the histogram logarithmically. Thus, the intensive thresholds and threshold distribution are

$$\alpha_t = \frac{\ln t}{\ln L} \tag{12}$$

and

$$f_t(\alpha_t) = \frac{\ln[L^2 tp(t)]}{\ln L} \tag{13}$$

in the limit of $L\to\infty$.

As the breakdown proceeds, τ grows from 0 to 1, and the current distribution evolves from a point $\alpha=-1$, $f=2$ to a multifractal curve $f(\alpha)$. At the same time, the threshold distribution evolves. This, since the thresholds belong to the bonds that burn out, new ones cannot be picked. Thus, there is a τ dependence in both f_t and α_t. However, while the distribution of currents becomes broader as the breakdown process evolves, the threshold distribution becomes narrower.

The rupture criterion, Eq. (1), in these variables becomes

$$\max\frac{i}{t} \to \max[\alpha_t(\tau)-\alpha(\tau)] . \tag{14}$$

We have no way of using this formulation to predict the detailed shape of $f(\alpha)$, since it is merely a rewriting of our starting point in different variables. However, this makes it possible to deduce some powerful statements on what kind of threshold distributions may allow for the evolvement of multifractality in the current distribution—and presumably on the scaling behavior seen in Refs. 2 and 3.

We have already introduced through Eqs. (12) and (13) the notion of a multifractal spectrum computed from a given distribution. Let us now show that when the distribution does not explicitly depend on the system size, its multifractal spectrum reduces to a very simple curve.

Suppose that the threshold distribution is bounded between $t_< \le t \le t_>$. Then, from extreme statistics,[20] we may estimate the smallest and largest thresholds, $t_<(L)$ and $t_>(L)$, that we expect to find among the L^2 bonds in the network,

$$\int_{t_<}^{t_<(L)} dt\, p(t) = \int_{t_>(L)}^{t_>} dt\, p(t) = \frac{1}{L^2} . \tag{15}$$

Quite generally, we have that $t_<(L)=t_<+\epsilon_<(L)$, and $t_>(L)=t_>-\epsilon_>(L)$, where both $\epsilon_>(L)$ and $\epsilon_>(L)$ tend to 0 as $L\to\infty$. Thus,

$$\frac{\ln[t_<(L)]}{\ln L} \to 0 \quad \text{if } t_< \neq 0 , \tag{16}$$

and

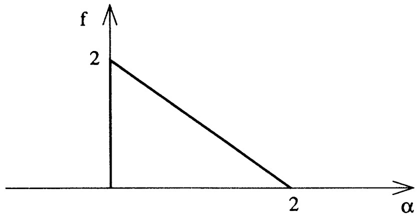

FIG. 2. The uniform distribution between 0 and 1 is expressed in the intensive variables α_t and f_t.

$$\frac{\ln[t_>(L)]}{\ln L} \to 0 \quad \text{if } t_> \neq 0 \; . \tag{17}$$

This result shows that unless either $t=0$ or $t=\infty$ is included in the threshold distribution, it is equivalent to *no disorder* in the limit $L\to\infty$; i.e., it is only a point $\alpha_t=0$, $f_t=2$.

Taking the logarithmic binning into account, a flat distribution between 0 and 1 gives an f_t-α_t curve as shown in Fig. 2. The "Weibull" distributions discussed in Ref. 3 have the form

$$p(t)\sim t^{m-1}e^{-t^m}, \quad 0\le t\le 1 \; . \tag{18}$$

This probability distribution, expressed in terms of $\ln t$ rather than t, has the form

$$p(\ln t)\sim t^m e^{-t^m} \; . \tag{19}$$

Now, using Eqs. (16) and (17), we find

$$f_t(\alpha_t)=\lim_{L\to\infty}\frac{\ln[L^2 p(\ln t)]}{\ln L} = 2-m\alpha_t-\lim_{L\to\infty}\left[\frac{L^{m\alpha_t}}{\ln L}\right] . \tag{20}$$

The last term on the right-hand side of this equation approaches 0 as $L\to\infty$, since $\alpha_t\le 0$. Using Eq. (15), we find that the range of α_t is from 0 to $2/m$.

Similarly, a power law on the interval $0\le t\le 1$,

$$p(t)\sim t^{\phi-1} \; , \tag{21}$$

gives

$$f_t(\alpha_t)=2-\phi\alpha_t, \quad 0\le\alpha_t\le\frac{2}{\phi} \; . \tag{22}$$

Comparing Eqs. (20) and (22), we see that the two distributions have exactly the same scale-invariant form when identifying $m=\phi$.

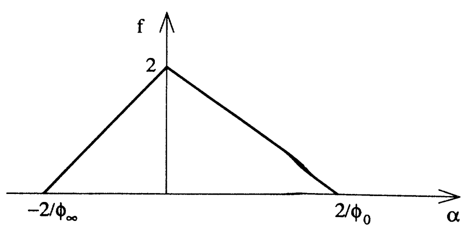

FIG. 3. The general scale-invariant spectrum of a size-independent threshold distribution in the intensive variables α_t and $f_t(\alpha_t)$.

In the general case, we can characterize the behavior of $p(t)$ close to 0 and ∞ by taking the limits

$$\lim_{t\to 0/\infty}\left[\frac{\ln[tp(t)]}{\ln t}\right]=\phi_{0/\infty} \; . \tag{23}$$

These two numbers, ϕ_0 and ϕ_∞, are enough to construct the multifractal spectrum of $p(t)$. It consists in three points of coordinates, $(-2/\phi_\infty,0)$, (0,2), and $(2/\phi_0,0)$, joined by straight lines (see Fig. 3). From this property it is obvious that very different distributions can share the same spectrum. For instance, an exponential distribution from 0 to ∞ has the same spectrum that a uniform distribution between 0 and 1 does. Both of them are characterized by $\phi_0=1$ and $1/\phi_\infty=0$ (Fig. 2).

IV. SOME RESULTS OBTAINED IN BREAKDOWN AND FRACTURE MODELS

It is apparently surprising to consider not only the small threshold part of the distribution but also the large threshold part. Indeed, if a very fragile bond can initiate the breakdown of the complete lattice, the role of very strong bonds is more subtle, since one very strong element can easily be avoided by the crack.

Let us give some arguments to understand how these strong bonds can play a determining role and obtain in addition a criterion for a nontrivial behavior. Let us consider here, for the sake of simplicity, a distribution of thresholds bounded by a nonzero lower value, and having a long tail up to infinity, say, $p(t)=\beta t^{-1-\beta}$ for $t\in[1,\infty]$. Let us first assume that this tail is unimportant. We would expect, due to the above-presented argument, that eventually after some transient stage, a single straight crack develops and finally disconnects the medium. This crack may start from a locally weak zone. Once the growth of this linear crack has started, then we known precisely where the crack is supposed to propagate. Let us call l the length of this crack, and let us suppose that the lattice size is infinite. We know that the current flowing at the tip of the crack, i_{tip}, scales as

$$i_{\text{tip}}\propto\sqrt{l} \; . \tag{24}$$

On the other hand, we can use the form of the distribution $p(t)$ to determine the value of the largest threshold, $t_{\max}$. The crack will encounter[20]

$$t_{\max}\propto l^{1/\beta} \; . \tag{25}$$

Therefore, by forming the ratio of these two quantities, we can write the scaling of the macroscopic breaking current $I(l)$ necessary to create a crack of length l:

$$I(l)\propto l^{1/\beta-1/2} \; . \tag{26}$$

From this result, we can distinguish two regimes according to the value of β. If $\beta>2$, then $I(l)$ will decrease with increasing l; otherwise it will increase. Let us note that we chose here a very particular scenario for the fracture process. Obviously, the current $I(l)$ is an upper bound for the breaking current.

This scenario should be compared with an alternative one, which is extremely simple; i.e., bonds break in increasing order of their threshold. This corresponds to a random dilution of the lattice. Since the distribution $p(t)$ is bounded below by a positive number (here 1), the breaking current is constant in the early stage of fracture, independent of the number of bonds broken.

When $\beta > 2$, the decrease of $I(l)$ with the number of bonds broken makes the straight-crack scenario much more favorable than the random dilution. We indeed expect in this case that the fracture will only produce a single straight crack; i.e., it will be identical to what is expected in a disorderless material.

On the contrary, for $\beta < 2$, $I(l)$ will increase according to the first scenario [Eq. (26)]. The physical meaning of this property is clear. Any linear crack will be arrested by a bond of very high strength, and thus the current has to be increased in order to follow the prescribed path. However, this increase is not realistic. We have indeed seen that the random-dilution hypothesis does not need any increase of the external current. Thus, the first scenario cannot be followed. It is difficult to say whether the random dilution will be a good approximation of the process or not, but we can use the fact that this scenario gives an upper bound on the breaking current. Thus we expect in all cases to see that the breaking current will decrease or remain constant as the number of broken bonds increases. Simultaneously, we obtain that the number of bonds at the final stage of rupture varies faster than L, where L is the system size. In the case of the random dilution, it goes as L^2, as we will show later.

Therefore, we see that the role of strong bonds may affect the development of the fracture, by arresting cracks and forcing the nucleation of new ones, until finally the crowding of microcracks will produce a local enhancement of the currents at the tips of some of them. Clearly this situation is much more difficult to analyze. However, we will show some data relative to numerical simulation of such distributions that suggest that the proportion of bonds broken when the fracture becomes unstable is finite and approaches a constant when the system size increases, in contrast to what is seen for the opposite situation, i.e., a power-law distribution close to zero threshold.

Let us again mention an example that shows the importance of strong bonds. It is the example of a simple hierarchical one-dimensional medium studied both analytically and numerically by Gabrielov and Newman.[21] The model consists in connecting two borders by strings grouped in pairs hierarchically. Each pair of strings is grouped into a bundle, which are themselves assembled by pairs, and this is at all levels. The grouping of strings and bundles is used to model the load sharing. At a given level, if two elements are not broken, they share the load equally, and if one is broken, the other carries the load entirely. Using this simple model it is possible to study the breaking of the structure, after having assigned at random local breaking thresholds according to some probability distribution $p(t)$. The result of Gabrielov and Newman[21] is the following: If the distribution $p(t)$ reaches zero at infinity faster than $1/t^2$, then the mean breaking strength of the structure decreases as $1/\ln[\ln(L)]$, where L is its size. It is easy to complement their result for distributions that behave as $t^{-1-\beta}$ at infinity for $\beta < 1$. In this case, the breaking strength of the medium goes as $L^{(1-\beta)/\beta}$.

We would like to emphasize in this example the fact that only the behavior of $p(t)$ close to infinity is important, and moreover, only the index ϕ_∞ introduced above [Eq. (23)] allows us to characterize the scaling behavior entirely, irrespective of rest of the distribution. In this simple example, since the structure is a one-dimensional array of bonds in parallel, only the large threshold part is relevant. If the strings had been placed in series, only the weak part would have contributed. For a two- (or more-) dimensional medium, both series and parallel contributions are mixed, and thus both sides of the distribution are relevant. Let us finally mention the parallel between this hierarchical model and the above-presented argument. The difference of limit value for β for a nontrivial regime comes from the load sharing rules, but besides that the spirit of the result is similar, since the localization of a single crack has a one-dimensional geometry.

Let us now see how the results obtained in previous studies of brittle fracture can be fitted into the picture. In Ref. 8, Kahng *et al.* studied the case of a uniform distribution, $p(t) = 1/w$ for $t \in [1-w/2, 1+w/2]$ They observed that, for all values of w less than 2, the uncontrolled fracture of the medium occurred after a finite number of bonds were broken, independent of the system size. Kahng *et al.* also noted that as w approached 2, a different behavior was found. Let us now rephrase this result in terms of scale-invariant distribution. The spectrum of the distribution for $w < 2$ is reduced to a point, as noted previously, since neither 0 nor ∞ are included in the distribution. We thus indeed expect to see the behavior of a disorderless material, as observed numerically and demonstrated theoretically.[8] For $w = 2$, $p(t)$ is a uniform distribution between 0 and 2, and thus Eq. (23) shows that $\phi_0 = 1$. We see clearly that the limit $w \to 2$ corresponds to a change in the scale-invariant part of the distribution as noted previously. We will come back to the behavior in this limit in the discussion of the results of Refs. 2 and 3.

In Ref. 16, a similar problem was addressed with equivalent conclusions. A different distribution was also studied, i.e., a power-law distribution close to the origin. This is exactly the case of a variable ϕ_0 and a fixed $1/\phi_\infty = 0$. In this case, the conclusion was that for ϕ_0 less than 2 a disorderless behavior was observed, but for ϕ_0 greater than 2 a more complex scaling behavior could be expected. Again, we see that naturally the criterion for determining the scaling behavior of the model lies exclusively on ϕ_0 (since ϕ_∞ is fixed).

In Ref. 9, Duxbury and co-workers studied the fracture of a randomly diluted medium. The distribution of breaking thresholds can be written as $p\delta(t-1) + (1-p)\delta(t)$, where δ is a Dirac distribution, and p is the fraction of present bonds. In this case, the scale-invariant part of the distribution cannot be obtained by the direct use of Eq. (23). However, since our interest is the effect of a change of scale, we can in this case apply a renormalization-group argument to study the evolution of this distribution as the scale changes. It is a classical result from percolation theory, that if p is equal to the percolation threshold p_c, then the distribution $p(t)$ is invariant under rescaling. For values of p larger than p_c (i.e., the lattice is not yet broken in the initial state) upon rescaling, the disorder will disappear, and the effective value of p defined at a scale L will converge to 1 as L goes to infinity. Therefore, away from the percolation threshold, the spectrum of the distribution is equivalent to no disorder at all; i.e., in our language $1/\phi_0=1/\phi_\infty=0$. Thus we expect that the scaling properties of the fracture of such systems is, at the limit of a large-size lattice, that of a disorderless material. Thus, it seems that the results of Duxbury and co-workers[9] apply to a first transient behavior.

In Refs. 2 and 3, various types of distributions were considered and studied numerically. All of them could be characterized by a finite ϕ_0 and $1/\phi_\infty=0$. In particular, the Weibull distribution considered in Ref. 3, $p(t)\propto t^{m-1}e^{-t^m}$, is characterized by $\phi_0=m$ and $1/\phi_\infty=0$. We have already mentioned [Eq. (2)] the type of scaling observed numerically for small values of ϕ_0. As ϕ_0 increased, up to 10 in Ref. 3, a progressive change of behavior was seen towards that of a disorderless material. Table I recalls some results obtained for $\phi_0=5$ and 10, and we see clearly the trend that the scaling exponents α and β approach a value of 1, as expected for a medium with no disorder. The fact that the change of behavior appears progressively is certainly a consequence of the finite size of the systems studied, and we believe that at the thermodynamic limit the transition is much more abrupt and probably occurs for a finite value of ϕ_0 as argued for in Ref. 16.

In Ref. 4, the conductor-superconductor transition in the presence of disorder was studied. Let us note that this problem cannot be compared directly to the fracture problem, but the same conclusions as the ones we have presented do apply in this case: Only ϕ_0 and ϕ_∞ determine the scaling properties, but these might be different from those of the fracture. The disordered superconductor was there, modeled by a network consisting of bonds having characteristics as shown in Fig. 1(b). When the current i through an element is larger than a threshold t, the bond acts as an ohmic resistor, but if the current drops below t it turns irreversibly into a superconductor. The disorder is introduced through a distribution of the thresholds $p(t)$. An external current I is lowered from a value high enough so that all elements are in the ohmic state to a low enough value so that all elements are superconducting. Three different threshold distributions were used: (1) The flat distribution discussed above, (2) an exponential distribution of the form

$$p(t)\sim e^{-t},\quad 0\le t<\infty\ , \tag{27}$$

and (3) a power law, $p(t)=\beta t^{-1-\beta}$ for $1\le t<\infty$, with the exponent $\beta=0.5$. With the analysis just presented, we see that the exponential and the uniform distribution are characterized by the same indices, $\phi_0=1$ and $1/\phi_\infty=0$, and indeed these two cases led to similar conclusions. The last case, of a power-law distribution led, in the early stage, to scaling laws comparable to those of fracture with a small ϕ_0.

From the different examples we mentioned, we see that all reported results on the brittle fracture of disordered media can be fitted into our analysis. We did not find any example of two distributions having the same scale-invariant part and different scaling behaviors. In the following we will show some additional examples that again fit into the framework. These examples will deal with annealed models and with single cracks simulations.

V. ANNEALED-DISORDER MODELS

In Ref. 10, a breakdown model with *annealed* disorder rather than *quenched* disorder was introduced. In the quenched-disorder models discussed so far, the disorder was introduced at the beginning of the breakdown process and not changed thereafter: The algorithm modeling the breakdown process is a deterministic one. In the annealed-disorder models, there is no disorder present in the network at the beginning of the breakdown process, but disorder is created by a *stochasticity* in the algorithm that is used to determine which bond is to break next. In particular, the algorithm used in Ref. 10 was the following: The setup is a two-dimensional network between two bus bars hooked up to an external voltage source. The current distribution is recalculated between each time a bond is broken by solving the Kirchhoff laws. Which bond to break next is then determined by choosing one with a probability proportional to the current it carries raised to a power η. It should be noted how close this model is to the fracture-growth model[14,15]—the only difference being that the growing cracks are not forced to be connected.

For small values of η, it was found that a *finite* fraction of bonds had to break in order to break the network apart. For $\eta=1$, and on a square lattice, this finite fraction is about 30% in the limit of $L\to\infty$. This percentage drops rapidly to zero with increasing η, and an argument along the lines of that presented in Ref. 16 hints at there being a transition in behavior for $\eta=2$ so that the fraction of broken bonds at a rupture becomes identical to 0 for $\eta>2$.[10] For $\eta>2$, this model and the scalar version of the fracture-growth model[14,15] become equivalent.

Seemingly, this model cannot provide an I-V characteristics of the breakdown process. The reason for this is that there is no absolute scale by which we may compare the currents flowing in the network at one stage in the process with the currents flowing at a different stage. In the quenched-disorder models, this scale is provided by the threshold distribution.

It is possible to *reconstruct*[11] a threshold distribution that with the quenched-disorder algorithms of Ref. 7 would produce the same breaking patterns as with the annealed-disorder algorithm. This we do by using the following algorithm: In the annealed model, assign to each bond j a number t_j that is set equal to 1. Then pick bonds with the annealed-disorder algorithm. Suppose now, that bond k was just picked. All t_j assigned to bonds *that have not yet broken* are then updated according to the formula

$$\frac{i_j}{t_j(\text{new})}=\min\left[\frac{i_j}{t_j(\text{old})},\frac{i_k}{t_k(\text{old})}\right]. \tag{28}$$

The idea behind this formula is to adjust upwards all bonds whose thresholds are such that the corresponding bonds should have broken *before* the one that actually broke. If we now use the quenched-disorder algorithm with this distribution of thresholds, we will reproduce exactly the same cluster as the annealed algorithm did.

After the network has broken apart, bonds will either have broken down or not. Only the reconstructed thresholds of these bonds that broke down have a meaning as thresholds in the sense of Eq. (1). These are those that set the scale leading to an I-V characteristics for the breakdown process.

The log-log histogram of the threshold distribution for $\eta=3$ is shown in Fig. 4. We see after a small transient regime (the first bonds broken, or the smallest thresholds) the distribution can be very well fitted by a power law with a small value of ϕ_∞. Respectively for $\eta=1$, 2, 3, and 4, the values of ϕ_∞ are 0.07, 0.16, 0.43, and 0.46. For the largest thresholds, we see a departure from this power-law behavior.

There is a question as to how to compare the reconstructed thresholds taken from one sample in the ensemble with those from other samples. The algorithm of Eq. (28) has a normalization built into it in that the smallest thresholds always will be equal to 1. This normalization leads to rather large fluctuations. We have in Fig. 4, however, shown the histogram as being taken when the second moment of the threshold distribution is fixed and set equal to 1. This leads to a nice data collapse for the larger thresholds—i.e., α_t near zero—while the finite-size effects makes the small thresholds—large negative values of α_t—decrease with increasing L. It is clear that the linear part of the histogram in Fig. 4 increase with increasing lattice size L.

This demonstrates that the reconstructed threshold distribution is scale invariant. In all cases, the exponent ϕ_∞ is smaller than 2 as expected from the above-presented argument, in order to have a scaling behavior different from the no-disorder case.

In Fig. 5 we show the distribution of thresholds based on the bonds that did *not* break down—i.e., the distribution from which the reconstructed thresholds are picked. We will, in the following, demonstrate why we expect there to be two power laws appearing in distribution of unbroken thresholds—one for small thresholds and one for large thresholds, why the exponents of these two power laws should be equal, and how these exponents are related to those governing the current distribution appearing in the network during the breakdown process.

It seems that there is a rather clear power law on the small threshold side of the histogram of Fig. 5, but whether there is a power law for large thresholds is less certain. It should be noted, however, that these histograms are known to converge very slowly to their asymptotic shapes, and the finite-size effect are very different for small and large values of α_t. For example, the normalization used keeping the second moment of the distribution constant introduces an asymmetry. The slope of the power law on the small-α_t side is about 1.5. A power law with slope -1.5 is not inconsistent with the behavior of the histogram on the large-α_t side.

The basis for the following discussion of the distribution of the unbroken thresholds is that this distribution lacks an absolute scale, which is present in the distribution of thresholds belonging to bonds that *have* broken. This may be seen by rewriting Eq. (28) as

$$\frac{t_j(n+1)}{t_k(n)}=\max\left[\frac{t_j(n)}{t_k(n)},\frac{i_j(n)}{i_k(n)}\right], \tag{29}$$

where n is the number of broken bonds so far. We now

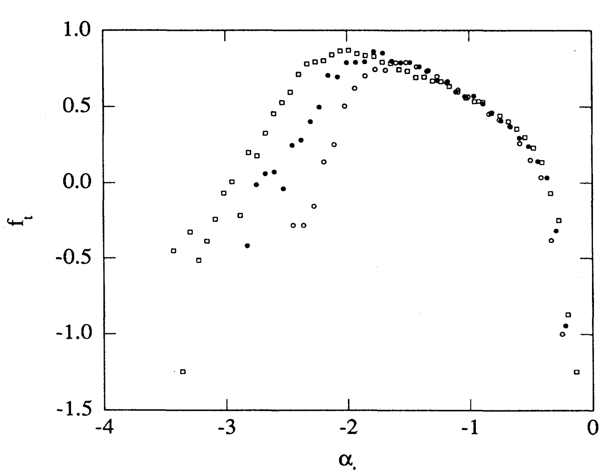

FIG. 4. The reconstructed thresholds, based on the bonds that broke throughout the breakdown process, for $\eta=3$ in intensive variables α_t and $f_t(\alpha_t)$ for lattice sizes $L=20$ (○), 30 (●), and 40 (□).

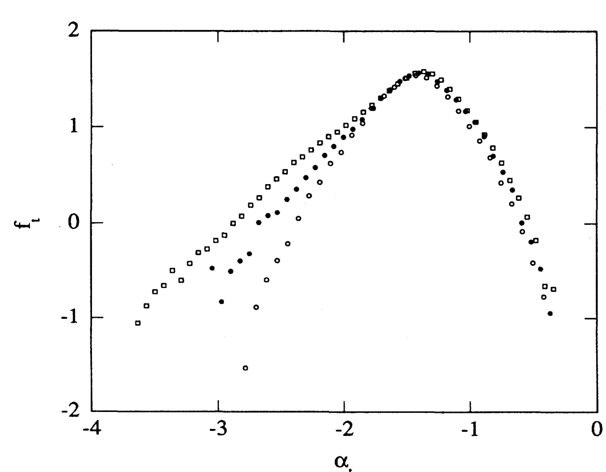

FIG. 5. The reconstructed thresholds, based on the bonds that did *not* break, for the same networks as in Fig. 4. We note that the maximum f_t value is not 2—the spatial dimension of the lattice—but a lower value. This is a finite-size effect. The maximum value of f_t approaches 2 in the limit $L\rightarrow\infty$.

introduce the ratio

$$r_j(n+1)=\frac{t_j(n+1)}{t_k(n)} . \tag{30}$$

Equation (29) then becomes the following if bond l broke at step $n-1$ and bond k at step n:

$$r_j(n+1)=\max\left[\frac{r_j(n)}{r_l(n)}, \frac{i_j(n)}{i_l(n)}\right] . \tag{31}$$

We now see that scaling $r_j(n)\rightarrow c(n)r_j(n)$ makes no difference in the way Eq. (31) updates the thresholds. In particular, we could choose

$$c(n)=\frac{1}{r_l(n)} , \tag{32}$$

where bond k broke at step n. This choice makes it particularly easy to analyze the relative distribution of thresholds belonging to unbroken bonds.

This choice of scale leads to the following updating rule for the thresholds:

$$r_{jk}(n+1)=\max\left[r_{jk}(n), \frac{i_j(n)}{i_l(n)}\right] . \tag{33}$$

The absolute values of the ratios between the thresholds will now be completely different from what Eq. (28) would give. However, the *relative* distribution between them will not be changed.

The updating rule of Eq. (33) gives rise to a distribution of ratios between the thresholds that is easy to analyze within the framework of extreme statistics.[20] We assume no correlations between t_j and i_j at a particular step in the breakdown process. Then Eq. (33) tells us that $r_j(n)$ is the largest ratio i_j/i_k—where k refers to bonds that have broken—that has ever appeared through the n first steps of the breakdown process. Thus, we have that[20]

$$p(r)=n[1-\Pi(r)]^{n-1}\pi(r) , \tag{34}$$

where π and Π are the distribution and the cumulative distribution of i_j/i_k throughout the breakdown process, and $p(r)$ is the distribution of rescaled ratios. Following Ref. 18, we will, in a moment, demonstrate that the distribution π is a power law,

$$\pi(i_j/i_k)\sim(i_j/i_k)^{-\beta-1} . \tag{35}$$

This leads to the cumulative distribution $P(r)$ in Eq. (34) being of the form[20]

$$P(r)\sim e^{-r^{-\beta}} , \tag{36}$$

which for large rescaled ratios r behaves as

$$P(r)\sim 1-r^{-\beta} , \tag{37}$$

i.e., a power law similar to that followed by i_j/i_k.

The behavior of $\pi(i_j/i_k)$ for small values of the argument is also that of a power law,

$$\pi(i_j/i_k)\sim(i_j/i_k)^{\beta-1} , \tag{38}$$

but with the sign of the exponent reversed. An identical argument that led to Eq. (37), now leads to

$$P(r)\sim 1-r^{\beta} \tag{39}$$

for the small-threshold behavior of the unbroken threshold distribution.

The natural question to ask now is where the exponent β appearing in Eqs. (35)–(39) comes from. There are two steps involved in showing this. The first one is based entirely on the discussion of Roux and Hansen,[18] where it was shown that a histogram of the currents in bonds picked in a breakdown process will show two power laws in the form of a wedge if the distribution develops towards a multifractal limit, as in Eqs. (8) and (9). The slopes of the two power laws do not show the symmetry indicated by Eqs. (37) and (39). We show such a histogram for $\eta=3$ in Fig. 6. It is, however, the *ratio* between two currents appearing in the network at each stage in the breakdown process that we are interested in. When the current distribution is multifractal, the distribution of possible ratios is also multifractal. A simple steepest-descent calculation shows this. The histogram of the multifractal current distribution is given by Eqs. (3) and (4). If we define the current ratios, $\rho=i_j/i_k$ as

$$\rho\sim L^{-\alpha_\rho} . \tag{40}$$

Then the histogram of the ratios at the breakdown point is given by

$$N_\rho(\rho)\sim L^{f_\rho(\alpha_\rho)}\sim L^{f(\alpha(\rho))+f(\alpha(\rho)-\alpha_\rho)-2} , \tag{41}$$

where $\alpha(\rho)$ solves the equation

$$\frac{df(\alpha)}{d\alpha}+\frac{df(\alpha-\alpha_\rho)}{d\alpha}=0 . \tag{42}$$

The function $f_\rho(\alpha_r)$ is symmetric about the α_ρ such that $f(\alpha(\rho))$ is maximum. Now, again following Ref. 18, the histogram of these ratios, collected throughout the entire

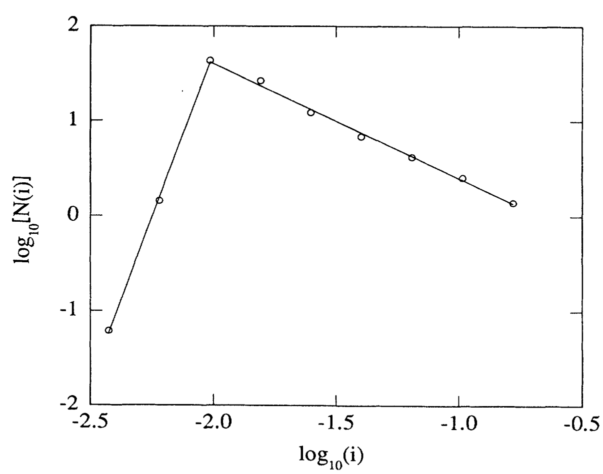

FIG. 6. A Histogram of the currents of the bonds broken in the annealed-disorder model of Ref. 13, when the total current flowing in the network was fixed equal to unity. The wedge shape indicates that the current distribution is developing towards a multifractal distribution. The exponents of the two power laws are +6.9 and −1.2 for $\eta=3$.

breakdown process must consist of two power laws and the exponents of these two power laws must be equal, since the underlying $f_p(\alpha_p)$ curve has a symmetry axis.

The distribution of reconstructed thresholds belonging to bonds that have broken is much more complicated than the distribution that we have just discussed. We note that it is the histogram of the rescaling factor defined in Eq. (32), which constitutes the histogram of the thresholds belonging to bonds that broke. This rescaling factor we took out of the problem in the subsequent discussion leading to the two power laws in the distribution of unbroken thresholds. Thus, there is no simple connection between the histogram of Fig. 5, and the one threshold that broke; see Fig. 4.

Once more, let us stress here the key role played by the multifractal representation which form the basis of the scale-invariance concept. Using the reconstructed threshold distribution, it is possible to study the I-V characteristic of the fracture process. This is shown in Fig. 7 for $\eta=1$.

The reconstruction algorithm obviously allows us to reproduce exactly the annealed disorder model with a quenched disorder, since it has been fitted to this purpose. However, it might very well be that spatial correlations are present in the distribution of thresholds. If it were so, then the reconstruction would be a purely academic exercise, since a random shuffling of the threshold would destroy the correspondence. It is thus extremely important to test the existence or lack of existence of spatial correlation. In order to do so, we first note that we cannot simply reshuffle the thresholds and run the quenched model. The reason is that the bonds that have not been broken in the annealed model are not given a real threshold but a lower bound on it. So we first extrapolate the result on the distribution of real thresholds, and extract a scale invariant part of the distribution. We chose the simplest distribution having the same spectrum, i.e., a power-law distribution from 1 to ∞.

Now, we can test the presence of correlations by a direct study of the quenched disorder model with the power-law distribution. Figure 8 shows the characteristic obtained in this case. We can see that it is extremely comparable to the one recorded during the reconstruction (Fig. 7). In addition, we also studied the number of bonds to be broken to reach an uncontrolled fracture of the medium. This number of bonds scales as L^2, where L is the lattice size as shown in Fig. 9. This scaling is identical to the one observed directly on annealed models with the corresponding η, see Fig. 10.

We may ask what happens in the limit of $\eta \to 0^+$. This limit is the *screened percolation* limit.[22] In this limit, bonds are broken at random as long as they belong to the current-carrying backbone. This problem is related to the usual percolation problem,[23] where bonds are removed—i.e., broken—at random whether they are part

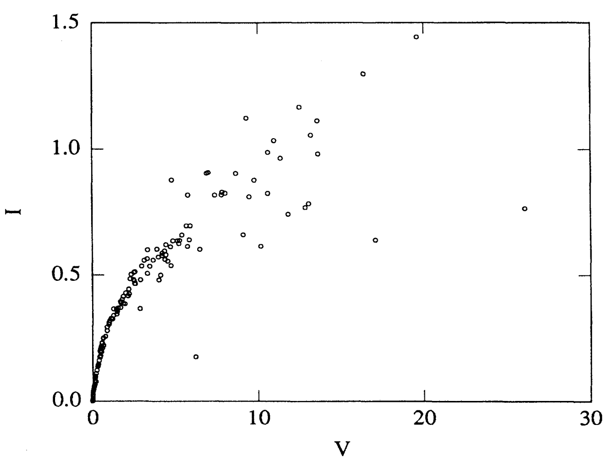

FIG. 7. The reconstructed I-V characteristics of the annealed model of Ref. 10, with $\eta=1$, and $L=20$.

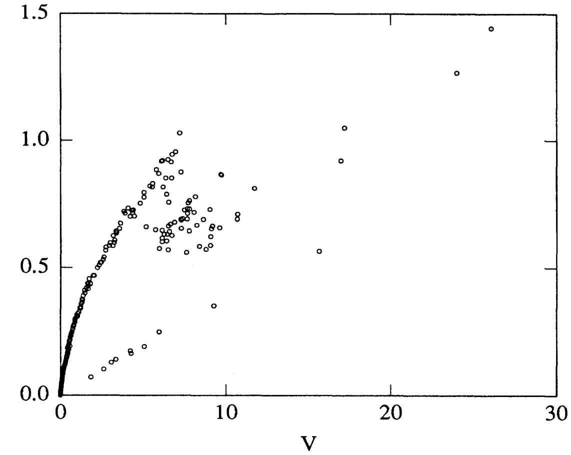

FIG. 8. The I-V characteristics of a quenched-disorder model where the thresholds are distributed as a power law $t^{-\beta-1}$, where $\beta=0.07$, and $t<1$. The value of β corresponds to the one found in the reconstruction algorithm, Eq. (28) for $\eta=1$. This characteristics is similar to that of Fig. 7. Since there is an undetermined scale factor both in V and I when comparing this figure with Fig. 7, we have normalized both axes in this figure to match those of Fig. 7.

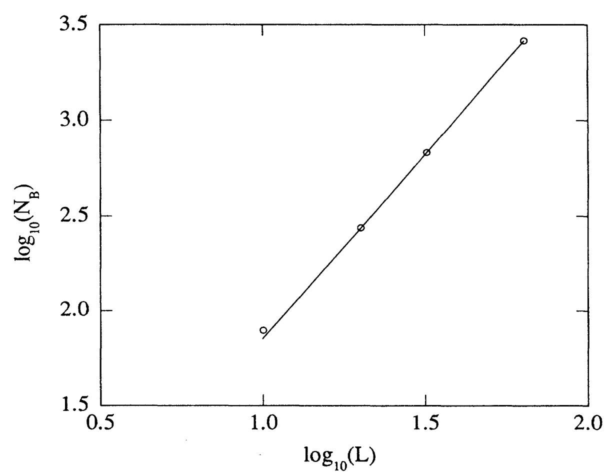

FIG. 9. Number of bonds broken, N_B, for the fracture of the annealed-disorder model plotted against the lattice size L, for $\eta=1$. The slope of the line, which is a least-squares fit based on the sizes $L=20$, 32, and 64, has a slope of 1.93. We expect that this slope will tend towards 2 in the limit $L \to \infty$.

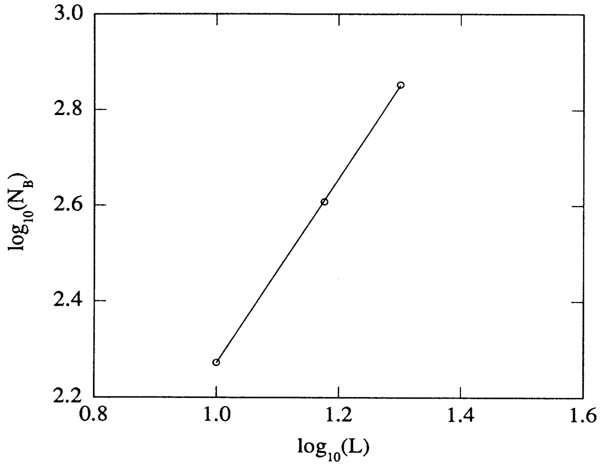

FIG. 10. Number of bonds broken for the fracture of the quenched disorder model, where the disorder is based on a power law, $t^{-\beta-1}$, where $\beta=0.07$ and $t<1$. This value of β corresponds to the one found in the reconstruction algorithm, Eq. (28) for $\eta=1$. The straight line is a least-squares fit based on all data points. The slope is 1.93. Presumably, this slope will, as in Fig. 9, tend towards 2 in the limit $L\rightarrow\infty$.

of the current-carrying backbone or not. In percolation there is a critical density of unbroken bonds, p_c, that marks the division between conduction or nonconduction in the limit $L\rightarrow\infty$. The density of the current-carrying backbone varies as

$$P_{BB}\sim(p-p_c)^{\beta_{BB}} \tag{43}$$

for $p\geq p_c$. P_{BB} is zero for $p<p_c$ The probability to break a bond in the limit $\eta\rightarrow 0^+$, r is related to p by

$$\frac{dr}{dp}\sim P_{BB}\ , \tag{44}$$

which gives

$$(r-r_c)\sim(p-p_c)^{1+\beta_{BB}} \tag{45}$$

after an integration. The critical density of broken bonds is $r_c=0.44$, while $p_c=\frac{1}{2}$ on the two-dimensional square lattice. Thus, in this limit the network breaks apart at a *finite* concentration of broken bonds also in this limit. This behavior is basically the same as found in the annealed-disorder model for other values of η that are larger than 0 and that form a natural limiting behavior.

In Ref. 22 it was argued that this *screened percolation* behavior would also be the limiting behavior of the quenched fuse models in the limit of and the infinitely broad threshold distribution, such as Eq. (21) in the limit $\phi\rightarrow 0$. This conclusion was reached by noting that in Eq. (1) if the threshold distribution is infinitely broad, the current distribution will never become broad enough to compete with the threshold distribution.

This limit can be reached in two ways: When ϕ_0 or ϕ_∞ or both go to 0, we reach the screened percolation limit. We note that when ϕ_0 decrease, the exponent γ [introduced in Eq. (2)] that gives the scaling of the number of bonds broken at fracture, increases, reaching 1.75 for $\phi_0=0.5$. This trend is consistent with the limit of screened percolation, where $\gamma=2$ and $\phi_0=0$. Considering the other way to reach the limit, i.e., $\phi_\infty\rightarrow 0$, we note that for small values of ϕ_∞ such as the one encountered in the reconstruction of annealed-disorder models, the exponent γ already has the value 2.

VI. A CONNECTION WITH THE DIELECTRIC-BREAKDOWN MODEL

As was discussed in the Introduction, there is an interesting connection between the annealed-disorder model of Ref. 11, which was discussed in the preceding section, and the dielectric-breakdown model.[13] This model in turn is a very close relative of the diffusion-limited aggregation (DLA) model, which has been the focus of a major research effort over the past few years.[24]

The interest in the dielectric-breakdown model in the present context is that it comes close to modeling the last stages of the breakdown processes of the models studied so far in the limit where we can consider the rest of the lattice as being homogeneous. These last stages of the breakdown process are governed by the growth of a single crack, which eventually eats through the network breaking it apart. In this part of the breakdown process no scaling behavior is found.[2,3] The dielectric-breakdown model, or rather the fracture-growth model[14,15] as we will show in a moment, models the unstable growth of a single crack by *demanding* that the crack is to be *singly connected.* This completely suppresses the initial stages of the rupture process where there is a competition between the disorder and the current distribution, leading to diffuse microcracking.

In the dielectric-breakdown model, bonds break down in a stochastic way, as in the model of Ref. 11, but with three important differences. (1) The bonds that are liable to break are *neighbors* of those that have already broken, (2) the bonds that "break" change their resistance from a finite amount to 0, rather than their conductance, and (3) the boundary conditions are different: The network is on all sides connected to grounded bus bars, while the cluster of "broken" bonds is kept at a constant voltage. It has long been known that in two dimensions the two dual situations we have here are equivalent.[12] The influence of the change of boundary conditions as described in (3) but with the connectedness criterion of (1) have been studied in Ref. 15. No change in the fractal dimension of the clusters that were grown were found whether one or the other boundary condition is used. Whether there is a connectedness requirement or not (1) is, however, a major difference between the two models.

The fracture-growth model and the dielectric-breakdown model produce fractal cracks, whose radii of gyration depend on the number of bonds belonging to the crack as a power law.[25] The exponent of this power law—being the inverse of the fractal dimension of the crack—goes from $\frac{1}{2}$ for $\eta=0$ to 1 in the limit $\eta\rightarrow\infty$. The first case is the *screened Eden model* where bonds on the surface are picked at random as long as they carry a current, while the second limit is the simple case where the crack grows in the direction of the largest gradient in the electric potential.

If we demand that only a singly connected crack is to be allowed in the *quenched-disorder* model, whose breaking criterion is given in Eq. (1), we also find cracks with well-defined fractal dimensions. This is much more surprising than in the annealed-disorder model, since the simple arguments presented to describe the various stages of the breakdown processes in the case where diffuse crack growth is allowed seem to indicate that there should be *no* well-defined fractal dimension but rather a crossover from *screened invasion percolation*[26] to a single straight-crack propagation: At the beginning of the breakdown process, the current distribution is narrow and the threshold distribution dominates in Eq. (1). Thus, the bonds that break are those on the surface of the already existing crack that have the smallest thresholds, as long as they carry a current. This is screened invasion percolation. Then follows a regime where the current distribution on the surface of the crack matches the threshold distribution in width. Subsequently, the current distribution dominates, and the growth happens where the current is largest. However, this argument neglects the fact that the sample of thresholds at any given stage of the breakdown process is *not* a random sample of the original threshold distribution. Indeed, these bonds belong to the surface of already existing cracks. This is already clear from the distinction between the Eden model and invasion percolation: If we between each growth step in the invasion percolation model picked the thresholds assigned to the surface bonds anew, we would be dealing with the Eden model. It is this biasing of the threshold distribution that prevents the current distribution from eventually dominating the breakdown process in the quenched-disorder model with a singly connected crack. The well-defined fractal dimension indicates that the competition between the threshold distribution and the current distribution enters a "steady state." This means that the width of the current distribution on the surface and the width of the threshold distribution become equal. It is known that the surface current distribution is multifractal.[27] This implies that the intensive current variable α introduced in Eq. (3) exists at intervals and is not just a single value. The threshold distribution on the surface must therefore, when expressed in the intensive variable α_t, Eq. (12), be defined at the same interval. This shows that the important part of the threshold distribution on the surface must be scale invariant. The surface threshold distribution is a *biased* subset of the entire threshold distribution. Thus, it is obvious that the entire distribution is as broad as the surface threshold distribution—remember that "broadness" as we have used it in this context is a statement about *scaling*. The surface threshold distribution is biased in that it will tend to pick out the smaller values from the entire distribution, thus making the surface distribution seem broader than the entire distribution. However, if the scale-invariant part of the entire distribution is equivalent to the no-disorder case, i.e., it has an α_t having just one value, then the surface distribution will also be equivalent to the no-disorder case. A linear crack will

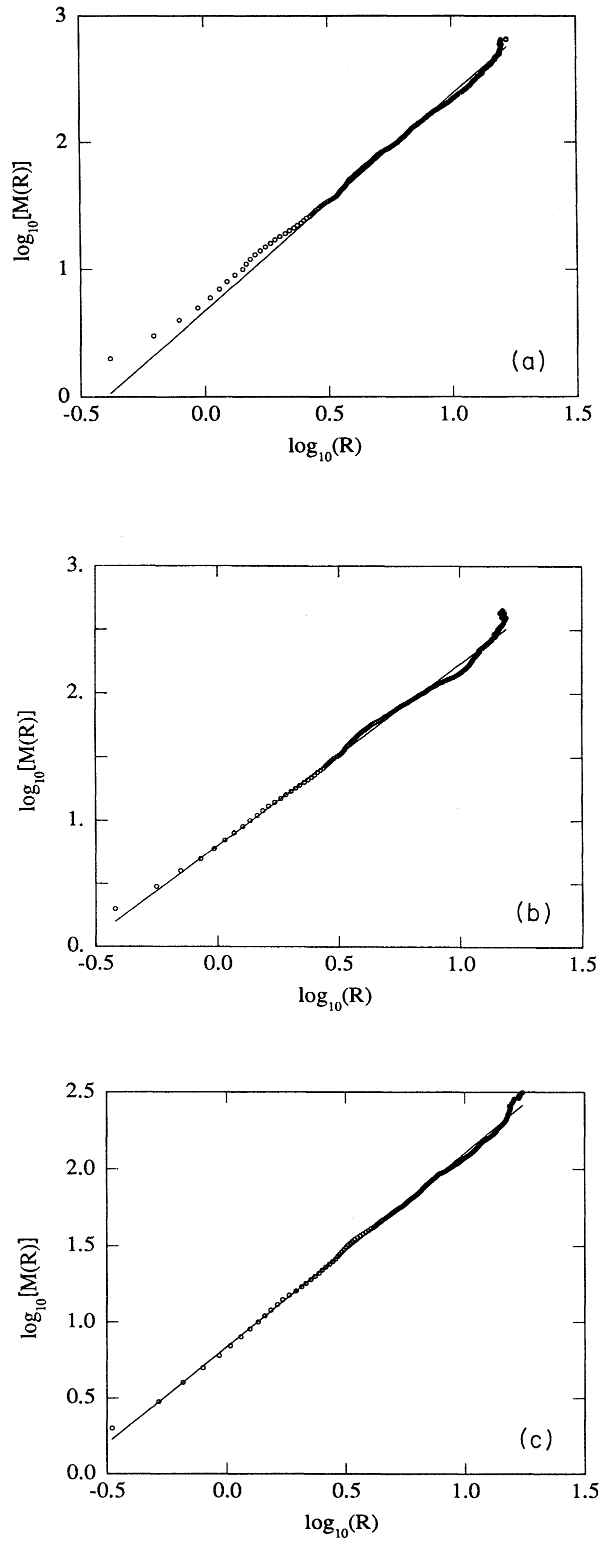

FIG. 11. The radius of gyration as a function of the number of bonds broken for a singly connected crack grown from the reconstructed threshold distribution with (a) $\eta=1$, (b) $\eta=2$, and (c) $\eta=3$. The straight lines are the values for the dielectric-breakdown model found by Amitrano (Ref. 28). Our data are based on averages over 50 64×64 lattices.

be the result. This demonstrates once more the importance of the concept of scale invariant distributions in order to find nontrivial scaling in the breakdown processes.

Let us make the assumption that if there is a threshold distribution that corresponds to the annealed-disorder model with the connectedness requirement, it should be the *same* as that of the annealed-disorder model without it.[11] We may argue for this by noting that the multifractal structure of the current distribution on the surface of a crack probably is dominated by the (fractal) surface of the crack in comparison to the effect of there being other cracks in the network. Furthermore, it is very likely that the *entire* current distribution is dominated by the multifractality of the current distribution on the surface of the cracks—it is here we are most likely to find both the largest and the smallest currents in the system. Since the threshold distribution reflects the current distribution as was argued for in Sec. V, the threshold distribution should get its main features from what happens on the surface of the cracks, and not be much influenced by the interactions between the cracks.

In this case too, we have to use the extrapolated distribution, with no upper limit as argued previously. The distribution selected are thus power laws from 1 to ∞. The fact that the model now explicitly introduces spatial connectedness, is a more sensitive test to the eventual necessity to introduce spatial correlation in the quenched disorder mapping. We check this by direct numerical simulations.

In Fig. 11 we show the result for the radius of gyration *versus* number of broken bonds when using the quenched-disorder algorithm of Eq. (1) with the added constraint that the crack is to be singly connected, and using the power-law threshold distribution gotten from the reconstruction algorithm for $\eta=1$, 2, and 3. The slopes of the straight lines in these plots are 1.70 ($\eta=1$), 1.43 ($\eta=2$), and 1.27 ($\eta=3$). These slopes are those determined by Amitrano for the dielectric-breakdown model.[28] The slopes measured in the dielectric-breakdown model, and the quenched-disorder model are indeed similar, thus suggesting that such a mapping between these two models is correct.[11]

With the discussion of scale-invariant disorder in connection with quenched disorder models where a singly connected crack is demanded, we note that the concept of scale invariant disorder is also relevant in these annealed models: The annealed disorder is equivalent to a scale-invariant quenched disorder, and the nontrivial scaling properties that are observed are determined by the two parameters ϕ_0 and ϕ_∞: tuning η is equivalent to tuning ϕ_0 and ϕ_∞.

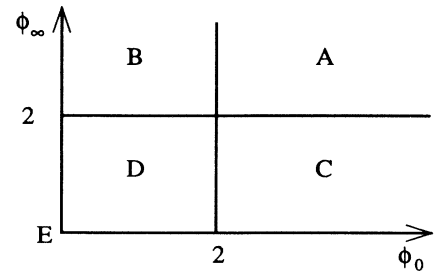

FIG. 12. Different scaling regimes of fracture of disordered brittle material are summarized in this chart, as a function of the two relevant parameters of the threshold distribution ϕ_0 and ϕ_∞. *A*: Disorderless regime with one single crack (it concerns Refs. 8 and 9); *B*: scaling regime with diffuse damage and localization (Refs. 2 and 3); *C*: diffuse damage; *D*: strong disorder case, *E*: (limit ϕ_0 or $\phi_\infty=0$) screened percolation limit (Ref. 22). The frontier *A*-*B*, $\phi_0=2$ is taken from Ref. 16, whereas the frontier *A*-*C*, $\phi_\infty=2$ comes from the argument developed in Sec. IV of this paper.

VII. CONCLUSION

The main purpose of the present paper was to analyze the determining factors that sets the scaling properties of the models of fracture of disordered brittle materials. We showed on many different examples that the behavior of the threshold distribution solely determined the scaling features. In addition, we were able to show that annealed disorder models could be mapped onto quenched disorder ones, thus extending the relevance of the recent analysis. Finally we introduced single-crack models and showed that again the same threshold distribution behaviors were relevant and that annealed- and quenched-disorder models could be compared. Figure 12 summarizes the main results obtained so far as a function of the two parameters ϕ_0 and ϕ_∞.

ACKNOWLEDGMENTS

We thank L. de Arcangelis, J. Feder, H. J. Herrmann, T. Jøssang, and P. Meakin for discussions on this subject. The numerical work was performed at the IBM 3090 computer of IBM Bergen Scientific Centre (Bergen, Norway). A.H. and E.L.H. were supported by the German-Norwegian Research Cooperation. S.R. was supported by an Action Thématique Programmée "Matériaux Hétérogènes" of the PIRMAT (Centre National de la Recherche Scientifique), The Laboratoire de Physique et Mécanique des Milieux Hétérogènes is Unité Associeé No. 857 du Centre National de la Recherche Scientifique.

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